Deconvolutional determination of the nonlinearity in a semilinear wave equation

Nicholas Hu

The University of California, Los Angeles (UCLA)

August 9, 2023

Introduction

Introduction

Consider the semilinear wave equation

$$\begin{cases} (\partial_{tt} - \Delta_x)u(t,x) = F(u(t,x)), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3; \\ u(0,\,\cdot\,) = u_0; \\ \partial_t u(0,\,\cdot\,) = u_1. \end{cases}$$

When the initial data (u_0, u_1) is “small”, the solution to this equation will “behave in the distant future or past” like the solution to a *linear* wave equation

$$\begin{cases} (\partial_{tt} - \Delta_x)u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3; \\ u(0,\,\cdot\,) = u_0^*; \\ \partial_t u(0,\,\cdot\,) = u_1^*. \end{cases}$$

This phenomenon is called “(small-data) scattering”.

Introduction

A commonly asked question in the study of nonlinear dispersive PDEs (e.g., NLS, NLW, Klein–Gordon) is:

"Is the nonlinearity determined by how it scatters solutions?"

Introduction

- Strong assumptions on the nonlinearity (e.g., analyticity) are often made to obtain a positive answer.
- (Sá Barreto–Uhlmann–Wang, 2020): quintic-type nonlinearities ($|F(u)| \approx |u|^5$) for the NLW equation in 3D; complicated argument with many assumptions on the nonlinearity.
- (Killip–Murphy–Vișan, 2023): power-type nonlinearities for the NLS equation in 2D; much simpler argument with few assumptions on the nonlinearity!
- We adapt the techniques of Killip, Murphy, and Vișan to the setting considered by Sá Barreto, Uhlmann, and Wang.

Introduction: admissible nonlinearities

Definition (Admissible nonlinearity)

A nonlinearity $F : \mathbb{R} \to \mathbb{R}$ is considered **admissible** if:

- $F(0) = 0$
- $|F(u) - F(v)| \lesssim (|u|^4 + |v|^4)|u - v|$ (so $|F(u)| \lesssim |u|^5$)
- $F(-u) = -F(u)$

The archetypal admissible nonlinearity is $F(u) = \pm|u|^4 u$. The corresponding equation is known as the **defocusing/focusing energy-critical** NLW equation because the rescaling $u(t,x) \mapsto \lambda u(\lambda^2 t, \lambda^2 x)$ (for $\lambda > 0$) preserves the **energy** of solutions,

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 \, dx \pm \frac{1}{6}\int_{\mathbb{R}^3} |u(t,x)|^6 \, dx.$$

Introduction: solutions of the NLW equation

The NLW equation can be written as

$$\partial_t \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ F(u(t)) \end{bmatrix},$$

so

$$e^{-\mathcal{A}t} \partial_t \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = e^{-\mathcal{A}t} \mathcal{A} \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} + e^{-\mathcal{A}t} \begin{bmatrix} 0 \\ F(u(t)) \end{bmatrix}.$$

Hence

$$\partial_t \left(e^{-\mathcal{A}t} \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} \right) = e^{-\mathcal{A}t} \begin{bmatrix} 0 \\ F(u(t)) \end{bmatrix}.$$

Integrating and rearranging, we obtain

$$\begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} u(0) \\ \partial_t u(0) \end{bmatrix} + \int_0^t e^{\mathcal{A}(t-s)} \begin{bmatrix} 0 \\ F(u(s)) \end{bmatrix} ds.$$

The **propagator** for the linear wave equation is

$$\mathcal{U}(t) := e^{\mathcal{A}t} = \exp \begin{bmatrix} 0 & t \\ t\Delta & 0 \end{bmatrix} = \begin{bmatrix} \cos(t|\nabla|) & \dfrac{\sin(t|\nabla|)}{|\nabla|} \\ -|\nabla|\sin(t|\nabla|) & \cos(t|\nabla|) \end{bmatrix},$$

where $|\nabla| = \sqrt{-\Delta}$.

