Synthetic division

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Consider the problem of dividing a polynomial a (called the **dividend**) by a *nonzero* polynomial b (called the **divisor**). Then there exist[1] *unique*[2] polynomials q (called the **quotient**) and r (called the **remainder**) such that $a = bq + r$, where r is of lesser degree than b or is zero.

We will restrict our attention to the nontrivial[3] case in which $a(x) = \sum_{k=0}^{n} a_k x^k$ has degree n and $b(x) = \sum_{k=0}^{m} b_k x^k$ has degree m, where $0 \le m \le n$. **Synthetic division** is a method for computing the coefficients of the quotient $q(x) = \sum_{k=0}^{n-m} q_k x^k$ and the remainder $r(x) = \sum_{k=0}^{m-1} r_k x^k$.

Monic linear divisors

We begin by illustrating the procedure for a monic linear divisor, that is, $b_m = 1$ with $m = 1$. In this case,

$$a_n x^n + \cdots + a_0 = (x + b_0)(q_{n-1} x^{n-1} + \cdots + q_0) + r_0.$$

Equating the coefficients of $x^n, \ldots, x^0$, we deduce that

$$\begin{aligned}
a_n &= q_{n-1} & &\implies q_{n-1} = a_n, \\
a_{n-1} &= q_{n-2} + b_0 q_{n-1} & &\implies q_{n-2} = a_{n-1} - b_0 q_{n-1}, \\
a_{n-2} &= q_{n-3} + b_0 q_{n-2} & &\implies q_{n-3} = a_{n-2} - b_0 q_{n-2}, \\
& & &\vdots \\
a_1 &= q_0 + b_0 q_1 & &\implies q_0 = a_1 - b_0 q_1, \\
a_0 &= b_0 q_0 + r_0 & &\implies r_0 = a_0 - b_0 q_0.
\end{aligned}$$

Synthetic division (also known as **Ruffini's rule** in this case) arranges the coefficients in a table as follows. First, the coefficients of the dividend and *the negative of* the trailing coefficient of the divisor are written.

$$\begin{array}{c|cccccc}
 & a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\
-b_0 & & & & & & \\
\hline
 & & & & & &
\end{array}$$

Next, the sum of the numbers in the first column on the right side of the table is written below the bar in the same column. This sum is then multiplied by the number on the left side of the table and the product is written in the next column.

$$\begin{array}{c|cccccc}
 & a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\
-b_0 & & -b_0 q_{n-1} & & & & \\
\hline
 & a_n = q_{n-1} & & & & &
\end{array}$$

These steps are then repeated in the next column.

$$\begin{array}{c|cccccc} & a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ -b_0 & & -b_0 q_{n-1} & -b_0 q_{n-2} & & & \\ \hline & a_n = q_{n-1} & a_{n-1} - b_0 q_{n-1} = q_{n-2} & & & & \end{array}$$

This process is continued until the end of the table is reached. By construction, the coefficients of the quotient and remainder appear in the bottom row!

$$\begin{array}{c|cccccc} & a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ -b_0 & & -b_0 q_{n-1} & -b_0 q_{n-2} & \cdots & -b_0 q_1 & -b_0 q_0 \\ \hline & q_{n-1} & q_{n-2} & q_{n-3} & \cdots & q_0 & r_0 \end{array}$$

This algorithm can also be used to efficiently evaluate $a(x_0)$ for a specific x_0. In this context, it is also known as **Horner's method**: by taking $b_0 = -x_0$, we see that $a(x) = (x - x_0)q(x) + r(x)$, so $a(x_0) = r_0$.

General divisors

For a general divisor, we have

$$a_n x^n + \cdots + a_0 = (b_m x^m + \cdots + b_0)(q_{n-m} x^{n-m} + \cdots + q_0) + r_{m-1} x^{m-1} + \cdots + r_0.$$

Equating the coefficients of $x^n, \ldots, x^0$, we deduce that

$$\begin{aligned} a_n &= b_m q_{n-m} & &\implies & q_{n-m} &= a_n / b_m, \\ a_{n-1} &= b_m q_{n-m-1} + b_{m-1} q_{n-m} & &\implies & q_{n-m-1} &= (a_{n-1} - b_{m-1} q_{n-m}) / b_m, \\ a_{n-2} &= b_m q_{n-m-2} + b_{m-1} q_{n-m-1} + b_{m-2} q_{n-m} & &\implies & q_{n-m-2} &= (a_{n-2} - b_{m-2} q_{n-m} - b_{m-1} q_{n-m-1}) / b_m, \\ & & &\vdots & & \\ a_m &= b_m q_0 + b_{m-1} q_1 + b_{m-2} q_2 + \cdots & &\implies & q_0 &= (a_m - \cdots - b_{m-2} q_2 - b_{m-1} q_1) / b_m, \\ a_{m-1} &= b_{m-1} q_0 + b_{m-2} q_1 + \cdots + r_{m-1} & &\implies & r_{m-1} &= a_{m-1} - \cdots - b_{m-2} q_1 - b_{m-1} q_0, \\ a_{m-2} &= b_{m-2} q_0 + \cdots + r_{m-2} & &\implies & r_{m-2} &= a_{m-2} - \cdots - b_{m-2} q_0, \\ & & &\vdots & & \\ a_0 &= b_0 q_0 + r_0 & &\implies & r_0 &= a_0 - b_0 q_0. \end{aligned}$$

