Analysis

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Calculus

Differential calculus

Derivatives

(Total) derivative of $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ at $\mathbf{x}$: linear map $d\mathbf{f}_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{f}(\mathbf{x}+\mathbf{h}) - [\mathbf{f}(\mathbf{x}) + d\mathbf{f}_{\mathbf{x}}(\mathbf{h})]\|}{\|\mathbf{h}\|} = 0$$

Jacobian (matrix) of $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ at $\mathbf{x}$: matrix representation of $d\mathbf{f}_{\mathbf{x}}$

Gradient of $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{x}$ (denoted $\nabla f(\mathbf{x})$): transpose of the Jacobian of f at $\mathbf{x}$; consequently, $\nabla f(\mathbf{x}) \cdot \mathbf{h} = df_{\mathbf{x}}(\mathbf{h})$ for all $\mathbf{h} \in \mathbb{R}^n$

Directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{x}$ along the *unit vector* $\mathbf{v}$:

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{x}+h\mathbf{v}) - f(\mathbf{x})}{h}$$

If f is differentiable at $\mathbf{x}$, then $\nabla_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v}$.

Partial derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{x}$ with respect to x_i:

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \nabla_{\mathbf{e}_i} f(\mathbf{x})$$

If f is differentiable at $\mathbf{x}$, then

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{bmatrix}.$$

Conversely, if all partials of a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ *exist in a neighbourhood of* $\mathbf{x}$ and are *continuous at* $\mathbf{x}$, then $\mathbf{f}$ is differentiable at $\mathbf{x}$.

Chain rule

If $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^k$ and $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable, then

$$d(\mathbf{f} \circ \mathbf{g})_{\mathbf{x}} = d\mathbf{f}_{\mathbf{g}(\mathbf{x})} \circ d\mathbf{g}_{\mathbf{x}}.$$

For instance, if $k = 1$, $y = f(\mathbf{u})$, and $\mathbf{u} = \mathbf{g}(\mathbf{x})$, then

$$\begin{bmatrix} \frac{\partial y}{\partial x_1} & \cdots & \frac{\partial y}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y}{\partial u_1} & \cdots & \frac{\partial y}{\partial u_m} \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_m}{\partial x_1} & \cdots & \frac{\partial u_m}{\partial x_n} \end{bmatrix},$$

so $\frac{\partial y}{\partial x_i} = \sum_{j=1}^m \frac{\partial y}{\partial u_j} \frac{\partial u_j}{\partial x_i}$.

Clairault's theorem

If the second partials of $f : \mathbb{R}^n \to \mathbb{R}$ exist and are continuous at $\mathbf{x}$, then the mixed partials of f are equal at $\mathbf{x}$, i.e.,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x})$$

for all $i, j \in [n]$.

Taylor series

Taylor's theorem

Suppose that $f : [a, b] \to \mathbb{R}$ is $n + 1$ times differentiable on (a, b) and $f^{(n)}$ is continuous on $[a, b]$, and let $x_0, x \in [a, b]$. Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

for some ξ between x_0 and x.

If $f^{(n)}$ is absolutely continuous on the closed interval between x_0 and x, then the remainder term is equal to

$$\int_{x_0}^{x} \frac{f^{(n+1)}(t)}{n!}(x - t)^n \, dt.$$

Critical points and extrema

First derivative test

If a is a critical point of $f : \mathbb{R} \to \mathbb{R}$ and f' changes sign at a, then a is a local minimum or maximum of f according as f' changes from nonpositive to nonnegative or vice-versa.

Second derivative test

If a is a critical point of $f : \mathbb{R}^n \to \mathbb{R}$ and the Hessian of f at a is positive definite (resp. negative definite), then a is a local minimum (resp. maximum) of f. If it is indefinite but invertible, then a is a saddle point of f.

Method of Lagrange multipliers

To find the extrema of $f : \mathbb{R}^n \to \mathbb{R}$ subject to the constraints $g_1(\mathbf{x}) = \cdots = g_M(\mathbf{x}) = 0$, set $\nabla f(\mathbf{x}) = \sum_{k=1}^{M} \lambda_k \nabla g_k(\mathbf{x})$ and $g_1(\mathbf{x}) = \cdots = g_M(\mathbf{x}) = 0$.

Integral calculus

Darboux and Riemann integrals

Partition of $[a, b]$: finite sequence $a = x_0 < x_1 < \cdots < x_n = b$

Lower/upper Darboux sum of $f : [a, b] \to \mathbb{R}$ with respect to the partition $\mathcal{P} = \{x_i\}_{i=0}^{n}$:

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} \left(\inf_{t \in [x_{i-1}, x_i]} f(t) \right)(x_i - x_{i-1})$$
$$U(f, \mathcal{P}) = \sum_{i=1}^{n} \left(\sup_{t \in [x_{i-1}, x_i]} f(t) \right)(x_i - x_{i-1})$$

Lower/upper Darboux integral of $f : [a, b] \to \mathbb{R}$:

$$\underline{\int_a^b} f(x)\,dx = \sup_{\mathcal{P}} L(f,\mathcal{P})$$

$$\overline{\int_a^b} f(x)\,dx = \inf_{\mathcal{P}} U(f,\mathcal{P})$$

If $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$, their common value is termed the **Darboux integral** of f and is denoted $\int_a^b f(x)\,dx$.

Tagged partition of $[a,b]$: partition $\mathcal{P} = \{x_i\}_{i=0}^n$ of $[a,b]$ with tags $t_i \in [x_{i-1}, x_i]$ for each $i \in [n]$

Riemann sum of $f : [a,b] \to \mathbb{R}$ with respect to the partition $\mathcal{P} = \{x_i\}_{i=0}^n$ with tags $\mathcal{T} = \{t_i\}_{i=1}^n$:

$$S(f,\mathcal{P},\mathcal{T}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$$

If there exists an $I \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of $[a,b]$ such that $|I - S(f,\mathcal{P},\mathcal{T})| < \varepsilon$ for all tagged partitions $\mathcal{P} \supseteq \mathcal{P}_\varepsilon$ (i.e., for all *refinements* of $\mathcal{P}_\varepsilon$), then I is called the **Riemann integral** of f and is denoted $\int_a^b f(x)\,dx$. The Riemann and Darboux integrals are equivalent.

A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann/Darboux integrable *if and only if* it is continuous a.e.

Multiple integrals

Fubini's theorem

Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be σ-finite measure spaces. If $f \in L^1(\mu \times \nu)$, then

$$\int_{X\times Y} f\,d(\mu\times\nu) = \int_X \int_Y f(x,y)\,d\nu(y)\,d\mu(x) = \int_Y \int_X f(x,y)\,d\mu(x)\,d\nu(y).$$

The conclusion of Fubini's theorem also holds if f is nonnegative and measurable (**Tonelli's theorem**). Hence if f is a measurable function on $X \times Y$ and any one of $\int |f|\,d(\mu\times\nu)$, $\iint |f|\,d\nu\,d\mu$, $\iint |f|\,d\mu\,d\nu$ is finite, then the iterated integrals of f are equal.

Change of variables

Let $\Omega \subseteq \mathbb{R}^n$ be open and $\mathbf{g} : \Omega \to \mathbb{R}^n$ be an injective C^1 function with $d\mathbf{g}_\mathbf{x}$ invertible for all $\mathbf{x} \in \Omega$. If $f \in L^1(\mathbf{g}(\Omega), d\mathbf{x})$, then

$$\int_{\mathbf{g}(\Omega)} f(\mathbf{x})\,d\mathbf{x} = \int_\Omega (f\circ\mathbf{g})(\mathbf{x})|\det d\mathbf{g}_\mathbf{x}|\,d\mathbf{x}.$$

Similarly, the conclusion of the change of variables theorem holds if f is nonnegative and measurable.

Coordinate system	Substitution	Jacobian determinant
Polar	$x = r\cos\theta$ $y = r\sin\theta$	r
Cylindrical	$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	r
Spherical	$x = r\cos\theta\,\sin\varphi$ $y = r\sin\theta\,\sin\varphi$ $z = r\cos\varphi$	$r^2\sin\varphi$

Differentiation under the integral sign

Leibniz's rule (differentiation under the integral sign)

Suppose that $f : X \times [a, b] \to \mathbb{C}$ and that $f(\cdot, y) \in L^1(\mu)$ for each $y \in [a, b]$. If $\frac{\partial f}{\partial y}$ exists and there is a $g \in L^1(\mu)$ such that $\left|\frac{\partial f}{\partial y}(x, y)\right| \leq g(x)$ for all x, y, then

$$\frac{d}{dy}\int_X f(x, y)\, d\mu(x) = \int_X \frac{\partial f}{\partial y}(x, y)\, d\mu(x).$$

Vector calculus

Vector differential operators

Gradient of $f : \mathbb{R}^3 \to \mathbb{R}$:

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

Curl of $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

Divergence of $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

By Clairault's theorem, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ and $\nabla \times (\nabla f) = \mathbf{0}$ provided that the requisite partials are continuous.