We therefore have the **Duhamel formula**

$$\begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = \mathcal{U}(t) \begin{bmatrix} u(0) \\ \partial_t u(0) \end{bmatrix} + \int_0^t \mathcal{U}(t-s) \begin{bmatrix} 0 \\ F(u(s)) \end{bmatrix} ds.$$

Introduction: solutions of the NLW equation

Definition (Solution)

A function $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is said to be a **(strong) global solution** of the NLW equation if $(u, \partial_t u) \in C_t^0 \dot{H}_x^1(K \times \mathbb{R}^3) \times C_t^0 L_x^2(K \times \mathbb{R}^3)$ and $u \in L_t^5 L_x^{10}(K \times \mathbb{R}^3)$ for all compact sets $K \subseteq \mathbb{R}$ and if u satisfies the Duhamel formula

$$\begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = \mathcal{U}(t) \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^t \mathcal{U}(t-s) \begin{bmatrix} 0 \\ F(u(s)) \end{bmatrix} \, ds.$$

Theorem (Strichartz estimates)

If $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is a global solution of the NLW equation, then

$$\|(u, \partial_t u)\|_{L_t^\infty \dot{H}_x^1 \times L_t^\infty L_x^2} + \|u\|_{L_t^5 L_x^{10}} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|F(u)\|_{L_t^1 L_x^2}.$$

Introduction: small-data scattering

Theorem (Small-data scattering)

Let F be an admissible nonlinearity for the NLW equation. Then there exists an $\eta > 0$ such that the NLW equation has a unique global solution u satisfying

$$\|(u, \partial_t u)\|_{L_t^\infty \dot{H}_x^1 \times L_t^\infty L_x^2} + \|u\|_{L_t^5 L_x^{10}} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}$$

whenever $(u_0, u_1) \in B_\eta$, where

$$B_\eta := \{(u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) : \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} < \eta\}.$$

(Continued on the next slide.)

Introduction: small-data scattering

Theorem (Small-data scattering, continued)

*This solution **scatters** in $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ as $t \to \pm\infty$, meaning that there exist (necessarily unique) asymptotic states $(u_0^\pm, u_1^\pm) \in \dot{H}^1 \times L^2$ for which*

$$\left\| \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} - \mathcal{U}(t) \begin{bmatrix} u_0^\pm \\ u_1^\pm \end{bmatrix} \right\|_{\dot{H}^1 \times L^2} \to 0 \quad \text{as } t \to \pm\infty.$$

In addition, for all $(u_0^-, u_1^-) \in B_\eta$, there exists a unique global solution u to the NLW equation and a unique asymptotic state $(u_0^+, u_1^+) \in \dot{H}^1 \times L^2$ for which the above holds.

Introduction: small-data scattering

The map

$$(u_0, u_1) \mapsto (u_0^+, u_1^+)$$

implicitly defined by this theorem (on some open ball $B_\eta \subseteq \dot{H}^1 \times L^2$) will be referred to as the **wave operator** and will be denoted W_F.

The map

$$(u_0^-, u_1^-) \mapsto (u_0^+, u_1^+)$$

is known as the **scattering operator** and will be denoted S_F.

Theorem (Determination of the nonlinearity)

Suppose that F and $\widetilde{F}$ are admissible nonlinearities for the NLW equation and that B_η and $B_{\widetilde{\eta}}$ are corresponding balls given by the small-data scattering theorem. If W_F and $W_{\widetilde{F}}$, or S_F and $S_{\widetilde{F}}$, agree on $B_\eta \cap B_{\widetilde{\eta}}$ (that is, the smaller of the two balls), then $F = \widetilde{F}$.

(We will only discuss the case where the wave operators agree as the case where the scattering operators agree can be treated similarly.)

Introduction: determination of the nonlinearity

<u>Proof</u> (outline)

- Small-data scattering and asymptotics for the wave (and scattering) operators

$$W_F = [\text{formula}] = [\text{approximate formula}] + [\text{error}]$$

- Reduction to a convolution equation

$$W_F = W_{\widetilde{F}} \implies H * w = \widetilde{H} * w$$

- Deconvolutional determination of the nonlinearity

$$H * w = \widetilde{H} * w \implies H = \widetilde{H} \implies F = \widetilde{F}$$

Small-data scattering

Small-data scattering

Theorem (Small-data scattering)

