Introduction to Shimura varieties and automorphic vector bundles
Niccolò Ronchetti
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1 Motivation

Let’s recall first the classical story. One considers the upper half plane \( \mathbb{H} \) and wants to study modular functions, or modular forms \( f : \mathbb{H} \rightarrow \mathbb{C} \) - these are holomorphic functions which have some sort of equivariant property in terms of ‘arithmetic groups’ i.e. finite order subgroups of \( \text{SL}_2(\mathbb{Z}) \). Notice indeed that \( \text{SL}_2(\mathbb{R}) \) acts on \( \mathbb{H} \) via fractional linear transformations (more on this later).

**Example 1.** If we consider the arithmetic group \( \Gamma = \text{SL}_2(\mathbb{Z}) \), then you know the usual picture for \( \Gamma \backslash \mathbb{H} \), and a modular form of full level can be thought of as a ‘nice’ function on this variety with corners.

More generally, one can consider the subgroups \( \Gamma_0(N) = \{ (a,b), N|c \} \) and then modular forms of level \( N \) are nice functions \( \mathbb{H} \rightarrow \mathbb{C} \) which have some \( \Gamma_0(N) \)-equivariance property. This property shouldn’t necessarily be ‘left \( \Gamma_0(N) \)-invariance’, but it could be that acting by \( \gamma \in \Gamma_0(N) \) to the left corresponds by twisting by some characters.

In other words, if one considers the complex manifold (possibly with corners) \( \Gamma_0(N) \backslash \mathbb{H} = Y_0(N) \), we may either be studying functions on it or sections of some nontrivial line bundle.

What makes this example so peculiar is that the compactification\(^1\) of \( Y_0(N) \), which we denote by \( X_0(N) \), has the structure of an algebraic variety over some number field, and is in fact the moduli space of pairs \( (E,P) \) where \( E \) is an elliptic curve and \( P \in E[N] \) is a point of order \( N \).

We conclude then that the spaces \( Y_0(N) \) and their compactifications \( X_0(N) \) are very special for two different (but obviously related) reasons: they are both natural spaces where modular forms live, and they are moduli spaces for objects of great arithmetic significance - elliptic curves.

Suppose we want to generalize this, for example we may want to study modular forms for groups other than \( \text{SL}_2 \), or we may want to find moduli spaces for higher dimensional abelian varieties: here comes the need for a more broader setup where one tries to ‘replicate’ the construction of \( \text{SL}_2(\mathbb{Z}) \sim \mathbb{H} \) - and this will lead to Shimura varieties.

2 Interpretation of complex points as cocharacters of \( G(\mathbb{C}) \)

The first question to ask is ‘why \( \mathbb{H} \) ?’ More precisely, why is \( \mathbb{H} \) exactly the right complex manifold, with an action of \( \text{SL}_2(\mathbb{Z}) \) that makes everything work out so well in the classical theory? Well,

\(^1\)The canonical way of compactifying these type of spaces is the Satake-Baily-Borel compactification, which we will not discuss.
that’s not a coincidence!

We have already remarked that $\text{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformation. This action is transitive, and the stabilizer of a point (equivalently, any point) is a maximal compact subgroup.

**Exercise.** Find a point $z \in \mathbb{H}$ whose stabilizer is the ‘standard’ maximal compact $\text{SO}_2(\mathbb{R})$ of $\text{SL}_2(\mathbb{R})$.

Fixing a point allows us then to view $\mathbb{H} = \text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$, and now we can see better why $\text{SL}_2(\mathbb{Z})$ is the arithmetic group that ‘plays well’ with $\mathbb{H}$.

We aim to replicate all of this in higher dimensional settings, and for more general reductive groups over a number field - in particular, classical matrix groups.

**Definition 1** (Hermitian symmetric domain). A Hermitian symmetric domain is a complex manifold isomorphic to a bounded open connected subset $D \subset \mathbb{C}^m$ where for every point $x \in D$ there exists an involutive automorphism $s_x \in \text{Aut}(D)$ where $x$ is an isolated fixed point.

**Example 2.** Our upper half plane $\mathbb{H}$ is isomorphic to the Poincare disk, and in fact we can make sure that any point $z \in \mathbb{H}$ is sent into he origin, which makes it clear that there exists an involutive automorphism $s_z$ with the required properties.