Conservative vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$: vector field such that $\mathbf{F} = \nabla\phi$ for some C^1 scalar field $\phi : \mathbb{R}^n \to \mathbb{R}$ (called a **scalar potential** for $\mathbf{F}$)

Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field with continuous partials. Then $\mathbf{F}$ is conservative *if and only if* $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$ (when $n = 2$) or $\nabla \times \mathbf{F} = \mathbf{0}$ (when $n = 3$).

Line and surface integrals

Line integral of a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ along C:

$$\int_C f(\mathbf{r})\, ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, dt,$$

where $\mathbf{r} : [a, b] \to C$ is a parametrization of C

Line integral of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ along C:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \hat{\mathbf{T}}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{r} : [a, b] \to C$ is a parametrization of C (and $\hat{\mathbf{T}}$ is the tangent vector to C)

Surface integral of a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ over S:

$$\iint_S f(\mathbf{r})\, dS = \iint_D f(\mathbf{r}(u, v)) \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\| du\, dv,$$

where $\mathbf{r} : D \to S$ is a parametrization of S

Surface integral of a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ over S:

$$\iint_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du\, dv,$$

where $\mathbf{r} : D \to S$ is a parametrization of S (and $\hat{\mathbf{n}}$ is the normal vector to S)

Integral theorems

Gradient theorem

Let C be a smooth curve parametrized by $\mathbf{r} : [a,b] \to C$, and let f be a scalar field with continuous partials on C. Then

$$f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = \int_C \nabla f \cdot d\mathbf{r}$$

If $\mathbf{F}$ is a conservative vector field, the gradient theorem implies that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path-independent (or equivalently, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C). Conversely, if $\mathbf{F}$ is continuous and $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path-independent in a domain D, then $\mathbf{F}$ is conservative on D.

Stokes' theorem

Let S be a piecewise smooth oriented surface bounded by a finite number of piecewise smooth simple closed positively-oriented curves, and let $\mathbf{F}$ be a vector field with continuous partials on S. Then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Gauss'/divergence theorem

Let V be a solid bounded by a piecewise smooth oriented surface, and let $\mathbf{F}$ be a vector field with continuous partials on V. Then

$$\iint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F}\, dV.$$

Green's theorem

Let D be a planar region bounded by a finite number of piecewise smooth simple closed positively-oriented curves, and let $\mathbf{F}$ be a vector field with continuous partials on D. Then

$$\int_{\partial D} F_x\, dx + F_y\, dy = \iint_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dx\, dy.$$

Green's theorem can be derived from Stokes' theorem by taking $\mathbf{F} = (F_x, F_y, 0)$ and identifying D with S (whose normal vector would be $\mathbf{e}_3$).

Real analysis

Fundamentals

Sequences

Bolzano-Weierstrass theorem

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Completeness of Euclidean space

Every Cauchy sequence in $\mathbb{R}^n$ converges.

Monotone convergence theorem

A monotonic sequence in $\mathbb{R}$ is convergent *if and only if* it is bounded.

More precisely, an increasing sequence that is bounded above converges to its supremum, whereas a decreasing sequence that is bounded below converges to its infimum.

Limit superior of a sequence $\{a_n\}_{n=1}^{\infty}$:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m\geq n} a_m = \inf_{n\in\mathbb{N}} \sup_{m\geq n} a_m.$$

Limit inferior of a sequence $\{a_n\}_{n=1}^{\infty}$:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{m\geq n} a_m = \sup_{n\in\mathbb{N}} \inf_{m\geq n} a_m.$$

Equivalently, the limit inferior and superior may be defined as the infimum and supremum, respectively, of the set of subsequential limits (in $\overline{\mathbb{R}}$) of $\{a_n\}$.

Series

As a consequence of the Cauchy criterion for sequences, we have

Cauchy's convergence test

$\sum a_n$ converges *if and only if* for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|\sum_{k=n}^{m} a_n| < \varepsilon$ for $m \geq n \geq N$.

In particular (taking $m = n$), the series converges only if $\lim_{n\to\infty} a_n = 0$.

Similarly, the monotone convergence theorem implies that a series of nonnegative terms converges *if and only if* the sequence of its partial sums is bounded.

Convergence of the geometric series

$\sum_{n=0}^{\infty} r^n$ converges to $1/(1-r)$ if $|r| < 1$ and diverges otherwise.

Convergence of the p-series

$\sum_{n=1}^{\infty} 1/n^p$ converges *if and only if* $p > 1$, and likewise for $\sum_{n=2}^{\infty} 1/[n(\ln n)^p]$.

Direct comparison test

If $a_n \in \mathcal{O}(b_n)$ and $\sum |b_n|$ converges, then $\sum |a_n|$ converges.

If $a_n \in \Omega(b_n)$ and $\sum |b_n|$ diverges, then $\sum |a_n|$ diverges.

Corollary:

Limit comparison test: if $a_n \in \Theta(b_n)$, then $\sum |a_n|$ converges *if and only if* $\sum |b_n|$ converges.

Cauchy condensation test

If $\{a_n\}$ is a *nonincreasing* sequence of *nonnegative* terms, then $\sum_{n=1}^{\infty} a_n \leq \sum_{k=0}^{\infty} 2^k a_{2^k} \leq 2\sum_{n=1}^{\infty} a_n$.

Thus, $\sum_{n=1}^{\infty} a_n$ converges *if and only if* $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Ratio test

If $\limsup_{n\to\infty} |a_{n+1}/a_n| < 1$, then $\sum |a_n|$ converges.

If $\liminf_{n\to\infty} |a_{n+1}/a_n| > 1$ or $|a_{n+1}/a_n| \geq 1$ for all sufficiently large n, then $\sum a_n$ diverges.

Root test

If $\limsup_{n\to\infty} |a_n|^{1/n} < 1$, then $\sum |a_n|$ converges.

If $\limsup_{n\to\infty} |a_n|^{1/n} > 1$, then $\sum a_n$ diverges.

Dirichlet's test

If the sequence of partial sums of $\sum a_n$ is bounded and $\{b_n\}$ decreases to zero (i.e., $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$), then $\sum a_n b_n$ converges.

Corollaries:

Alternating series test: if $c_{2k-1} \geq 0, c_{2k} \leq 0$ for all $k \in \mathbb{N}$ and $\{|c_n|\}$ decreases to zero, then $\sum c_n$ converges. [1]

Abel's test: if $\sum a_n$ converges and $\{b_n\}$ is bounded and monotone, then $\sum a_n b_n$ converges. [2]

Continuity, compactness, and connectedness

Continuous function f between topological spaces X, Y: the preimage under f of each open set in Y is an open set in X

Function f between topological spaces X, Y is **continuous** at $x \in X$: the preimage under f of each neighbourhood of $f(x)$ is a neighbourhood of x (f is continuous *if and only if* f is continuous at each $x \in X$)

Compact topological space X: every open cover of X has a finite subcover

Connected topological space X: *not* **disconnected** (i.e., *not* the union of two disjoint nonempty open sets)

In Euclidean space, we have the following characterizations:

Continuity in Euclidean space

$f : \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $\mathbf{x}$ *if and only if*
$\forall \varepsilon > 0\ \exists \delta > 0 : \forall \mathbf{y} \in \mathbb{R}^n\ (\|\mathbf{y} - \mathbf{x}\| < \delta \implies \|f(\mathbf{y}) - f(\mathbf{x})\| < \varepsilon)$.

Heine-Borel theorem

A subset of $\mathbb{R}^n$ is compact *if and only if* it is closed and bounded.

Connectedness in $\mathbb{R}$

A subset of $\mathbb{R}$ is connected *if and only if* it is an **interval** (i.e., a set I such that $\forall x, y \in I\ (x \le z \le y \implies z \in I)$).

In general, continuous maps preserve both compactness and connectedness. Two important consequences for real-valued functions are:

Extreme value theorem

Let X be a nonempty compact topological space and $f : X \to \mathbb{R}$ be continuous. Then f is bounded and attains both its infimum and supremum on X.