Let F be an admissible nonlinearity for the NLW equation. Then there exists an $\eta > 0$ such that the NLW equation has a unique global solution u satisfying

$$\|(u, \partial_t u)\|_{L_t^\infty \dot{H}_x^1 \times L_t^\infty L_x^2} + \|u\|_{L_t^5 L_x^{10}} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}$$

whenever $(u_0, u_1) \in B_\eta$, where

$$B_\eta := \{(u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) : \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} < \eta\}.$$

Small-data scattering

Proof (sketch)

Consider the nonempty complete metric space (X, d), where

$$X := \{u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} : \\ (u, \partial_t u) \in C_t^0 \dot{H}_x^1 \times C_t^0 L_x^2,\ u \in L_t^5 L_x^{10}, \\ \|(u, \partial_t u)\|_{L_t^\infty \dot{H}_x^1 \times L_t^\infty L_x^2} + \|u\|_{L_t^5 L_x^{10}} \leq 2C \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}\}$$

for some constant $C > 0$ and

$$d(u, v) := \|(u, \partial_t u) - (v, \partial_t v)\|_{L_t^\infty \dot{H}_x^1 \times L_t^\infty L_x^2} + \|u - v\|_{L_t^5 L_x^{10}}.$$

Define a map Φ on X using the Duhamel formula,

$$\begin{bmatrix} (\Phi(u))(t) \\ (\partial_t \Phi(u))(t) \end{bmatrix} := \mathcal{U}(t) \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^t \mathcal{U}(t-s) \begin{bmatrix} 0 \\ F(u(s)) \end{bmatrix} ds.$$

Small-data scattering

Proof (sketch)

Show that Φ is a contraction on (X, d) whenever $(u_0, u_1) \in B_\eta$ and η is sufficiently small. By the Banach fixed point theorem, we then have

$$\begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = \begin{bmatrix} (\Phi(u))(t) \\ (\partial_t \Phi(u))(t) \end{bmatrix} = \mathcal{U}(t) \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^t \mathcal{U}(t-s) \begin{bmatrix} 0 \\ F(u(s)) \end{bmatrix} ds.$$

for some unique $u \in X$, meaning that u is a solution!

Small-data scattering

Theorem (Small-data scattering, continued)

*This solution **scatters** in $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ as $t \to \pm\infty$, meaning that there exist (necessarily unique) asymptotic states $(u_0^\pm, u_1^\pm) \in \dot{H}^1 \times L^2$ for which*

$$\left\| \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} - \mathcal{U}(t) \begin{bmatrix} u_0^\pm \\ u_1^\pm \end{bmatrix} \right\|_{\dot{H}^1 \times L^2} \to 0 \quad \textit{as } t \to \pm\infty.$$

In addition, for all $(u_0^-, u_1^-) \in B_\eta$, there exists a unique global solution u to the NLW equation and a unique asymptotic state $(u_0^+, u_1^+) \in \dot{H}^1 \times L^2$ for which the above holds.

Small-data scattering

Proof (sketch)

WLOG, consider $t \to +\infty$. We want to show that

$$(u(t), \partial_t u(t)) \approx \mathcal{U}(t)(u_0^+, u_1^+) \quad \text{in } \dot{H}^1 \times L^2 \text{ as } t \to +\infty.$$

Since $\mathcal{U}(t)$ is unitary on $\dot{H}^1 \times L^2$ for all t and $\mathcal{U}(t)^{-1} = \mathcal{U}(-t)$, this is equivalent to

$$\mathcal{U}(-t)(u(t), \partial_t u(t)) \approx (u_0^+, u_1^+) \quad \text{in } \dot{H}^1 \times L^2 \text{ as } t \to +\infty.$$

We found that

$$\begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = \mathcal{U}(t) \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^t \mathcal{U}(t-s) \begin{bmatrix} 0 \\ F(u(s)) \end{bmatrix} \, ds,$$

so we expect (and indeed, it can be shown) that

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^\infty \mathcal{U}(-s) \begin{bmatrix} 0 \\ F(u(s)) \end{bmatrix} \, ds = \begin{bmatrix} u_0^+ \\ u_1^+ \end{bmatrix}.$$

Small-data scattering

This argument shows that the wave operator is given by

$$W_F\left(\begin{bmatrix}u_0\\u_1\end{bmatrix}\right) = \begin{bmatrix}u_0\\u_1\end{bmatrix} + \int_0^\infty \mathcal{U}(-t)\begin{bmatrix}0\\F(u(t))\end{bmatrix}\,dt,$$

where u is the solution of the NLW equation with initial data (u_0, u_1).