**Fact 1.** Every Hermitian symmetric domain $D$ has an Hermitian structure preserved by all the automorphisms (in particular, our involutions preserve it).

This ‘Hermitian structure’ is an Hermitian form at the tangent space at every point, varying smoothly in the point. In this sense, Hermitian manifolds (= complex manifold with a Hermitian structure) are the complex equivalent of Riemannian (real) manifolds.

**Theorem 2.** Every Hermitian symmetric domain arises as $G^{\text{ad}}(\mathbb{R})^0/K$, where $G$ is a connected, simple, algebraic group over $\mathbb{R}$ of adjoint type, and $K$ is a maximal compact subgroup of $G^{\text{ad}}(\mathbb{R})^0$ (the identity component of the group of real points).

Moreover, suppose $G$ is defined over $\mathbb{Q}$, and let $\Gamma \subset G(\mathbb{Q})$ be a torsion-free subgroup. Then the complex manifold $S = \Gamma \backslash G^{\text{ad}}(\mathbb{R})^0/K$ has a canonical structure of a quasi-projective algebraic variety (defined over some number field).

We now build to the definition of a Shimura variety. As described above, one of the technical steps is that rather than a pair like $(\text{SL}_2(\mathbb{Z}) \sim \mathbb{H})$, we do not want to mention the Hermitian symmetric space, and just express everything in terms of the group, so that we need to generalize the isomorphism $\mathbb{H} = \text{SL}_2(\mathbb{R})/K$.

Unfortunately, these technical issues force the following ‘unesthetic’ definition, which is nonetheless quite powerful and from which we will easily get our preferred definition of Shimura variety.

**Definition 2** (Connected Shimura datum). Let $G/\mathbb{Q}$ be a semisimple group and $X$ be a $G^{\text{ad}}(\mathbb{R})^0$-conjugacy class of homomorphisms $S : \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow G^{\text{ad}}_{\mathbb{R}}$ such that

1. Composing with $G^{\text{ad}}_{\mathbb{R}} \longrightarrow \text{GL}(g)$ for every $h \in X$ yields a Hodge structure of type $\{(-1, 1), (0, 0), (1, -1)\}$ on $g$. In other words, the homomorphism $h : \mathbb{G}_m \longrightarrow \text{GL}(g)$ induces a decomposition in eigenspaces

$$g \otimes_{\mathbb{R}} \mathbb{C} \cong g^{-1,1} \oplus g^{0,0} \oplus g^{1,-1}$$

such that $g^{p,q} = \overline{g^{q,p}}$ and that $z \in S(\mathbb{R}) \cong \mathbb{C}^*$ acts on $g^{p,q}$ as $z^{p} \overline{z}^{-q}$.  


2. For each $h \in X$, conjugation by $h(i)$ is a Cartan involution of $G_{\mathbb{R}}$.

3. $G^{\text{ad}}$ has no factors defined over $\mathbb{Q}$ whose real points are compact.

**Remark.** (i) The third conditional is technical, and is added because it implies (via the strong approximation theorem) that (for every simple factor of the simply connected cover of $G$) the $\mathbb{Q}$-points are dense in the $\mathbb{A}_f$-points.

1. Obviously one only needs to check conditions 1 and 2 on a single representative in the conjugacy class.

2. My impression is that condition 1 on $hPq$ inducing that precise type of Hodge decomposition is quite stringent, basically because if $G$ is large one would expect that a torus (even though a non-split one, like our $S$) has many different eigenspaces.

Fix now $h \in X$ and let $K$ be its stabilizer under the $G(\mathbb{R})^0$-action by conjugation. Then clearly

$$G(\mathbb{R})^0/K \rightarrow X \quad g \mapsto ghg^{-1}$$

is a bijection, and hence we can endow $X$ with the structure of a real analytic manifold. In fact, this can be upgraded to a homomogeneous complex structure - and the latter is even unique once we fix which way $i \in \mathbb{C}$ should act on the tangent space of $X$ at $h$. $X$ becomes our Hermitian symmetric domain.