Intermediate value theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. If $c \in \mathbb{R}$ is strictly between $f(a)$ and $f(b)$, then there exists an $x \in (a, b)$ such that $f(x) = c$.

Uniformly continuous function f between metric spaces $(X, d_X), (Y, d_Y)$:
$\forall \varepsilon > 0\ \exists \delta > 0 : \forall x, y \in X\ (d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon)$

Let X, Y be metric spaces with X compact. If $f : X \to Y$ is continuous, then it is uniformly so.

Advanced calculus

(Cauchy's) mean value theorem

If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then there exists a $c \in (a, b)$ such that $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$. (The 'ordinary' mean value theorem has $g(x) = x$.)

Mean value theorem for integrals

If $f : [a, b] \to \mathbb{R}$ is continuous, then it attains its mean value at some $c \in (a, b)$, i.e., $\int_a^b f(x)\,dx = f(c)(b - a)$.

Darboux's theorem

Let $f : [a, b] \to \mathbb{R}$ be differentiable (but *not necessarily* continuously so). If $c \in \mathbb{R}$ is strictly between $f'(a)$ and $f'(b)$, then there exists an $x \in (a, b)$ such that $f'(x) = c$.

Hölder's inequality

Suppose f, g are measurable functions on some measure space. If $p \in [1, \infty]$ and q is the conjugate exponent to p (i.e., $1/p + 1/q = 1$), then $\|fg\|_1 \le \|f\|_p \|g\|_q$.

Corollary:

Cauchy-Schwarz inequality: $|\langle f, g\rangle| \le \|f\|_2 \|g\|_2$ for $f, g \in L^2(\mu)$. [3]

Inverse function theorem

Let $\Omega \subseteq \mathbb{R}^n$ be open and $\mathbf{f} : \Omega \to \mathbb{R}^n$ be C^1. If $d\mathbf{f}_{\mathbf{x}}$ is invertible at some $\mathbf{x} \in \Omega$, then $\mathbf{f}$ is a local C^1 diffeomorphism at $\mathbf{x}$ (i.e., there exists an open set $U \ni \mathbf{x}$ such that $\mathbf{f}\restriction_U$ is bijective, C^1, and has a C^1 inverse). Moreover, $(d\mathbf{f}^{-1})_{\mathbf{f}(\mathbf{x})} = (d\mathbf{f}_{\mathbf{x}})^{-1}$.

Corollary:

If $\Omega, \mathbf{f}$ are as above and $d\mathbf{f}_{\mathbf{x}}$ is invertible at every $\mathbf{x} \in \Omega$, then $\mathbf{f}$ is an open map. [4]

Implicit function theorem

Let $\Omega \subseteq \mathbb{R}^{n+m}$ be open and $\mathbf{f} : \Omega \to \mathbb{R}^m$ be C^1, and suppose that $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ for some $(\mathbf{a}, \mathbf{b}) \in \Omega$ (regarding Ω as a subset of $\mathbb{R}^n \times \mathbb{R}^m$). Write $d\mathbf{f}_{(\mathbf{a},\mathbf{b})} = [\, X \quad Y \,]$ with $X \in \mathbb{R}^{m \times n}, Y \in \mathbb{R}^{m \times m}$.

If Y is invertible, then there exists an open set $U \ni \mathbf{a}$ for which there is a unique C^1 map $\mathbf{g} : U \to \mathbb{R}^m$ satisfying $\mathbf{g}(\mathbf{a}) = \mathbf{b}$ and $\mathbf{f}(\mathbf{x}, \mathbf{g}(\mathbf{x})) = \mathbf{0}$ for all $\mathbf{x} \in U$. Moreover, $d\mathbf{g}_{\mathbf{a}} = -Y^{-1}X$.

Contraction mappings

Contraction mapping f on a metric space (X, d): there exists a $\rho \in [0, 1)$ such that $d(f(x), f(y)) \leq \rho d(x, y)$ for all $x, y \in X$ (the minimal ρ is called the **Lipschitz constant** of f)

Banach fixed point theorem

Let f be a contraction mapping on a nonempty *complete* metric space. Then f admits a unique fixed point (a point x such that $f(x) = x$). Moreover, fixed-point iteration starting from any point converges to it (more precisely, if x_* is the fixed point of f, x_0 is arbitrary, and $x_n := f(x_{n-1})$, then $x_n \to x_*$ with $d(x_*, x_{n+1}) \leq \rho d(x_*, x_n)$).

Sequences and series of functions

In this subsection, all functions are assumed to be real- or complex-valued functions defined on a subset of a metric space, unless otherwise indicated.

Pointwise and uniform convergence

$f_n \to f$ **pointwise** on E: $\forall x \in E \; (f_n(x) \to f(x))$

$f_n \to f$ **uniformly** on E: $\|f - f_n\|_\infty = \sup_{x \in E} |f(x) - f_n(x)| \to 0$

Weierstrass M-test

If $\|f_n\|_\infty \leq M_n$ for each n and $\sum M_n < \infty$, then $\sum f_n$ converges absolutely and uniformly.

Uniform limit theorem

If $f_n \to f$ uniformly, then $\lim_{x \to a} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to a} f_n(x)$. Consequently, the uniform limit of a sequence of continuous functions is continuous.

Dini's theorem

If a *monotone* sequence of *continuous* functions converges *pointwise* on a *compact* topological space, then the convergence is uniform.

If X, Y are metric spaces with Y *complete*, then $B(X, Y)$ (bounded functions) and $C_b(X, Y)$ (continuous bounded functions) are complete with respect to the uniform metric $d(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$.

Note that if X is compact, $C(X, Y) = C_b(X, Y)$ is complete.

Differentiation and integration of sequences of functions

Differentiation of a sequence of functions

Suppose that $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ such that $\{f_n(x_0)\}$ *converges for some* $x_0 \in [a, b]$. If $\{f_n'\}$ converges uniformly, then $\{f_n\}$ converges uniformly to a function f and $f' = \lim_{n \to \infty} f_n'$.

Dominated convergence theorem

Suppose that $\{f_n\} \subseteq L^1$ with $f_n \to f$ a.e. If there exists a $g \in L^1$ such that $|f_n| \leq g$ a.e. for each n, then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

Corollary:

If $\{f_n\} \subseteq L^1$ and $\sum \int |f_n| < \infty$, then ($\sum f_n$ is defined a.e.,) $\sum f_n \in L^1$ and $\int \sum f_n = \sum \int f_n$. [5]

The Arzelà–Ascoli theorem

Pointwise bounded family $\mathcal{F}$ of functions on E: $\forall x \in E\ (\sup_{f\in\mathcal{F}} |f(x)| < \infty)$

Uniformly bounded family $\mathcal{F}$ of functions on E: $\sup_{f\in\mathcal{F},\, x\in E} |f(x)| < \infty$

Equicontinuous family $\mathcal{F}$ of functions on E:
$\forall \varepsilon > 0\ \exists \delta > 0 : \forall f \in \mathcal{F}\ \forall x, y \in E\ (d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon)$ (cf. uniform continuity of f)

Relatively compact subspace Y of a topological space X: $\overline{Y}$ is compact; if X is a metric space, this is equivalent to every sequence in Y having a *subsequence converging in* X

Arzelà–Ascoli theorem

Let K be a compact metric space. A family $\mathcal{F} \subseteq C(K)$ is relatively compact *if and only if* it is pointwise bounded and equicontinuous.

In fact, for an equicontinuous family, pointwise and uniform boundedness are equivalent, so "pointwise" may be replaced by "uniformly" above. [6]

The Weierstrass approximation theorem

Subalgebra $\mathcal{A} \subseteq C(K, \mathbb{F})$, where K is a compact Hausdorff space and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$: set of functions closed under scalar ($\mathbb{F}$) multiplication, addition, and multiplication

$S \subseteq C(K, \mathbb{F})$ **separates points**: for all $x, y \in K$ with $x \neq y$, there exists an $f \in S$ such that $f(x) \neq f(y)$

$S \subseteq C(K, \mathbb{F})$ **vanishes nowhere**: for all $x \in K$, there exists an $f \in S$ such that $f(x) \neq 0$

$S \subseteq C(K, \mathbb{C})$ is **self-adjoint**: S is closed under complex conjugation

Stone-Weierstrass theorem (real version)

Let K be a compact Hausdorff space and $\mathcal{A}$ be a subalgebra of $C(K, \mathbb{R})$ that *separates points* and *vanishes nowhere*. Then $\mathcal{A}$ is dense in $C(K, \mathbb{R})$.

Corollary:

Weierstrass approximation theorem: polynomials are dense in $C([a, b], \mathbb{R})$.