The **Born approximation** to W_F is

$$W_F\left(\begin{bmatrix}u_0\\u_1\end{bmatrix}\right) \approx \begin{bmatrix}u_0\\u_1\end{bmatrix} + \int_0^\infty \mathcal{U}(-t)\begin{bmatrix}0\\F(u_{\text{lin}}(t))\end{bmatrix}\,dt,$$

where

$$\begin{bmatrix}u_{\text{lin}}(t)\\\partial_t u_{\text{lin}}(t)\end{bmatrix} := \mathcal{U}(t)\begin{bmatrix}u_0\\u_1\end{bmatrix}.$$

Small-data scattering

Corollary (Small-data asymptotics for the wave operator)

Suppose that F is an admissible nonlinearity for the NLW equation and that B_η is a corresponding ball given by the small-data scattering theorem. If u_{lin} denotes the solution of the linear wave equation with initial data $(u_0, u_1) \in B_\eta$, then (in $\dot{H}^1 \times L^2$) we have

$$W_F\left(\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}\right) = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^\infty \mathcal{U}(-t) \begin{bmatrix} 0 \\ F(u_{\text{lin}}(t)) \end{bmatrix} \, dt + \mathcal{O}\left(\left\| \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right\|_{\dot{H}^1 \times L^2}^9\right).$$

Proof

Comparing the formula for the wave operator to that of its Born approximation, we see that we need to prove that

$$\left\| \int_0^\infty \mathcal{U}(-t) \begin{bmatrix} 0 \\ F(u(t)) - F(u_{\mathrm{lin}}(t)) \end{bmatrix} \, dt \right\|_{\dot{H}^1 \times L^2} \lesssim \left\| \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right\|_{\dot{H}^1 \times L^2}^9,$$

which we will do by duality.

Proof

Fix some $(v_0, v_1) \in \dot{H}^1 \times L^2$ and let v_{lin} denote the solution of the linear wave equation with initial data (v_0, v_1). Then

$$\begin{aligned}
&\left\langle \int_0^\infty \mathcal{U}(-t) \begin{bmatrix} 0 \\ F(u(t)) - F(u_{\mathrm{lin}}(t)) \end{bmatrix} dt, \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\rangle_{\dot{H}^1 \times L^2} \\
&\quad = \int_0^\infty \left\langle \begin{bmatrix} 0 \\ F(u(t)) - F(u_{\mathrm{lin}}(t)) \end{bmatrix}, \mathcal{U}(t) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\rangle_{\dot{H}^1 \times L^2} dt \\
&\quad = \int_0^\infty \left\langle \begin{bmatrix} 0 \\ F(u(t)) - F(u_{\mathrm{lin}}(t)) \end{bmatrix}, \begin{bmatrix} v_{\mathrm{lin}}(t) \\ \partial_t v_{\mathrm{lin}}(t) \end{bmatrix} \right\rangle_{\dot{H}^1 \times L^2} dt \\
&\quad = \int_0^\infty \left\langle F(u(t)) - F(u_{\mathrm{lin}}(t)), \partial_t v_{\mathrm{lin}}(t) \right\rangle_{L^2} dt.
\end{aligned}$$

Small-data scattering

Proof

By Hölder's inequality, the properties of F, the estimates for u, and the Strichartz estimates, we have

$$\left|\int_0^\infty \big\langle F(u(t)) - F(u_{\mathrm{lin}}(t)), \partial_t v_{\mathrm{lin}}(t)\big\rangle_{L^2}\, dt\right|$$
$$\leq \|F(u) - F(u_{\mathrm{lin}})\|_{L_t^1 L_x^2} \cdot \|\partial_t v_{\mathrm{lin}}\|_{L_t^\infty L_x^2}$$
$$\lesssim (\|u\|_{L_t^5 L_x^{10}}^4 + \|u_{\mathrm{lin}}\|_{L_t^5 L_x^{10}}^4)\|u - u_{\mathrm{lin}}\|_{L_t^5 L_x^{10}} \cdot \|\partial_t v_{\mathrm{lin}}\|_{L_t^\infty L_x^2},$$

