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A proof of Zarantonello's lemma

Note: The following lemma appears in a number of texts by Saad [4, 5], who refers to Rivlin [3] for its proof. However, Rivlin's book does not appear to contain this lemma – at least, not explicitly in the form presented by Saad. A search for this lemma in other publications [1, 2] reveals that it was originally introduced in a slightly different form by Varga [6], who in turn attributes it to a *personal communication* with Zarantonello and provides a rather complicated proof of the lemma. What follows is a simple proof of the lemma as it appears in Saad's books.

Lemma (Zarantonello). *Let $r > 0$ and $\gamma \in \mathbb{C}$ be a point not enclosed by $C_r = \{z \in \mathbb{C} : |z| = r\}$ (i.e., $|\gamma| \geq r$). Then*

$$\min_{p \in \Pi_k,\, p(\gamma)=1} \max_{z \in C_r} |p(z)| = \left(\frac{r}{|\gamma|}\right)^k,$$

where Π_k denotes the set of polynomials of degree at most k, with the minimum attained by $p(z) = (z/\gamma)^k$.

Proof. We will prove that if $p \in \Pi_k$ and $M := \max_{z \in C_r} |p(z)|$, then for all $|z| \geq r$,

$$|p(z)| \leq \frac{M}{r^k} |z|^k,$$

from which the lemma will follow by setting $z = \gamma$ and taking the infimum over all such p with $p(\gamma) = 1$. Let $\tilde{p}$ be the polynomial defined by $\tilde{p}(w) = w^k p(r^2/w)$. For $|w| \leq r$, we have

$$|\tilde{p}(w)| \leq \max_{w \in C_r} |\tilde{p}(w)| = r^k \max_{w \in C_r} |p(r^2/w)| = M r^k,$$

where the inequality follows from the maximum modulus principle. Now if $|z| \geq r$, then $w := r^2/z$ satisfies $|w| \leq r$, so $|\tilde{p}(w)| \leq M r^k$. But $|\tilde{p}(w)| = (r^2/|z|)^k |p(z)|$, whence $|p(z)| \leq M|z|^k/r^k$. ∎

References

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