Stone-Weierstrass theorem (complex version)

Let K be a compact Hausdorff space and $\mathcal{A}$ be a *self-adjoint* subalgebra of $C(K, \mathbb{C})$ that *separates points* and *vanishes nowhere*. Then $\mathcal{A}$ is dense in $C(K, \mathbb{C})$.

Corollary:

Trigonometric polynomials ($\operatorname{span}\{z^k : k \in \mathbb{Z}\}$) are dense in $C(\mathbb{T})$ (where $\mathbb{T}$ is the unit circle in $\mathbb{C}$).

(The Stone-Weierstrass theorems continue to hold if K is merely locally compact Hausdorff and "C" is replaced by "C_0", the continuous functions vanishing at infinity.)

Power series

Power series: series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where $z_0 \in \mathbb{C}$, $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$

Radius of convergence of a power series: $r = [\limsup_{n\to\infty} |a_n|^{1/n}]^{-1} = \liminf_{n\to\infty} |a_n|^{-1/n}$ (we also have $r = [\lim_{n\to\infty} |a_{n+1}/a_n|]^{-1}$ if this limit exists)

Convergence of power series

A power series converges *absolutely* and *compactly* (i.e., uniformly on compact subsets) in $B_r(z_0)$, its **disc of convergence**.

Differentiation and integration of power series

A power series may be differentiated and integrated term-by-term in $B_r(z_0)$; the resulting series have the same radii of convergence as the original.

Abel's theorem

If a *real* power series centred at $x_0 \in \mathbb{R}$ converges at $x_0 \pm r$ (i.e., at an endpoint of its interval of convergence), then it is also continuous at $x_0 \pm r$.

Complex analysis

Fundamentals

Elementary functions

Complex exponential: $e^z = \sum_{n=0}^{\infty} z^n/n!$

Complex sine and cosine: $\cos z = (e^{iz} + e^{-iz})/2, \; \sin z = (e^{iz} - e^{-iz})/2i$

Euler's formula: $e^{iz} = \cos z + i \sin z$

de Moivre's formula: $(\cos z + i \sin z)^n = \cos(nz) + i \sin(nz)$ for all $n \in \mathbb{Z}$

$e^z = 1 \iff z = 2\pi i k, \; k \in \mathbb{Z}$

Complex hyperbolic sine and hyperbolic cosine: $\cosh z = (e^z + e^{-z})/2, \; \sinh z = (e^z - e^{-z})/2i$

n^{th} **root** (*n-valued*): $\sqrt[n]{z} = \sqrt[n]{|z|} e^{i \arg(z)/n}$

Complex logarithm (*multi-valued*): $\ln z = \ln|z| + i \arg z$

Complex power: $z^w = e^{w \ln z}$ (if $w \in \mathbb{Q}$ with $w = m/n$ in lowest terms, then z^w is *n-valued*; otherwise, it has countably infinitely many values)

Differentiability and holomorphicity

In the following definitions, Ω denotes an open subset of $\mathbb{C}$.

$f : \Omega \to \mathbb{C}$ **(complex-)differentiable** at $z_0 \in \Omega$ [7] : $f'(z_0) = \lim_{h \to 0} [f(z_0 + h) - f(z_0)]/h$ exists

$f : \Omega \to \mathbb{C}$ **holomorphic** on Ω: f is differentiable at every $z_0 \in \Omega$

$f : \Omega \to \mathbb{C}$ **holomorphic** at $z_0 \in \Omega$ [7] : f is differentiable in a neighbourhood of z_0 (or equivalently, f is holomorphic on a neighbourhood of z_0)

f **entire**: f is holomorphic on $\mathbb{C}$

Wirtinger derivatives of $f : \Omega \to \mathbb{C}$:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

where $x = \Re(z)$ and $y = \Im(z)$

If f is viewed as a function of two real variables x, y with $z = x + iy, \overline{z} = x - iy$, then $\partial f/\partial z = df/dz = (\partial f/\partial x)(\partial x/\partial z) + (\partial f/\partial y)(\partial y/\partial z)$ and likewise $\partial f/\partial \overline{z} = df/d\overline{z}$.

Cauchy-Riemann equations

If f is complex-differentiable at z_0, then $(\partial f/\partial \overline{z})(z_0) = 0$ and $f'(z_0) = (\partial f/\partial z)(z_0)$.

Writing $f = u + iv$ and $z_0 = x_0 + iy_0$, the first part of the conclusion reads

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0),$$

which are the eponymous equations.

Conversely, if the partials of u and v *exist in a neighbourhood of* (x_0, y_0), are *continuous at* (x_0, y_0), and satisfy the Cauchy-Riemann equations, then f is complex-differentiable at z_0. [8]

Analytic functions

From the section on power series, we know that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is holomorphic on its disc of convergence with $f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}$. (Thus, a power series is infinitely differentiable on its disc of convergence.)

$f : \Omega \to \mathbb{C}$ **analytic** at $z_0 \in \Omega$: there exists a power series centred at z_0 converging to f in a neighbourhood of z_0; in other words, f is "locally given by a power series"

$f : \Omega \to \mathbb{C}$ **analytic** on Ω: f is analytic at every $z_0 \in \Omega$

$f_n \to f$ **compactly** on Ω: $f_n \to f$ uniformly on all compact subsets of Ω

$f_n \to f$ **locally uniformly** on Ω: every $z_0 \in \Omega$ has a neighbourhood on which $f_n \to f$ uniformly

For open sets $\Omega \subseteq \mathbb{C}$, compact and locally uniform convergence are equivalent. A consequence of Morera's theorem is the following:

Locally uniform limit of analytic functions

If $\{f_n\}$ is a sequence of analytic functions on an open set $\Omega \subseteq \mathbb{C}$ converging locally uniformly (or equivalently, compactly) to f, then f is analytic.

Complex integration

Contour integration

Integral of a continuous function $f : \Omega \to \mathbb{C}$ along a rectifiable curve γ:

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt,$$

where $z : [a, b] \to \gamma$ is a parametrization of γ

Integrals along **contours** comprised of successive rectifiable curves are then defined in the obvious way. We allow contours to include "curves" consisting of a single point.

Let $\Omega \subseteq \mathbb{C}$ be open and suppose that $F : \Omega \to \mathbb{C}$ is continuously complex-differentiable. Then for any contour $\Gamma \subseteq \Omega$ from z_1 to z_2,

$$\int_\Gamma F'(z)\,dz = F(z_2) - F(z_1).$$

Corollary:

If, moreover, Ω is connected and $F' = 0$, then F is constant on Ω.

Domain $\Omega \subseteq \mathbb{C}$: an open connected subset of $\mathbb{C}$; equivalently (by openness), an open *path-connected* subset of $\mathbb{C}$

Thus, the corollary above may be stated as: "if $F' = 0$ on a domain Ω, then it is constant on Ω". [9]

Suppose that f is continuous on a domain Ω. Then f has an antiderivative on Ω *if and only if* $\int_\Gamma f(z)\,dz$ is path-independent (for $\Gamma \subseteq \Omega$), which in turn is equivalent to $\int_\Gamma f(z)\,dz = 0$ for all closed contours Γ.

Cauchy's integral theorem

Simply connected set $\Omega \subseteq \mathbb{C}$: Ω is path-connected and any two curves in Ω with common endpoints are **fixed-endpoint homotopic** (i.e., can be continuously deformed into each other while keeping both endpoints fixed) [10]

Deformation invariance theorem

Suppose that f is holomorphic on an open set $\Omega \subseteq \mathbb{C}$. If $\gamma_0, \gamma_1 \subseteq \Omega$ are rectifiable curves that are fixed-endpoint homotopic (or are closed rectifiable curves that are homotopic as such), then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

As every closed curve in a simply connected domain is null-homotopic (that is, homotopic to a point), we obtain:

Cauchy's integral theorem

Suppose that f is holomorphic in a *simply connected* domain Ω. Then for any *closed* contour $\Gamma \subseteq \Omega$,

$$\int_{\Gamma} f(z)\,dz = 0.$$

Cauchy's integral formula

Suppose that f is holomorphic in an open set $\Omega \subseteq \mathbb{C}$ that contains the closure of a disc D. Then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

for all $z_0 \in D$. (By the above, the integral may equivalently be taken over any closed curve in $\Omega \setminus \{z_0\}$ that is homotopic to ∂D.)