where

$$\|u\|_{L_t^5 L_x^{10}}^4 \lesssim \|(u_0, u_1)\|_{\dot H^1 \times L^2}^4,$$
$$\|u_{\mathrm{lin}}\|_{L_t^5 L_x^{10}}^4 \lesssim \|(u_0, u_1)\|_{\dot H^1 \times L^2}^4,$$
$$\|u - u_{\mathrm{lin}}\|_{L_t^5 L_x^{10}} \lesssim \|F(u)\|_{L_t^1 L_x^2} \lesssim \|u\|_{L_t^5 L_x^{10}}^5 \lesssim \|(u_0, u_1)\|_{\dot H^1 \times L^2}^5,$$
$$\|\partial_t v_{\mathrm{lin}}\|_{L_t^\infty L_x^2} \leq \Big\| \|(v_{\mathrm{lin}}, \partial_t v_{\mathrm{lin}})\|_{\dot H_x^1 \times L_x^2} \Big\|_{L_t^\infty} = \|(v_0, v_1)\|_{\dot H^1 \times L^2}.$$

Reduction to a convolution equation

Reduction to a convolution equation

Proposition (Reduction to a convolution equation)

Suppose that F and $\widetilde{F}$ are admissible nonlinearities for the NLW equation. For $\tau \in \mathbb{R}$, define

$$H(\tau) := F'(e^\tau)e^{-4\tau} + F(e^\tau)e^{-5\tau}$$

and define $\widetilde{H}(\tau)$ analogously. Then $H, \widetilde{H} \in L^\infty(\mathbb{R})$, and under the hypotheses of the main theorem, we have

$$H * w = \widetilde{H} * w,$$

where

$$w(\tau) := [\textit{some function in } L^1(\mathbb{R})].$$

Reduction to a convolution equation

The proof of this proposition involves considering a specific solution u_{lin} of the linear wave equation with initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$.

For $\alpha, \varepsilon > 0$, we define

$$u_{\text{lin}}^{\alpha,\varepsilon}(t,x) := \alpha u_{\text{lin}}((\alpha/\varepsilon)^2 t, (\alpha/\varepsilon)^2 x),$$

which solves the linear wave equation with initial data

$$(u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon}) := (u_{\text{lin}}^{\alpha,\varepsilon}(0), \partial_t u_{\text{lin}}^{\alpha,\varepsilon}(0)).$$

Under this rescaling,

$$\|(u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon})\|_{\dot{H}^1 \times L^2} = \varepsilon \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}.$$

In particular, if F is an admissible nonlinearity for the NLW equation and B_η is a corresponding ball given by the small-data scattering theorem, then $(u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon}) \in B_\eta$ for all $\varepsilon \ll \eta$.

Reduction to a convolution equation

This solution will also have the property that

$$u_{\mathrm{lin}}(t,x) = \partial_t v_{\mathrm{lin}}(t,x),$$

where v_{lin} is itself a solution of the linear wave equation with initial data $(v_0, v_1) \in \dot{H}^1 \times L^2$.

For $\alpha, \varepsilon > 0$, we define $v_{\mathrm{lin}}^{\alpha,\varepsilon}$ so that

$$u_{\mathrm{lin}}^{\alpha,\varepsilon}(t,x) = \partial_t v_{\mathrm{lin}}^{\alpha,\varepsilon}(t,x).$$

Then $v_{\mathrm{lin}}^{\alpha,\varepsilon}$ solves the linear wave equation with initial data

$$(v_0^{\alpha,\varepsilon}, v_1^{\alpha,\varepsilon}) := (v_{\mathrm{lin}}^{\alpha,\varepsilon}(0), \partial_t v_{\mathrm{lin}}^{\alpha,\varepsilon}(0)).$$

Under this rescaling,

$$\|(v_0^{\alpha,\varepsilon}, v_1^{\alpha,\varepsilon})\|_{\dot{H}^1 \times L^2} = (\alpha/\varepsilon)^{-2} \varepsilon \|(v_0, v_1)\|_{\dot{H}^1 \times L^2}.$$

Proof (of the reduction)

