Weak convergence implies strong convergence in $\ell^1(\mathbb{N})$

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Perhaps my favorite homework problem from 245B was to establish the following:

**Proposition 0.1.** If $f_n \rightharpoonup f$ in $\ell^1(\mathbb{N})$, then $f_n \rightarrow f$ strongly in $\ell^1(\mathbb{N})$.

**Proof.** By subtracting $f$ from $f_n$ we may assume WLOG that $f_n \rightarrow 0$. For ease of notation we write

$$\langle f, g \rangle := \sum_{m \in \mathbb{N}} f(m)g(m)$$

for $f \in \ell^1(\mathbb{N})$ and $g \in \ell^\infty(\mathbb{N})$. By this we do not mean an $\ell^2$ inner product (though on the intersection of $\ell^1$ and $\ell^\infty$ it will agree with that inner product, except for a complex conjugate somewhere). Our assumption is that $\langle f_n, g \rangle \rightarrow 0$ for any $g \in \ell^\infty(\mathbb{N})$.

Taking $g$ to be the $k$th standard basis vector $\delta_k$ we see in particular that

$${f_n}(k) = \langle f_n, \delta_k \rangle \rightarrow 0$$

for each $k \in \mathbb{N}$.

We prove the contrapositive. Assume $\|f_n\|_1 \neq 0$. Then we have $\epsilon > 0$ and a subsequence $f_{n_k}$ such that $\|f_{n_k}\|_1 > \epsilon$ for all $k \in \mathbb{N}$. We will use this bad subsequence to make a bad $g \in \ell^\infty(\mathbb{N})$.

Since $f_{n_1} \in \ell^1(\mathbb{N})$, there exists $M_1 > 0$ such that

$$\sum_{m \geq M_1} |f_{n_1}(m)| < \epsilon/100.$$

Note this means that $\sum_{m \leq M_1} |f_{n_1}(m)| > .99\epsilon$. Set $n_{k_1} = n_1$, the first element of a sub-subsequence $n_{k_j}$.

With this $M_1$ fixed, it follows from (1) that there is $n_{k_2} > n_{k_1}$ such that

$$\sum_{m < M_1} |f_{n_{k_2}}(m)| < \epsilon/100$$

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(since $M_1$ is finite we can take $n_{k_1}$ large enough that each of the $f_{n_{k_1}}(m)$ for $0 \leq m < M_1$ is sufficiently small). Now again since $f_{n_{k_2}} \in \ell^1(\mathbb{N})$, there exists $M_2 > M_1$ such that

$$\sum_{m \geq M_2} |f_{n_{k_2}}(m)| < \epsilon/100.$$ 

It follows that

$$\sum_{M_1 \leq m < M_2} |f_{n_{k_2}}(m)| > .98\epsilon.$$ 

We continue inductively, constructing a subsequence $f_{n_{k_j}}$ and a sequence $M_j \in \mathbb{R}_+$ such that for each $j \geq 2$,

$$\sum_{M_{j-1} \leq m < M_j} |f_{n_{k_j}}(m)| > .98\epsilon$$

(each time using the pointwise convergence of $f_{n_{k_{j-1}}}$ to choose $n_{k_j}$ large enough that $f_{n_{k_j}}$ has at most $\epsilon/100$ of mass near 0, and using that $f_{n_{k_j}} \in \ell^1(\mathbb{N})$ to choose $M_j$ sufficiently large that the tail has mass at most $\epsilon/100$).

We can use this subsequence with packets of mass in the ranges $\{M_j, \ldots, M_{j+1} - 1\}$ to construct a bad sequence $g \in \ell^\infty(\mathbb{N})$. Define

$$g(m) = \frac{f_{n_{k_j}}(m)}{|f_{n_{k_j}}(m)|}$$

for $M_j \leq m < M_{j+1}$ for each $j \geq 1$ (and set it to zero on the remaining coordinates $m < M_1$). Then we have $||g||_\infty = 1$, and for each $j \geq 2$,

$$|\langle f_{n_{k_j}} g \rangle| \geq \left| \sum_{M_{j-1} \leq m < M_j} f_{n_{k_j}}(m)g(m) \right| - \left| \sum_{m < M_{j-1}} f_{n_{k_j}}(m)g(m) \right| - \left| \sum_{m \geq M_j} f_{n_{k_j}}(m)g(m) \right|$$

$$\geq \left( \sum_{M_{j-1} \leq m < M_j} |f_{n_{k_j}}(m)| \right) - ||g||_\infty \left( \sum_{m \notin \{M_{j-1}, \ldots, M_j - 1\}} |f_{n_{k_j}}(m)| \right)$$

$$\geq .98\epsilon - .01\epsilon - .01\epsilon = .96\epsilon.$$ 

Hence $f_n$ does not converge weakly to $f$, which concludes the proof by contrapositive. \qed

**Remark 0.2.** Jim Ralston told me this argument is a variant of the “traveling hump” method. We were able to use weak convergence and a lower bound on the $\ell^1$ mass of the elements of the subsequence to track a traveling packet with mass at least $.98\epsilon$ on its journey out to infinity (viewing $m$ as a spatial coordinate and $n$ as a time coordinate).