Corollary:

Cauchy's inequalities/estimates: if z_0 is the centre of the above disc D and r is its radius, then

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n}\|f\|_{\infty,\,\partial D}.$$

Analyticity of holomorphic functions

Suppose that f is holomorphic in an open set $\Omega \subseteq \mathbb{C}$ and that $\overline{B_r(z_0)} \subseteq \Omega$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

for all $z \in B_r(z_0)$.

Thus, holomorphicity and analyticity are equivalent (both "at a point" and "on an open set").

Liouville's theorem

If f is bounded and entire, then f is constant.

Fundamental theorem of algebra

Every complex polynomial of degree $n \geq 0$ has *exactly* n roots in $\mathbb{C}$ counted with multiplicity.

Identity theorem

If f and g are holomorphic in a domain Ω and agree on a nonempty open subset of Ω (or, more generally, on a subset having a limit point in Ω), then $f = g$ throughout Ω.

Morera's theorem

If f is continuous on a domain Ω (not necessarily simply connected) and $\int_T f(z)\,dz = 0$ for every triangle $T \subseteq \Omega$, then f is holomorphic on Ω.

The complex logarithm

Existence of the complex logarithm on a simply connected domain

Suppose that Ω is a simply connected domain containing 1 and excluding 0. For all $z \in \Omega$, define

$$\log_\Omega(z) = \int_\gamma \frac{dw}{w},$$

where $\gamma \subseteq \Omega$ is any rectifiable curve from 1 to z, which is well-defined by Cauchy's integral theorem. Then $\log_\Omega$ is holomorphic in Ω with $(\log_\Omega)'(z) = 1/z$ and $e^{\log_\Omega(z)} = z$ for all $z \in \Omega$, and $\log_\Omega$ agrees with the real logarithm near 1.

The **principal branch** of the complex logarithm has $\Omega = \mathbb{C} \setminus (-\infty, 0]$.

More generally, if f is a nowhere vanishing holomorphic function on a nonempty simply connected domain Ω, there exists a holomorphic function g such that $g' = f'/f$ and $e^g = f$ on Ω:

$$g(z) = \int_\gamma \frac{f'(w)}{f(w)}\,dw + C,$$

where $\gamma \subseteq \Omega$ is a rectifiable curve from a fixed point $z_0 \in \Omega$ to z and $C \in \mathbb{C}$ is any constant satisfying $e^C = f(z_0)$.

Residue theory

Zeroes, singularities, and residues

Zero of f: a point $z_0 \in \mathbb{C}$ at which f vanishes

By the identity theorem, the zeroes of a nontrivial holomorphic function must be isolated.

Zeroes of holomorphic functions

If f is holomorphic in a domain Ω and has a zero at $z_0 \in \Omega$ (but does not vanish identically), then there exists a unique $n \in \mathbb{N}$ and a neighbourhood U of z_0 in which $f(z) = (z - z_0)^n g(z)$ for some non-vanishing holomorphic function g on U.

The number n above is called the **multiplicity/order** of the zero; a **simple zero** is a zero of multiplicity 1.

Deleted/punctured neighbourhood of $z_0 \in \mathbb{C}$: $B_r(z_0) \setminus \{z_0\}$ for some $r > 0$

Isolated/point singularity of f: a point $z_0 \in \mathbb{C}$ at which f is undefined but about which f is defined in a deleted neighbourhood; these are categorized into three types:

- **Removable singularity**: f may be defined at z_0 such that it is holomorphic in a neighbourhood of z_0
- **Pole**: $1/f$ *defined to be zero at* z_0 is holomorphic in a neighbourhood of z_0
- **Essential singularity**: neither a removable singularity nor a pole

Riemann's theorem (on removable singularities)

Suppose that f is holomorphic in an open set $\Omega \subseteq \mathbb{C}$ except at z_0, where it has an isolated singularity. If f is bounded in a deleted neighbourhood of z_0, then z_0 is a removable singularity of f.

Corollary:

If f and z_0 are as above, then z_0 is a pole of f *if and only if* $\lim_{z \to z_0} |f(z)| = \infty$.

Casorati-Weierstrass theorem (on essential singularities)

Suppose that f is holomorphic in an open set $\Omega \subseteq \mathbb{C}$ except at z_0, where it has an essential singularity. Then the image under f of any deleted neighbourhood of z_0 is dense in $\mathbb{C}$.

Poles and residues

If f has a pole at z_0, then there exists a unique $n \in \mathbb{N}$ and a neighbourhood U of z_0 in which $f(z) = (z - z_0)^{-n} h(z)$ for some non-vanishing holomorphic function h on U.

The number n above is called the **order/multiplicity** of the pole; a **simple pole** is a pole of order 1. If f has a zero of multiplicity n at z_0, then $1/f$ has a pole of order n at z_0, and vice versa (defining $(1/f)(z_0) = 0$).

If f has a pole of order n at z_0, then

$$f(z) = \left[\sum_{k=-n}^{-1} a_k (z - z_0)^k \right] + G(z),$$

where G is a holomorphic function in a neighbourhood of z_0.

The sum above is called the **principal part** of f at z_0, and a_{-1} is called the **residue** of f at z_0 and is denoted $\operatorname{res}_{z_0} f$ (or $\operatorname{res}_{z=z_0} f(z)$).

Computation of residues

If f has a pole of order n at z_0, then

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \cdot \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)].$$

Residue theorem

Suppose that f is holomorphic in an open set $\Omega \subseteq \mathbb{C}$ except at a finite set of poles P. Then for any positively-oriented simple closed contour $\Gamma \subseteq \Omega$ enclosing P,

$$\int_\Gamma f(z)\, dz = 2\pi i \sum_{z_0 \in P} \operatorname{res}_{z_0} f.$$

Laurent series

If f is holomorphic in the annulus $A_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$ (where $0 \le r < R < \infty$) centred at z_0, then it admits a unique **Laurent series** expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

therein that converges absolutely and compactly. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz,$$

where Γ is a simple closed contour in the annulus enclosing z_0 (cf. Cauchy's integral formula).

Classification of isolated singularities using Laurent series

Suppose that f has an isolated singularity at z_0 and that it admits a Laurent series expansion $\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ in $A_{0,R}(z_0)$. If $m = -\inf\{n \in \mathbb{Z} : a_n \neq 0\}$, then z_0 is a

- removable singularity *if and only if* $m = 0$.
- pole of order m *if and only if* $0 < m < \infty$.
- essential singularity *if and only if* $m = \infty$.

Evaluation of integrals

Trigonometric integrals

Given an integral of the form

$$\int_I R(\cos\theta, \sin\theta)\, d\theta,$$

where $R(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$ and I is an interval of length 2π, the substitution $z = e^{i\theta}$ yields

$$\int_{|z|=1} R\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right) \frac{dz}{iz}\,.$$

Principal value integrals

Cauchy principal value of $\int_{-\infty}^{\infty} f(x)\,dx$:

$$\text{p. v.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to\infty} \int_{-r}^{r} f(x)\,dx$$

(If the improper integral – that is, $\int_{-\infty}^{0} f(x)\,dx + \int_0^\infty f(x)\,dx$ – exists, it is equal to its principal value.)

If P, Q are polynomials with $\deg(Q) \geq \deg(P) + 2$, then

$$\lim_{r\to\infty} \int_{C_r^+} \frac{P(z)}{Q(z)}\,dz = 0,$$

where $C_r^+ = \{re^{i\theta} : \theta \in [0,\pi]\}$. (The same is true for $C_r^- = \{re^{-i\theta} : \theta \in [0,\pi]\}$.)

Jordan's lemma

If $k > 0$ and P, Q are polynomials with $\deg(Q) \geq \deg(P) + 1$, then

$$\lim_{r\to\infty} \int_{C_r^+} e^{ikz} \frac{P(z)}{Q(z)}\,dz = 0.$$

(The same is true for C_r^- when $k < 0$.)

Cauchy principal value of $\int_a^b f(x)\,dx$, where f is discontinuous at $c \in (a,b)$:

$$\text{p. v.} \int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+} \left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx\right]$$

(If the improper integral – that is, $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ – exists, it is equal to its principal value.)

Small arc/indentation lemma

If z_0 is a *simple pole* of f, then

$$\lim_{r \to 0^+} \int_{A_r} f(z)\, dz = (\theta_2 - \theta_1) i \cdot \operatorname{res}_{z_0} f,$$

where $A_r = \{z_0 + re^{i\theta} : \theta \in [\theta_1, \theta_2]\}$.

Integrals involving (multi-valued) complex powers

Integrals over $[0, \infty)$ involving non-integral powers are occasionally encountered.

