Types of complex square matrices

Underlined properties are equivalent
Italicized properties are equivalent for normal matrices
Red properties are equivalent for Hermitian matrices
Dashed borders denote sets of real matrices
(Properties involving $\mathbf{x}$ and/or $\mathbf{y}$ are understood to hold for *all complex vectors* $\mathbf{x}$ and/or $\mathbf{y}$)

Skew-symmetric
$A^T = -A$

Determinant is nonnegative
Diagonal entries are zero

Skew-Hermitian
$A^* = -A$

Eigenvalues are imaginary
Diagonal entries are imaginary

Diagonalizable
$A = MDM^{-1}$

Eigenvalues are semisimple
Basis of eigenvectors

Non-normal
$AA^* \neq A^*A$

Oblique projection
$P^2 = P,\ \mathrm{im}(P)^\perp \neq \ker(P)$

Eigenvalues are in $\{0, 1\}$

Symmetric
$A^T = A$

Orthogonally diagonalizable

Hermitian
$A^* = A$

Eigenvalues are real
$\mathbf{x}^*A\mathbf{x} \in \mathbb{R}$
Diagonal entries are real

Normal
$AA^* = A^*A$

Unitarily diagonalizable
Orthonormal basis of eigenvectors
Eigenvals. are moduli of sing. vals.
Eigenspaces are orthogonal
$\|A\mathbf{x}\| = \|A^*\mathbf{x}\|$
$\|A\| = \rho(A)$

Unitary
$U^* = U^{-1}$

Eigenvalues have unit modulus
$\langle U\mathbf{x}, U\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$
$\|U\mathbf{x}\| = \|\mathbf{x}\|$ (isometry)
Determinant has unit modulus
Columns/rows are orthonormal bases

Orthogonal projection
$P^2 = P,\ \mathrm{im}(P)^\perp = \ker(P)$

Eigenvalues are in $\{0, 1\}$

Rotation
$R^T = R^{-1},\ \det(R) = 1$

Orthogonal
$Q^T = Q^{-1}$

Determinant is ± 1

Positive semidefinite
$\mathbf{x}^*A\mathbf{x} \geq 0$

Eigenvalues are nonnegative
Principal minors are nonnegative
Gram matrix of a set of vectors
Pos. semidef. square root exists

Negative semidefinite
$\mathbf{x}^*A\mathbf{x} \leq 0$

Eigenvalues are nonpositive
Even-order principal minors are positive;
odd-order principal minors are negative

Householder reflection
$I - 2\mathbf{v}\mathbf{v}^T,\ \|\mathbf{v}\| = 1$

Symmetric and involutory
Eigenvalues are ± 1
-1 is a simple eigenvalue
Determinant is -1

Positive definite
$\mathbf{x}^*A\mathbf{x} > 0,\ \mathbf{x} \neq \mathbf{0}$

Eigenvalues are positive
Leading principal minors are positive
Gram matrix of a lin. indep. set of vectors
Principal submatrices are pos. def.
Inverse exists and is pos. def.

Negative definite
$\mathbf{x}^*A\mathbf{x} < 0,\ \mathbf{x} \neq \mathbf{0}$

Eigenvalues are negative
Even-order leading principal minors are positive;
odd-order leading principal minors are negative

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