Random regular digraphs:
Singularity and discrepancy

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Introduction

Invertibility of rrd matrices

Ideas of the proof

rrd matrix ensemble

Circular law universality class

rrd matrix ensemble

- $n$ large, $d \in [1, n/2]$

\[
\mathcal{M}_d := \left\{ n \times n \text{ matrices, entries } \in \{0, 1\}, \text{ constraint: each row and column has } d \text{ 1s} \right\}
\]

$d = 1$: permutation matrices

General $d$: adjacency matrices of $d$-regular directed graphs.

- $M \in \mathcal{M}_d$ uniform random. “Random regular digraph (rrd) matrix”

Random matrix theory questions:

Is $M$ invertible with high probability?

Limiting spectral distribution?

- non-Hermitian, discrete distribution, dependent entries.

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- Random matrix theory questions:
  
  Is $M$ invertible with high probability?
  
  Limiting spectral distribution?

- non-Hermitian, discrete distribution, dependent entries.
Consider $M_{\pm}$ having iid uniform $\pm 1$ entries.

Q: Is $M_{\pm}$ invertible with high probability?
A: Yes. $P(\det M_{\pm} = 0) \leq c^{-n}$ for some $c < 1$ (Kahn-Komlós-Szemerédi ’95). Subsequent improvements by Tao-Vu ’05, Bourgain-Vu-Wood ’09.

Q: What is the limiting spectral distribution?
A: The spectrum of $M_{\pm}$ is governed by the circular law:

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(\frac{1}{\sqrt{n}}M_{\pm})} \rightarrow \frac{1}{\pi} 1_{B_{\mathbb{C}}(0,1)} dxdy \quad a.s.$$ 

From Tao-Vu ’08 universality principle: the circular law holds for any ensemble with entries that are iid with finite second moment.
The circular law universality class
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iid, finite 2nd moment (Tao-Vu '08)

- Bernoulli
- Ginibre
The circular law universality class

- iid, heavier tails (Bordenave, Caputo, Chafaï '10)
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Circular law

Dependent entries?
The circular law universality class

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Circular law

- Uniform doubly-stoch.
  (Nguyen '12)

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- Exchangeable array + moment hypothesis (Adamczak, Chafaï, Wolff '14)
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- rrd matrix $d \to \infty$ (conjectured)

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Circular law

- Uniform doubly-stoch. (Nguyen '12)
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- Dependent entries?

Oriented Kesten-McKay law

- (conjectured) \( d \to \infty \)
- \( d \) fixed

- Sum of \( d \) iid Haar unitaries (Basak, Dembo '12)
Invertibility

Proofs follow Girko’s *Hermitization* strategy.
- Requires lower bounds on $\sigma_n(M - zI)$ for $z \in \mathbb{C}$.
- Related problem: show $\sigma_n(M) > 0$, i.e. $M$ is non-singular, whp.

Is the rrd matrix $M$ invertible with high probability?
- $d = 1$: easy...
- $d = 2$?
  Exercise: $M$ is *singular* a.a.s.!
- $3 \leq d \leq n/2$, unclear...
**Main result**

**Theorem (C. ’14)**

Assume \( n^{\frac{1}{2}+\epsilon} \ll_{\epsilon} d \leq \frac{n}{2} \) for \( \epsilon \in (0, \frac{1}{2}] \). Then

\[
P(\det M = 0) = O(d^{-c})
\]

for some \( c > 0 \) absolute.

**Conjecture**

There are constants \( C, c > 0 \) such that for any \( d \in [3, n/2] \),

\[
P(\det M = 0) \leq Cn^{-c}.
\]
Key ideas

1. Small ball estimate

2. Inject independence

3. Tools to avoid some “bad events”:
   - Separately rule out “structured” null vectors
   - Discrepancy properties of the digraph
Proofs of upper bounds on $\sigma_1(M) = ||M||_{op}$ reduce to an application of concentration of measure.

Proofs of lower bounds on $\sigma_n(M) = ||M^{-1}||_{op}^{-1}$ reduce to the application of anti-concentration or “small ball” estimates.

**Theorem (Anti-concentration for random walks, Erdős ’40s)**

Let $R = (\xi_1, \ldots, \xi_n)$ vector of iid uniform signs, and $x \in \mathbb{R}^n$. Then for any $a \in \mathbb{R}$,

$$P(R \cdot x = a) = P\left(\sum_{j=1}^{n} \xi_j x(j) = a\right) \ll |spt(x)|^{-1/2}$$

where $spt(x) := \{j : x(j) \neq 0\}$.

Nice hammer. Where is the nail?
Injecting independence

- Suppose $M \sim \mu$, non-product distribution, but enjoys several “local symmetries”.
- We want to bound $\mathbb{P}(P \text{ holds for } M)$.
- Form a coupled pair $(M, \tilde{M})$ of $\mu$-distributed vectors, with

$$\tilde{M} = \Phi_\omega(M)$$

where $\Phi_\omega$ is a random $\mu$-preserving transformation.
- Now we can replace $M$ with $\tilde{M}$:

$$\mathbb{P}(P \text{ holds for } M) = \mathbb{P}(P \text{ holds for } \tilde{M}) = \mathbb{E}\mathbb{P}(P \text{ holds for } \tilde{M}|M)$$

and proceed using only the randomness we’ve added in.
Where is the nail?

- Toy problem: want to bound
  \[ P(R_1 \in \text{span}(R_3, \ldots, R_n)). \]
  - Conditional on \( R_3, \ldots, R_n \), pick \( u \in \text{span}(R_3, \ldots, R_n) \perp \) a unit vector.
  Then
  \[ P(R_1 \in \text{span}(R_3, \ldots, R_n)) \leq P(R_1 \cdot u = 0). \]
- How is \( R_1 \) distributed under this conditioning?
Conditional on $R_3, \ldots, R_n$, the only randomness is in the choice of sets $Ex(1, 2), Ex(2, 1)$.

Let $\pi : Ex(1, 2) \to Ex(2, 1)$ uniform random bijection.

Conditional on $\pi$, independently resample the $2 \times 2$ minors $M_{(1,2) \times (j, \pi(j))}$. 
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Let $\pi : Ex(1, 2) \rightarrow Ex(2, 1)$ uniform random bijection.

Conditional on $\pi$, independently resample the $2 \times 2$ minors $M_{(1,2) \times (j, \pi(j))}$.  

\[
\begin{bmatrix}
1 & \cdots & 1 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & 0 & 0 & \cdots \\
2 & \cdots & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 1 & 0 & \cdots \\
\vdots & & & & & & & & & & \vdots & & \vdots 
\end{bmatrix}
\]
In the randomness of the resampling, $R_1 \cdot u$ is a random walk with steps $u(j) - u(\pi(j))$. (Found the nail!)

Key technical proposition: normal vectors $u$ have small level sets.

Combining this fact with the randomness of $\pi$ guarantees many nonzero steps.
Discrepancy properties

- For this to work, we need control on the codegree $|Co(1, 2)|$.
- Using a “reflection” coupling with Chatterjee’s method of exchangeable pairs for concentration of measure, can show codegrees are concentrated.
- Using this with another coupling, get discrepancy properties:

  For $A, B \subset [n]$ and $\varepsilon \geq 0$,

  $$\Pr \left( \left| \frac{e(A, B)}{|A||B|} - 1 \right| \geq \varepsilon \right) \leq 2 \exp \left( - \frac{c\varepsilon^2}{1 + \varepsilon} \frac{d}{n} |A||B| \right).$$
Thank you