- For such integrals, it is efficacious to integrate back and forth *along the nonnegative real branch cut of the logarithm*, with shrinking circular indentations around nonnegative singularities and an expanding circular contour closing the loop
- The residue theorem remains applicable (albeit not directly)
- The values of the power (say, z^w) "below" the branch cut will be $e^{2\pi i w}$ times its values "above" the branch cut

Meromorphic functions

Extended complex plane ($\mathbb{C}_\infty$): $\mathbb{C} \cup \{\infty\}$, where $z/0 = \infty$ for $z \neq 0$ and $z/\infty = 0$ for $z \neq \infty$ (addition and multiplication by ∞ also yield ∞, except for $\infty - \infty$ and $0 \cdot \infty$, which are undefined as usual)

($\mathbb{C}_\infty$ is the *one-point compactification* of $\mathbb{C}$; viz., its open sets are the open subsets of $\mathbb{C}$ together with sets of the form $\mathbb{C}_\infty \setminus K$ for K compact in $\mathbb{C}$.)

Isolated singularity of f **at** ∞: isolated singularity of $z \mapsto f(1/z)$ at 0

Isolated singularities of entire functions at ∞

If f is an entire function on $\mathbb{C}$, then f has an isolated singularity at ∞ and

- ∞ is a removable singularity of f *if and only if* f is constant.
- ∞ is a pole of order n of f *if and only if* f is a polynomial of degree n.
- ∞ is an essential singularity of f *if and only if* f is non-polynomial.

$f : \Omega \setminus S \to \mathbb{C}$ **meromorphic** on $\Omega \subseteq \mathbb{C}$: f is holomorphic on Ω except for a *closed discrete* set S of removable singularities and poles (or equivalently, f can be extended to a holomorphic function $\tilde{f} : \Omega \to \mathbb{C}_\infty$ with $\tilde{f} \not\equiv \infty$)

$f : \mathbb{C} \setminus S \to \mathbb{C}$ **meromorphic** on $\mathbb{C}_\infty$: f is meromorphic on $\mathbb{C}$ and is holomorphic or has a pole at ∞ (or equivalently, f can be extended to a holomorphic function $\tilde{f} : \mathbb{C}_\infty \to \mathbb{C}_\infty$)

Meromorphic functions on $\mathbb{C}_\infty$

The meromorphic functions on $\mathbb{C}_\infty$ are the rational functions.

The argument principle and Rouché's theorem

Argument principle

Suppose that f is meromorphic in an open set $\Omega \subseteq \mathbb{C}$. Then for any positively-oriented simple closed contour $\Gamma \subseteq \Omega$ on which f is nonvanishing and nonsingular,

$$\int_\Gamma \frac{f'(z)}{f(z)}\, dz = 2\pi i(Z - P),$$

where Z and P are, respectively, the numbers of zeroes and poles of f inside Γ, counted with multiplicities.

Rouché's theorem (symmetric, meromorphic version)

Suppose that f and g satisfy the hypotheses of the argument principle (for the same Ω and Γ) and moreover that $|f+g| < |f| + |g|$ on Γ. Then $Z_f - P_f = Z_g - P_g$, where Z_f and P_f (resp. Z_g and P_g) are, respectively, the numbers of zeroes and poles of f (resp. g) inside Γ, counted with multiplicities.

Corollary:

Rouché's theorem (asymmetric, holomorphic version): if f and h satisfy the hypotheses of the argument principle with "meromophic" replaced by "holomorphic" and, in addition, $|h| < |f|$ on Γ, then $Z_f = Z_{f+h}$. [11] [12]

The following result can be derived from Rouché's theorem.

Open mapping theorem

If f is nonconstant and holomorphic on a domain Ω, then f is an open map.

Corollary:

Maximum modulus principle: under the same hypotheses, f cannot attain a maximum in Ω. [13]

Furthermore, if $\overline{\Omega}$ is compact and f is also continuous on $\overline{\Omega}$, then f attains its maximum (over $\overline{\Omega}$) on $\partial\Omega = \overline{\Omega} \setminus \Omega$. Hence $\|f\|_{\infty,\,\Omega} \leq \|f\|_{\infty,\,\partial\Omega}$.

Harmonic functions

Harmonic function $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$: $u \in C^2$ and $\Delta u = \nabla \cdot \nabla u = 0$

We now identify subsets of $\mathbb{R}^2$ with those of $\mathbb{C}$.

If f is holomorphic on $\Omega \subseteq \mathbb{C}$, then $\Re(f)$ and $\Im(f)$ are harmonic on Ω.

If u is harmonic on a simply connected domain $\Omega \subseteq \mathbb{R}^2$, then there exists a holomorphic function f on Ω such that $\Re(f) = u$. Moreover, $\Im(f)$ is unique up to an additive (real) constant and is called a **harmonic conjugate** of u.

To find a harmonic conjugate of u, one can take f to be an antiderivative of $(\partial u/\partial x) - i(\partial u/\partial y)$, which is holomorphic by the Cauchy-Riemann equations.

Mean value property

If f is holomorphic in $B_R(z_0)$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

for any $r < R$.

Corollary:

Harmonic functions also possess the mean value property.

If u is harmonic in a simply connected domain and v is a harmonic conjugate of u, the maximum modulus principle applied to e^{u+iv} (whose modulus is e^u) implies that the conclusions of the maximum modulus principle apply to the extrema of harmonic functions.

Conformal maps

Biholomorphic functions

Let $U, V \subseteq \mathbb{C}_\infty$ be open sets.

Biholomorphic function (or **biholomorphism**) $\phi : U \to V$: ϕ is bijective and holomorphic

Biholomorphic functions are also called **conformal maps**. [14]

If $f : U \to V$ is *injective* and holomorphic, then f' is nonvanishing on U. Thus, f^{-1} *defined on* $f(U)$ is holomorphic.

In particular, the inverse of a biholomorphism is a biholomorphism.

U and V are said to be **biholomorphically equivalent** (or **conformally equivalent** [14]) if there exists a biholomorphism between them. (Note that this is indeed an equivalence relation by the result above.)

Local injectivity and preservation of angles [14]

If $f : U \to V$ is holomorphic at z_0 and $f'(z_0) \neq 0$, then f is *locally injective* at z_0. Moreover, f preserves the angles of directed smooth curves through z_0.

Automorphisms and Möbius transformations

Automorphism of U: biholomorphic function from U to itself [15] ; the set of all automorphisms on U is denoted $\mathrm{Aut}(U)$ and is a group under composition

Automorphisms of the unit disc

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $\alpha \in \mathbb{D}$, define

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$$

and $\phi_{\theta,\alpha} = e^{i\theta}\psi_\alpha$. Then

$$\mathrm{Aut}(\mathbb{D}) = \{\phi_{\theta,\alpha} : \theta \in [0, 2\pi),\ \alpha \in \mathbb{D}\}.$$

The maps ψ_α are called **Blaschke factors**. In other words, the automorphisms of the unit disc are the Blaschke factors (modulo complex signs). Moreover, we see that those that fix the origin are rotations.

This can be proved using the following lemma.

Schwarz's lemma

Suppose that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and that $f(0) = 0$. Then:

- $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$
- $|f'(0)| \leq 1$
- If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D} \setminus \{0\}$ or $|f'(0)| = 1$, then $f(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$.

Möbius (or **linear fractional**) **transformation**: function of the form $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$

We define $f(\infty) = \infty$ if $c = 0$ and $f(\infty) = a/c$ otherwise, so that f is an automorphism of $\mathbb{C}_\infty$.

Möbius transformations compose a group under composition, wherein

$$f^{-1}(z) = \frac{dz - b}{-cz + a}\,.$$

Fixed points of Möbius transformations

Every *non-identity* Möbius transformation has exactly two fixed points (counted with multiplicity).

Corollary:

If two Möbius transformations agree at three distinct points, then they are identical. [16]

Given distinct points $z_2, z_3, z_4 \in \mathbb{C}_\infty$, the Möbius transformation with $z_2 \mapsto 1$, $z_3 \mapsto 0$, and $z_4 \mapsto \infty$ is

$$z \mapsto \frac{z_2 - z_4}{z_2 - z_3} \cdot \frac{z - z_3}{z - z_4}.$$

Cross-ratio of distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$:

$$(z_1, z_2; z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4} = \frac{z_2 - z_4}{z_2 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_4}$$

As a consequence of this definition, the Möbius transformation $w = T(z)$ that satisfies $T(z_2) = w_2, T(z_3) = w_3, T(z_4) = w_4$ may be found by solving $(w, w_2; w_3, w_4) = (z, z_2; z_3, z_4)$ for w.