Observe that

$$
\begin{aligned}
&\int_0^\infty \left\langle F(u_{\mathrm{lin}}^{\alpha,\varepsilon}(t)), u_{\mathrm{lin}}^{\alpha,\varepsilon}(t) \right\rangle_{L^2} dt \\
&\quad = \int_0^\infty \left\langle F(u_{\mathrm{lin}}^{\alpha,\varepsilon}(t)), \partial_t v_{\mathrm{lin}}^{\alpha,\varepsilon}(t) \right\rangle_{L^2} dt \\
&\quad = \int_0^\infty \left\langle \begin{bmatrix} 0 \\ F(u_{\mathrm{lin}}^{\alpha,\varepsilon}(t)) \end{bmatrix}, \mathcal{U}(t) \begin{bmatrix} v_0^{\alpha,\varepsilon} \\ v_1^{\alpha,\varepsilon} \end{bmatrix} \right\rangle_{\dot{H}^1 \times L^2} dt \\
&\quad = \left\langle \int_0^\infty \mathcal{U}(-t) \begin{bmatrix} 0 \\ F(u_{\mathrm{lin}}^{\alpha,\varepsilon}(t)) \end{bmatrix} dt, \begin{bmatrix} v_0^{\alpha,\varepsilon} \\ v_1^{\alpha,\varepsilon} \end{bmatrix} \right\rangle_{\dot{H}^1 \times L^2}.
\end{aligned}
$$

Proof

Since $W_F((u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon})) = W_{\widetilde{F}}((u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon}))$ (for all $\varepsilon \ll \eta, \widetilde{\eta}$), we have

$$\int_0^\infty \mathcal{U}(-t) \begin{bmatrix} 0 \\ F(u_{\text{lin}}^{\alpha,\varepsilon}(t)) \end{bmatrix} dt$$

$$= \int_0^\infty \mathcal{U}(-t) \begin{bmatrix} 0 \\ \widetilde{F}(u_{\text{lin}}^{\alpha,\varepsilon}(t)) \end{bmatrix} dt + \mathcal{O}\left(\left\| \begin{bmatrix} u_0^{\alpha,\varepsilon} \\ u_1^{\alpha,\varepsilon} \end{bmatrix} \right\|_{\dot{H}^1 \times L^2}^9 \right).$$

Hence

$$\int_0^\infty \left\langle F(u_{\text{lin}}^{\alpha,\varepsilon}(t)), u_{\text{lin}}^{\alpha,\varepsilon}(t) \right\rangle_{L^2} dt$$

$$= \int_0^\infty \left\langle \widetilde{F}(u_{\text{lin}}^{\alpha,\varepsilon}(t)), u_{\text{lin}}^{\alpha,\varepsilon}(t) \right\rangle_{L^2} dt + \mathcal{O}(\varepsilon^9) \left\| \begin{bmatrix} v_0^{\alpha,\varepsilon} \\ v_1^{\alpha,\varepsilon} \end{bmatrix} \right\|_{\dot{H}^1 \times L^2}$$

$$= \int_0^\infty \left\langle \widetilde{F}(u_{\text{lin}}^{\alpha,\varepsilon}(t)), u_{\text{lin}}^{\alpha,\varepsilon}(t) \right\rangle_{L^2} dt + \mathcal{O}_\alpha(\varepsilon^{12}).$$

Proof

On the other hand, if $G(u) := F(u)u$, then

$$
\begin{aligned}
&\int_0^\infty \langle F(u_{\text{lin}}^{\alpha,\varepsilon}(t)), u_{\text{lin}}^{\alpha,\varepsilon}(t) \rangle_{L^2} \, dt \\
&\quad = \int_0^\infty \int_{\mathbb{R}^3} G(u_{\text{lin}}^{\alpha,\varepsilon}(t)) \, dx \, dt \\
&\quad = \int_0^\infty \int_{\mathbb{R}^3} \int_0^{u_{\text{lin}}^{\alpha,\varepsilon}(t)} G'(\lambda) \, d\lambda \, dx \, dt \\
&\quad = \int_0^\infty G'(\lambda) \int_0^\infty \int_{\mathbb{R}^3} 1_{\{\lambda < u_{\text{lin}}^{\alpha,\varepsilon}(t,x)\}}(t, x, \lambda) \, dx \, dt \, d\lambda \\
&\quad = \int_0^\infty G'(\lambda) \int_0^\infty \int_{\mathbb{R}^3} 1_{\{\lambda < u_{\text{lin}}(t,x)\}}((\alpha/\varepsilon)^2 t, (\alpha/\varepsilon)^2 x, \lambda/\alpha) \, dx \, dt \, d\lambda.
\end{aligned}
$$