The synthetic division procedure can therefore be generalized as follows. When computing a coefficient of the quotient, we must divide by b_m after the summation step and multiply by the negatives of all the trailing coefficients of the divisor in the multiplication step. On the other hand, when computing a coefficient of the remainder, no further divisions or multiplications are performed. The table can be expanded to accommodate the additional numbers used:

$$\begin{array}{c|ccccccccc} & a_n & a_{n-1} & a_{n-2} & \cdots & a_m & a_{m-1} & a_{m-2} & \cdots & a_0 \\ -b_0 & & & & & & & & & \\ \vdots & & & & & & & & & \\ -b_{m-2} & & & & & & & & & \\ -b_{m-1} & & & & & & & & & \\ \hline & & & & & & & & & \\ /b_m & & & & & & & & & \end{array}$$

For the first coefficient, we compute:

$$
\begin{array}{r|cccc}
 & a_n & a_{n-1} & a_{n-2} & \cdots \\
-b_0 & & & & \\
\vdots & & & & \cdots \\
-b_{m-2} & & & -b_{m-2}q_{n-m} & \\
-b_{m-1} & & -b_{m-1}q_{n-m} & & \\
\hline
 & a_n & & & \\
/b_m & a_n/b_m = q_{n-m} & & &
\end{array}
$$

For the next coefficient, we compute:

$$
\begin{array}{r|cccc}
 & a_n & a_{n-1} & a_{n-2} & \cdots \\
-b_0 & & & & \\
\vdots & & & & \cdots \\
-b_{m-2} & & & -b_{m-2}q_{n-m} & \cdots \\
-b_{m-1} & & -b_{m-1}q_{n-m} & -b_{m-1}q_{n-m-1} & \\
\hline
 & a_n & a_{n-1} - b_{m-1}q_{n-m} & & \\
/b_m & a_n/b_m = q_{n-m} & (a_{n-1} - b_{m-1}q_{n-m})/b_m = q_{n-m-1} & &
\end{array}
$$

Eventually, we obtain:

$$
\begin{array}{r|ccccccccc}
 & a_n & a_{n-1} & a_{n-2} & \cdots & a_m & a_{m-1} & a_{m-2} & \cdots & a_0 \\
-b_0 & & & & \cdots & \cdots & \cdots & \cdots & \cdots & -b_0q_0 \\
\vdots & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \\
-b_{m-2} & & & -b_{m-2}q_{n-m} & \cdots & -b_{m-2}q_2 & -b_{m-2}q_1 & -b_{m-2}q_0 & & \\
-b_{m-1} & & -b_{m-1}q_{n-m} & -b_{m-1}q_{n-m-1} & \cdots & -b_{m-1}q_1 & -b_{m-1}q_0 & & & \\
\hline
 & \cdots & \cdots & \cdots & \cdots & \cdots & r_{m-1} & r_{m-2} & \cdots & r_0 \\
/b_m & q_{n-m} & q_{n-m-1} & q_{n-m-2} & \cdots & q_0 & & & &
\end{array}
$$

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1. If a is of lesser degree than b or is zero, then we can trivially take $q = 0$ and $r = a$. Otherwise, suppose that a has degree n and b has degree m (where $0 \leq m \leq n$) and write $a(x) = \sum_{k=0}^{n} a_k x^k$ and $b(x) = \sum_{k=0}^{m} b_k x^k$. Then $a'(x) = a(x) - (a_n/b_m)x^{n-m}b(x)$ has degree *less than* n or is zero and hence, by induction on n, can be written as $a' = bq' + r$, where r is of lesser degree than b or is zero. Thus, if $q(x) = (a_n/b_m)x^{n-m} + q'(x)$, then $a = bq + r$ as claimed. (Alternatively, the derivation of synthetic division itself constitutes a proof of existence!) ↩ ↩

2. If $a = bq' + r'$, where r' is also of lesser degree than b or is zero, then $r' - r = b(q - q')$ is of lesser degree than b or is zero. This is only possible if $q - q' = 0$, which implies that $r' - r = 0$ as well. ↩

3. For the trivial cases, see the note [1] above. ↩