Preservation of cross-ratios

If T is a Möbius transformation, then

$$(z_1, z_2; z_3, z_4) = (T(z_1), T(z_2); T(z_3), T(z_4))$$

for any distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$.

Generalized circle in $\mathbb{C}_\infty$: a circle ($\{z \in \mathbb{C}_\infty : |z - z_0| = r\};\ z_0 \in \mathbb{C}, r > 0$) or a line ($\{z_0 + tw : t \in \mathbb{R}\} \cup \{\infty\};\ z_0, w \in \mathbb{C}$)

For any three distinct points in $\mathbb{C}_\infty$, there is a unique generalized circle passing through them.

Preservation of generalized circles

Möbius transformations map generalized circles to generalized circles.

Symmetric points $z, z' \in \mathbb{C}_\infty$ with respect to the generalized circle C: all generalized circles passing through z and z' intersect C orthogonally

Preservation of symmetric points

If T is a Möbius transformation and z, z' are symmetric with respect to C, then $T(z), T(z')$ are symmetric with respect to $T(C)$.

If C is the circle $z_0 + re^{i[0,2\pi)}$, then $z' = z_0 + r^2/\overline{z - z_0}$.

If C is the line $z_0 + \mathbb{R}e^{i\theta}$, then $z' = z_0 + e^{2i\theta}\overline{z - z_0}$.

The automorphisms of the unit disc can also be written as Möbius transformations [17]:

Automorphisms of the unit disc

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then

$$\mathrm{Aut}(\mathbb{D}) = \left\{ z \mapsto \frac{az + b}{\bar{b}z + \bar{a}} : a, b \in \mathbb{C};\ |a|^2 - |b|^2 = 1 \right\}.$$

Indeed, the automorphisms of several important domains are groups of Möbius transformations:

Automorphisms of the upper half-plane

Let $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. Then

$$\operatorname{Aut}(\mathbb{H}) = \left\{ z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C};\ ad - bc = 1 \right\}.$$

Automorphisms of the complex plane

$$\operatorname{Aut}(\mathbb{C}) = \{az + b : a, b \in \mathbb{C};\ a \neq 0\}.$$

Automorphisms of the extended complex plane

$$\operatorname{Aut}(\mathbb{C}_\infty) = \left\{ z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C};\ ad - bc \neq 0 \right\}.$$

In other words, the automorphisms of the extended complex plane are the Möbius transformations.

The Riemann mapping theorem

Riemann mapping theorem

Let U be a nonempty simply connected domain that is not all of $\mathbb{C}$. Then for any $a \in U$, there exists a *unique* conformal map $\phi : U \to \mathbb{D}$ satisfying $f(a) = 0$ and $f'(a) \in (0, \infty)$. (Hence U is conformally equivalent to $\mathbb{D}$.)

Conformal maps

In the table below, $\operatorname{Log}_\theta$ *denotes the logarithm with branch cut* $\{re^{i\theta} : r \in [0, \infty)\}$.

Domain	Range	Map	Inverse
$\mathbb{D}$	$\mathbb{H}$	$i\frac{1-z}{1+z}$	$\frac{i-w}{i+w}$
$\mathbb{H}$	$\{w \in \mathbb{C} : \arg(w) \in (0, \theta)\};\ \theta \in (0, 2\pi)$ (sector)	$z^{\theta/\pi}$	$w^{\pi/\theta}$
$\mathbb{D} \cap \mathbb{H}$	$\{w \in \mathbb{C} : \Re(w) > 0, \Im(w) > 0\}$ (first quadrant)	$\frac{1+z}{1-z}$	$\frac{w-1}{w+1}$
$\mathbb{H}$	$\{w \in \mathbb{C} : \Im(w) \in (0, \pi)\}$ (horizontal strip)	$\operatorname{Log}_{-\pi/2}(z)$	e^w
$\mathbb{D} \cap \mathbb{H}$	$\{w \in \mathbb{C} : \Re(w) < 0, \Im(w) \in (0, \pi)\}$ (horizontal half-strip)	$\operatorname{Log}_{-\pi/2}(z)$	e^w
$\mathbb{D} \cap \mathbb{H}$	$\{w \in \mathbb{C} : \Re(w) \in (-\pi/2, \pi/2), \Im(w) > 0\}$ (vertical half-strip)	$\operatorname{Log}_{-\pi/2}(z/i)/i$	ie^{iw}
$\mathbb{D} \cap \mathbb{H}$	$\mathbb{H}$	$-\frac{z+1/z}{2}$	*omitted*
$\mathbb{H} \setminus \mathbb{D}$	$\mathbb{H}$	$\frac{z+1/z}{2}$	*omitted*

For example, $\sin = (z \mapsto -\frac{z+1/z}{2}) \circ (w \mapsto ie^{iw})$, so $\sin$ maps the vertical half-strip $\{w \in \mathbb{C} : \Re(w) \in (-\pi/2, \pi/2), \Im(w) > 0\}$ to $\mathbb{H}$.

Harmonic analysis

Fourier series

In this subsection, 1-periodic functions on $\mathbb{R}$ are identified with functions on $\mathbb{T}$ ($\cong \mathbb{R}/\mathbb{Z}$), the unit circle in the complex plane.

Fourier coefficients of $f \in L^1(\mathbb{T})$: $\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}\,dx$, where $n \in \mathbb{Z}$

Fourier series of $f \in L^1(\mathbb{T})$: $f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}$

The N^{th} partial sum of the Fourier series of f, i.e., $\sum_{|n|\le N} \hat{f}(n)e^{2\pi inx}$ is denoted $S_N f(x)$.

Uniform convergence of Fourier series

If $f \in C(\mathbb{T})$ and $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$, then $S_N f \to f$ uniformly.

If $f \in C(\mathbb{T})$ and f' exists and is *piecewise continuous*, then $S_N f \to f$ uniformly.

Pointwise convergence of Fourier series

If $f \in L^1(\mathbb{T})$ and $|f(x+t) - f(x)| \lesssim |t|$ for all t in a neighbourhood of 0, then $S_N f(x) \to f(x)$. (In particular, if f is differentiable, then the hypothesized bound holds.)

If $f \in BV(\mathbb{T})$, then $S_N f(x) \to [f(x^-) + f(x^+)]/2$.

L^2 convergence of Fourier series

If $f \in L^2(\mathbb{T})$, then $S_N f \to f$ in L^2.

Corollaries:

If $f, g \in L^2(\mathbb{T})$, then $\langle f, g\rangle = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}$.

Parseval's identity: if $f \in L^2(\mathbb{T})$, then $\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$.

The Fourier transform

Fourier transform of $f \in L^1(\mathbb{R}^n)$: $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi\cdot x}\,dx$

Inverse Fourier transform of $f \in L^1(\mathbb{R}^n)$: $\check{f}(x) = \int_{\mathbb{R}^n} f(\xi)e^{2\pi ix\cdot\xi}\,d\xi$

Properties of the Fourier transform

- If $f \in L^1(\mathbb{R}^n)$, then $\|\hat{f}\|_\infty \le \|f\|_1$ and $\hat{f}$ is uniformly continuous.
- If $A \in \mathrm{GL}_n(\mathbb{R})$, then $\widehat{f \circ A} = |\det(A)|^{-1}\hat{f} \circ A^{-T}$.
- If $f_\varepsilon(x) = f(\varepsilon x)$ and $f^\varepsilon(x) = f(x/\varepsilon)/\varepsilon^n$, then $\widehat{f_\varepsilon} = \hat{f}^\varepsilon$ and $\widehat{f^\varepsilon} = \hat{f}_\varepsilon$.
- If $f \in C^N(\mathbb{R}^n)$ and $D^\alpha f \in L^1(\mathbb{R}^n)$ for all multiindices with $|\alpha| \le N$, then $\widehat{D^\alpha f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi)$.
- If $f \in L^1(\mathbb{R}^n)$ is compactly supported, then $\hat{f} \in C^\infty(\mathbb{R}^n)$ with $[(-2\pi ix)^\alpha f(x)]^\wedge = D^\alpha \hat{f}$.
- **Riemann-Lebesgue lemma**: if $f \in L^1(\mathbb{R}^n)$, then $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.
- $(e^{-\pi|x|^2})^\wedge = e^{-\pi|\xi|^2}$ (hence the Fourier transform of a Gaussian is a Gaussian).