Reduction to a convolution equation

Proof

Thus, if

$$m(\lambda) := \left|\{(t,x) \in (0,\infty) \times \mathbb{R}^3 : u_{\text{lin}}(t,x) > \lambda\}\right|,$$

then

$$\begin{aligned}
\int_0^\infty \langle F(u_{\text{lin}}^{\alpha,\varepsilon}(t)), u_{\text{lin}}^{\alpha,\varepsilon}(t)\rangle_{L^2}\,dt
&= \int_0^\infty G'(\lambda)(\alpha/\varepsilon)^{-8} m(\lambda/\alpha)\,d\lambda \\
&= \frac{\varepsilon^8}{\alpha^8}\int_{-\infty}^\infty G'(e^\tau)e^\tau m(e^{\tau-\log\alpha})\,d\tau \qquad (\lambda =: e^\tau) \\
&= \cdots = \frac{16\pi\varepsilon^8}{3\alpha^2}(H * w)(\log 2\alpha),
\end{aligned}$$

where $H(\tau) = G'(e^\tau)e^{-5\tau}$ and $w(\tau) := \frac{12}{\pi}e^{-6\tau}m(e^{-(\tau-\log 2)})$.

Reduction to a convolution equation

Proof

Finally, given a $\tau_0 \in \mathbb{R}$, let $\alpha := \frac{1}{2}e^{\tau_0}$ so that $\tau_0 = \log 2\alpha$. Combining the above, we deduce that

$$(H * w)(\tau_0) = (\widetilde{H} * w)(\tau_0) + \mathcal{O}(\varepsilon^4).$$

Taking $\varepsilon \to 0$, we arrive at the conclusion.

Reduction to a convolution equation

Task

Find a solution u_{lin} of the linear wave equation with initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$ for which:

- $u_{\text{lin}}(t, x) = \partial_t v_{\text{lin}}(t, x)$ (where v_{lin} is a solution of the linear wave equation with initial data $(v_0, v_1) \in \dot{H}^1 \times L^2$)
- $w(\tau) = \frac{12}{\pi} e^{-6\tau} m(e^{-(\tau - \log 2)})$ is computable/analyzable (where $m(\lambda) = \left|\{(t, x) \in (0, \infty) \times \mathbb{R}^3 : u_{\text{lin}}(t, x) > \lambda\}\right|$)

Approach

- Consider *radially symmetric* solutions, whose radial rescalings satisfy a 1D linear wave equation.
- Use *d'Alembert's formula* to write the general solution of this equation and search for a suitable particular solution.

Reduction to a convolution equation

The radially symmetric solution

$$u_{\mathrm{lin}}(t,x) := \frac{f(r-t)-f(r+t)}{r}, \quad r := |x|$$

formed from the triangular function $f(s) := \max\{1-|s|, 0\}$ works.

After some computation, we find that

$$w(\tau) = \left(e^{-3\tau} - \frac{4e^{-6\tau}}{(e^{-\tau}+1)^3}\right) 1_{(0,\infty)}(\tau).$$

Deconvolutional determination of the nonlinearity

Deconvolutional determination of the nonlinearity

Now that we have $H * w = \widetilde{H} * w$, we seek to formally "deconvolve" with w to conclude that $H = \widetilde{H}$, from which it will follow that $F = \widetilde{F}$.

The tool that will enable us to do so is the following formulation of Wiener's L^1 Tauberian theorem.

Theorem (Wiener's Tauberian theorem)

Let $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. If $f * g = 0$ and $\widehat{f}$ has no zeroes, then $g = 0$.

Proposition

Let w be as defined previously. Then $\widehat{w}$ has no zeroes.

Deconvolutional determination of the nonlinearity

Task

For the solution u_{lin} found previously, ensure that $\widehat{w}$ has no zeroes. In our case,

$$w(\tau) = \left(e^{-3\tau} - \frac{4e^{-6\tau}}{(e^{-\tau}+1)^3} \right) 1_{(0,\infty)}(\tau).$$

Approach

- Decompose w as $w = w_0 + w_1$ (since $\widehat{w}$ does not seem to be explicitly computable).
- Compute $\widehat{w_0}$, which has no zeroes, and show that $\widehat{w_1}$ remains sufficiently small.

