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Closure operators

Definition. Let S be a set. A **closure operator** on S is a function $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ such that for all $A, B \subseteq S$,

- $A \subseteq \mathrm{cl}(A)$ (cl is *extensive*)
- $A \subseteq B \implies \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ (cl is *increasing*)
- $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ (cl is *idempotent*).

We call $\mathrm{cl}(A)$ the **closure** of A; a set $C \subseteq S$ that is the closure of some subset of S is called a **closed set** of cl.

These three defining conditions are in fact equivalent to a single biconditional, which is sometimes easier to verify.

Theorem 1. A function $f : \mathcal{P}(S) \to \mathcal{P}(S)$ is a closure operator on S *if and only if* $A \subseteq f(B) \iff f(A) \subseteq f(B)$ for all $A, B \subseteq S$.

Proof. Let $A, B \subseteq S$, and suppose first that f is a closure operator. Then $A \subseteq f(B) \implies f(A) \subseteq f(f(B))$ (since f is increasing) $\implies f(A) \subseteq f(B)$ (since f is idempotent). Conversely, $f(A) \subseteq f(B) \implies A \subseteq f(B)$ (since f is extensive).

On the other hand, if f satisfies the biconditional above, then $f(A) \subseteq f(A) \implies A \subseteq f(A)$, so f is extensive. Using extensivity, we obtain $A \subseteq B \implies A \subseteq f(B) \implies f(A) \subseteq f(B)$, so f is increasing. Finally, $f(A) \subseteq f(A) \implies f(f(A)) \subseteq f(A)$, where the reverse inclusion holds by extensivity, so f is idempotent. ■

The following characterization of closed sets will be used extensively, and is frequently taken as the *definition* of a closed set.

Proposition. A set $A \subseteq S$ is closed *if and only if* it is equal to its own closure, i.e., $\mathrm{cl}(A) = A$.

Proof. If A is closed, then $A = \mathrm{cl}(B)$ for some $B \subseteq S$, whence $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(B)) = \mathrm{cl}(B) = A$ (using idempotence). The converse follows immediately from the definition of a closed set. ■

Corollary. S is a closed set.

Proof. Clearly $\mathrm{cl}(S) \subseteq S$, and the reverse inclusion follows from extensivity. ■

Moore collections

Definition. Let S be a set. A collection $\mathcal{C} \subseteq \mathcal{P}(S)$ of subsets of S is called a **Moore collection** if any intersection of elements in $\mathcal{C}$ is also in $\mathcal{C}$.

The following two theorems exhibit a 'duality' between closure operators and Moore collections.

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Theorem 2. Let cl be a closure operator on S. If $\mathcal{C}$ is the collection of all closed sets of cl, then $\mathcal{C}$ is a Moore collection, and for any $A \subseteq S$,

$$\mathrm{cl}(A) = \bigcap \{C \in \mathcal{C} : C \supseteq A\}.$$

In other words, any intersection of closed sets is also closed, and cl(A) is the intersection of all closed sets containing A.

Proof. Let $\{C_i\}_{i \in I} \subseteq \mathcal{C}$ be a collection of closed sets, and define $C = \cap_{i \in I} C_i$. If $x \in \mathrm{cl}(C)$ and $i \in I$, then $C \subseteq C_i \implies \mathrm{cl}(C) \subseteq \mathrm{cl}(C_i) \implies x \in \mathrm{cl}(C_i) = C_i$, so $\mathrm{cl}(C) \subseteq C$. As the reverse inclusion holds by extensivity, C is equal to its own closure, and hence is closed. Thus $\mathcal{C}$ is a Moore collection.

Now if $A \subseteq S$, let B denote the set on the right-hand side above. Clearly $A \subseteq B$ (an intersection of sets containing A must also contain A), so $\mathrm{cl}(A) \subseteq \mathrm{cl}(B) = B$, since B is an intersection of closed sets. On the other hand, $\mathrm{cl}(A) \in \mathcal{C}$ and $\mathrm{cl}(A) \supseteq A$, so $B \subseteq \mathrm{cl}(A)$. ■

Remark. cl(A) is therefore the "smallest closed set containing A", in that it is contained in any other closed set containing A.

Theorem 3. Let $\mathcal{C} \subseteq \mathcal{P}(S)$ be a Moore collection. Then the function $f : \mathcal{P}(S) \to \mathcal{P}(S)$ given by

$$f(A) := \bigcap \{C \in \mathcal{C} : C \supseteq A\}$$

is a closure operator on S, and the collection of closed sets of f is $\mathcal{C}$.

Proof. By Theorem 1, it suffices to prove that $A \subseteq f(B) \iff f(A) \subseteq f(B)$ for all $A, B \subseteq S$ to show that f is a closure operator on S. Suppose first that $A \subseteq f(B)$. Then $f(B) \in \mathcal{C}$, as it is an intersection of elements of the Moore collection $\mathcal{C}$, and $f(B) \supseteq A$, so $f(A) \subseteq f(B)$. On the other hand, if $f(A) \subseteq f(B)$, we must also have $A \subseteq f(B)$, since $A \subseteq f(A)$. Hence f is a closure operator on S.

Now if $C \subseteq S$ is closed, then $C = f(C) \in \mathcal{C}$ since $\mathcal{C}$ is a Moore collection. Conversely, if $C \in \mathcal{C}$, then $C \supseteq C \implies f(C) \subseteq C$, and the reverse inclusion holds by extensivity. Thus the closed sets of f are exactly the sets of $\mathcal{C}$. ■

Thus, any closure operator is determined by a Moore collection (namely, the collection of its closed sets), and any Moore collection is determined by a closure operator.

Examples

If V is a vector space, it is easily verified that the collection of all subspaces of V is a Moore collection. The closure operator that this collection defines is none other than the *span* of a set of vectors. In other words, if $A \subseteq V$, then span(A) is *the intersection of all subspaces of V containing A*, or equivalently, *the smallest subspace of V containing A*.

Other examples are tabulated below:

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Set	**Moore collection**	**Closure operator**	**Notation**
A vector space V	Subspaces of V	Span of A	$\operatorname{span}(A)$
A topological space X	Closed sets of X	(Topological) closure of A	$\overline{A}$
A group G	Subgroups of G	Subgroup generated by A	$\langle A \rangle$
A group G	Normal subgroups of G	Normal closure of A	$\langle A^G \rangle$
A ring R	Subrings of R	Subring generated by A	—
A ring R	Ideals of R	Ideal generated by A	(A)
All subsets of a set X	σ-algebras on X	σ-algebra generated by A	$\sigma(A)$
$\mathbb{R}^n$	Convex subsets of $\mathbb{R}^n$	Convex hull of A	$\operatorname{conv}(A)$
$\mathbb{R}^n$	Affine subsets of $\mathbb{R}^n$	Affine hull of A	$\operatorname{aff}(A)$
$\mathbb{R}^n$	Convex cones in $\mathbb{R}^n$ containing the origin	Conical hull of A	$\operatorname{cone}(A)$
All binary relations on a set X	Transitive binary relations on X	Transitive closure of A	A^+
All words over an alphabet Σ	Languages containing the empty string that are closed under string concatenation	Kleene star of A	A^*