Schwartz space $\mathcal{S}(\mathbb{R}^n)$: $\{f \in C^\infty(\mathbb{R}^n) : \forall\alpha,\beta\ (|f|_{\alpha,\beta} < \infty)\}$, where $|f|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)|$ (these are seminorms for multiindices α, β)

We say that a sequence $\{f_n\} \subseteq \mathcal{S}(\mathbb{R}^n)$ converges to $f \in \mathcal{S}(\mathbb{R}^n)$ *if and only if* $|f - f_n|_{\alpha,\beta} \to 0$ for each pair of multiindices α, β.

Properties of Schwartz space

- The following are equivalent to $f \in \mathcal{S}(\mathbb{R}^n)$:
  - $(1+|x|)^N D^\beta f$ is bounded for each $N \in \mathbb{Z}_{\geq 0}$ and multiindex β.
  - $\lim_{|x|\to\infty} x^\alpha D^\beta f = 0$ for each pair of multiindices α, β.
  - $f \in C^\infty(\mathbb{R}^n)$ and $\|x^\alpha D^\beta f\|_1 < \infty$ for each pair of multiindices α, β.
- $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.
- $\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ for all $p \in [1, \infty]$.
- $\mathcal{S}(\mathbb{R}^n)$ is closed under multiplication by polynomials, differentiation, multiplication, and *the Fourier transform*.

Convolution of $f, g : \mathbb{R}^n \to \mathbb{C}$: $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$

Properties of convolution

- **Young's convolution inequality**: if $p, q, r \in [1, \infty]$ and $1/p + 1/q = 1 + 1/r$, then $\|f * g\|_r \leq \|f\|_p \|g\|_q$ for all $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$.
- If $\phi \in C_c^\infty(\mathbb{R}^n)$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $\phi * f \in C^\infty(\mathbb{R}^n)$ and $D^\alpha(\phi * f) = (D^\alpha \phi) * f$.
- $\mathcal{S}(\mathbb{R}^n)$ is closed under convolution.
- If $f, g \in L^1(\mathbb{R}^n)$, then $\widehat{f * g} = \hat{f}\,\hat{g}$; if $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{fg} = \hat{f} * \hat{g}$.
- Let $\{\phi^\varepsilon\} \subseteq \mathcal{S}(\mathbb{R}^n)$ be an **approximate identity** [18].
  - If $f \in C_0(\mathbb{R}^n)$, then $\phi^\varepsilon * f \to f$ uniformly as $\varepsilon \to 0$.
  - If $f \in L^p(\mathbb{R}^n)$ with $p \in [1, \infty)$, then $\phi^\varepsilon * f \to f$ in L^p as $\varepsilon \to 0$.
  - If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, there exists a sequence $\{g_n\} \subseteq C_c^\infty(\mathbb{R}^n)$ such that if $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}^n)$, then $g_n \to f$ in L^p. (If $f \in C_0(\mathbb{R}^n)$ also, then $g_n \to f$ uniformly.)

Fourier duality

If $f, g \in L^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} \hat{f}\, g\,dx = \int_{\mathbb{R}^n} f \hat{g}\,dx$.

Fourier inversion theorem

If $f, \hat{f} \in L^1(\mathbb{R}^n)$, then $f = \check{\hat{f}}$ a.e. (or equivalently, $\hat{\hat{f}}(x) = f(-x)$ for a.e. x).

Corollaries:

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} = 0$, then $f = 0$ (a.e.) (viz., the Fourier transform is *injective* on $L^1(\mathbb{R}^n)$).

The Fourier transform is an *automorphism* of $\mathcal{S}(\mathbb{R}^n)$ (as a TVS).

Plancherel's theorem

If $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} f\overline{g}\,dx = \int_{\mathbb{R}^n} \hat{f}\,\overline{\hat{g}}\,dx$. Furthermore, there exists a unique bounded operator $\mathcal{F}$ on $L^2(\mathbb{R}^n)$ that agrees with the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$; $\mathcal{F}$ is unitary and agrees with the Fourier transform on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

1. Take $a_n = (-1)^{n+1}$ and $b_n = |c_n|$ in Dirichlet's test. ↩

2. Let $b = \lim_{n\to\infty} b_n$ and $d_n = \pm(b_n - b)$ according as $\{b_n\}$ is nonincreasing or nondecreasing. By Dirichlet's test, $\sum a_n d_n$ converges, whence $\sum a_n b_n = b \sum a_n \pm \sum a_n d_n$. ↩

3. Take $p = q = 2$ in Hölder's inequality and note that $|\langle f, g\rangle|^2 = |\int f\bar{g}\, d\mu|^2 \le (\int |f\bar{g}|\, d\mu)^2 = \|f\bar{g}\|_1^2$. ↩

4. It suffices to show that $\mathbf{f}(\Omega)$ is open. By the inverse function theorem, for each $\mathbf{x} \in \Omega$, there exists an open set $U_{\mathbf{x}} \ni \mathbf{x}$ such that $\mathbf{f}(U_{\mathbf{x}})$ is open (using continuity of the local inverse). Therefore $\mathbf{f}(\Omega) = \cup_{\mathbf{x}\in\Omega} \mathbf{f}(U_{\mathbf{x}})$ is open. ↩

5. By the monotone convergence theorem, $\int \sum |f_n| = \sum \int |f_n| < \infty$, so $g = \sum |f_n| \in L^1$. Hence $\sum f_n$ is defined a.e. and is integrable. The conclusion then follows from the dominated convergence theorem applied to the sequence of partial sums. ↩

6. That uniform boundedness implies pointwise boundedness is evident. As for the converse, let $\varepsilon > 0$ be given and $\delta > 0$ be as in the definition of equicontinuity. Cover K with finitely many balls of radius δ and thence combine the pointwise bounds for the centres of the balls with the assumption of equicontinuity. ↩

7. Some authors use the term "holomorphic at z_0" in place of what we refer to as "differentiable at z_0". Under this convention, $f(z) = |z|^2$, for instance, is holomorphic at 0. ↩ ↩

8. The first two conditions imply that u and v are $\mathbb{R}^2$-differentiable at (x_0, y_0) (see Derivatives); thence the Cauchy-Riemann equations entail that f is $\mathbb{C}$-differentiable at z_0. ↩

9. Fix $z_0 \in \Omega$. For any $z \in \Omega$, let Γ be a piecewise smooth curve from z_0 to z (which exists by path-connectedness). Then $0 = \int_\Gamma F'(z)\, dz = F(z) - F(z_0)$. ↩

10. More precisely, two curves $\gamma_0, \gamma_1 : [0,1] \to \Omega$ with common endpoints $a = \gamma_0(0) = \gamma_1(0)$ and $b = \gamma_0(1) = \gamma_1(1)$ are said to be **fixed-endpoint homotopic** if there exists a continuous function $H : [0,1]^2 \to \Omega$ such that $H(\cdot, 0) = \gamma_0$, $H(\cdot, 1) = \gamma_1$, and $H(0,t) = a$, $H(1,t) = b$ for all $t \in [0,1]$ (note that γ_0, γ_1 may need to be reparametrized for this definition to be applicable in general). Homotopy may similarly be defined between closed curves, which are permitted to have different initial points. ↩

11. In fact, it is not necessary that h be nonvanishing on Γ; we have only assumed that it is for brevity of exposition. ↩

12. Apply the symmetric, meromorphic version to f and $g = -(f+h)$, noting that $g \neq 0$ on Γ since $|h| < |f|$. ↩

13. If f attained a maximum at $z_0 \in \Omega$, the image under f of a neighbourhood of z_0 would be open, and would therefore contain values of greater moduli than $f(z_0)$ – a contradiction. ↩

14. We adopt the convention under which "biholomorphic" and "conformal" are synonymous. However, the term "conformal" is sometimes used to refer to maps $\phi : U \to V$ that are holomorphic with ϕ' nonvanishing on U. This condition is *strictly weaker than* biholomorphicity, as evidenced by functions such as $\phi(z) = e^z$ (with $U = \mathbb{C}, V = \mathbb{C} \setminus \{0\}$). Yet other definitions take "conformal" to mean *injective* and holomorphic, which is a stronger requirement than in the aforementioned definition but still weaker than biholomorphicity. ↩ ↩ ↩

15. If $U = \mathbb{C}_\infty$, we define $\mathrm{Aut}(U)$ to be the group of *meromorphic* bijections from $\mathbb{C}_\infty$ to itself. ↩

16. If the transformations are denoted S and T, then $T^{-1}S$ has three distinct fixed points and is therefore the identity. ↩

17. Make the change of variables $a = e^{i\theta/2}/\sqrt{1-|\alpha|^2}$, $b = \alpha a$. ↩

18. That is, $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi\, dx = 1$ and $\phi^\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$. ↩