Proof (of main theorem)

We know that $H * w = \widetilde{H} * w$. Wiener's Tauberian theorem and the nonvanishing of $\widehat{w}$ imply that $H = \widetilde{H}$.

Retracing the definitions of H and $\widetilde{H}$, we conclude that $F = \widetilde{F}$ (recall that $H(\tau) = G'(e^{\tau})e^{-5\tau}$, where $G(u) = F(u)u$).

Future work

Future work

Consider the Schrödinger equation in 1D with a (Schwartz) *potential* and a *cubic-type nonlinearity:*

$$\begin{cases} i\partial_t u(t,x) = (-\Delta_x + V(x))u(t,x) + F(u(t,x)), \quad (t,x) \in \mathbb{R} \times \mathbb{R}; \\ u(0,\,\cdot\,) = u_0. \end{cases}$$

Can the potential be determined from the scattering behaviour?

Future work: background

Let us first consider the *linear* Schrödinger equation $i\partial_t u(t,x) = (-\Delta_x + V(x))u(t,x)$. The **stationary states** of this equation are functions of space that solve the time-independent Schrödinger equation

$$\underbrace{(-\Delta + V)}_{H} f = k^2 f \quad \text{for some } k.$$

For $k \in \mathbb{R} \setminus \{0\}$, let $f_1(\,\cdot\,; k)$ and $f_2(\,\cdot\,; k)$ denote the solutions of the latter that satisfy $f_1(x; k) \sim e^{+ikx}$ as $x \to +\infty$ and $f_2(x; k) \sim e^{-ikx}$ as $x \to -\infty$ (called **Jost solutions**). Then there exist functions T, R_1, and R_2 (called **transmission** and **reflection coefficients**) such that

$$\begin{aligned} f_1(x;k) &\sim \tfrac{1}{T(k)} e^{+ikx} + \tfrac{R_2(k)}{T(k)} e^{-ikx} && \text{as } x \to -\infty, \\ f_2(x;k) &\sim \tfrac{1}{T(k)} e^{-ikx} + \tfrac{R_1(k)}{T(k)} e^{+ikx} && \text{as } x \to +\infty. \end{aligned}$$

Future work: background

In this setting, scattering behaviour is encoded by the **scattering matrix**

$$S(k) := \begin{bmatrix} T(k) & R_2(k) \\ R_1(k) & T(k) \end{bmatrix}.$$

It is known that the scattering matrix is determined by $R := R_1$ (or R_2) and the eigenvalues $-\beta_n^2 < \cdots < -\beta_1^2 < 0$ of H. These *together with* the constants $\|f_1(x; i\beta_j)\|_{L_x^2}^{-2}$ defined by the corresponding eigenfunctions determine the potential (Faddeev, 1958).

However, S *alone* does not determine the potential (when H has eigenvalues) (Deift, 1978)!

Future work: background

Theorem (Deift–Trubowitz, 1979)

Let $\beta > \beta_n$ *and define* $g_\alpha := f_1(\,\cdot\,; i\beta) + \alpha f_2(\,\cdot\,; i\beta)$ *for* $\alpha > 0$. *Then the reflection coefficient for the potential*

$$V_\alpha := V - 2(\log g_\alpha)''$$

is $R_\alpha(k) := -\frac{k+i\beta}{k-i\beta} R(k)$ *and the eigenvalues of*

$$H_\alpha := -\Delta + V_\alpha$$

are $-\beta^2 < -\beta_n^2 < \cdots < -\beta_1^2 < 0$.

In particular, starting from the "vacuum potential" $V = 0$, one can construct a family of "reflectionless potentials".

Future work: ideas

- The nonlinearity might actually allow us to glean more information about the potential because, for instance, varying the amplitude of the initial data changes the solution nonlinearly.
- In the focusing cubic case ($F(u) = -|u|^2 u$), we have access to *soliton* solutions. By scaling these so that they are sufficiently tall/narrow/fast, we might be able to "probe" the potential.

Thank you for your attention!