Here’s yet another way of viewing elements of our hermitian symmetric domain. Let’s identify the base change $S_\mathbb{C} \cong G_m \times G_m$ by imposing that $S(\mathbb{R}) \cong \mathbb{C}^* \hookrightarrow \mathbb{C}^* \times \mathbb{C}^* \cong S(\mathbb{C})$ embeds as $x \mapsto (z, iz)$. Then for each $h \in X$ we can base change $h_\mathbb{C} : S_\mathbb{C} \rightarrow G^{\text{ad}}_\mathbb{C}$.

From the Hodge decomposition $V \otimes_\mathbb{R} \mathbb{C} \cong \bigoplus V^{p,q}$ one can associate a Hodge filtration $F^pV = \bigoplus_{i \geq p} V^i$. By categorical nonsense (under the word Tannakian), this filtration will arise from a homomorphism $\mu : G_m \rightarrow G$ where $F^pW = \bigoplus_{i \geq p} W^i$ is the subspace where $\mu$ acts by $t^i$ for $i \geq p$.

We associate to $h$ the cocharacter

$$\mu_h : G_m \rightarrow G^{\text{ad}}_\mathbb{C} \quad z \mapsto h_\mathbb{C}(z, 1)$$

It turns out that this $\mu_h$ is exactly (up to conjugacy I guess?) the cocharacter $\mu$ defining the Hodge filtration as above.

**Remark.** The upshot is that we can think of points of $X$, our Hermitian symmetric domain, as cocharacters $\mu_x : G_m \rightarrow G^{\text{ad}}_\mathbb{C}$ which canonically determine the Hodge filtration. The condition on the type of the Hodge decomposition then implies that knowing the Hodge filtration is the same as knowing the whole decomposition.

Let’s finally construct the Shimura variety. Suppose given the data $(G, X)$ as above. For each compact open subgroup $K \subset G(\mathbb{A}_f)$ we obtain a congruence subgroup $\Gamma = G(\mathbb{Q}) \cap K$. Let’s confuse $\Gamma$ and its image into $G^{\text{ad}}(\mathbb{Q})^0$ to relax notation.

If $\Gamma$ is torsion-free, the quotient $\Gamma \backslash X$ is a locally symmetric algebraic variety. Consider the projective system $(\Gamma \backslash X)$ as we vary across torsion-free $\Gamma$s: this admits an action of $G^{\text{ad}}(\mathbb{Q})^0$ as

$$\Gamma \backslash X \rightarrow g^{-1}\Gamma g \backslash X \quad x \mapsto g^{-1}x$$

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For practical purposes, a Cartan involution is an involution which fixes a maximal compact subgroup and acts like the inversion on a maximal torus. Think $X \mapsto (X^{-1})^T$ in the case of a general linear group.
**Fact 3.** This map is holomorphic and in fact algebraic!

**Definition 3** (Shimura variety). The Shimura variety \( \text{Sh}(G, X) \) is the projective system \((\Gamma \backslash X)\), or its limit, together with the \( G^{\text{ad}}(\mathbb{Q})^0 \)-action described above.

**Remark.** In fact Milne mentions that the action of \( G^{\text{ad}}(\mathbb{Q})^0 \) can be upgraded to an action of the completion of this group under the topology where a neighborhood basis of the identity (in \( G^{\text{ad}}(\mathbb{Q})^0 \)) is given by the images of the congruence subgroups.

Now suppose \( G \) is simply connected. Then strong approximation gives that \( G(\mathbb{Q})K = G(A_f) \), and connectedness of \( G(\mathbb{R}) \) implies that the map

\[
\Gamma \backslash X \longrightarrow G(\mathbb{Q}) \backslash X \times G(A_f)/K \quad x \mapsto (x, 1)
\]

is a bijection. We recover then a somewhat more appealing adelic description.

Consider now the complex points \( \text{Sh}(G, X) (\mathbb{C}) = \varprojlim \Gamma \backslash X = G(\mathbb{Q}) \backslash X \times G(A_f) \).

Here the \( G(A_f) \)-action on the right is evident. We sometimes call the operator \( T(g) \) induced by \( g \in G(A_f) \) a Hecke operator.

**Remark.** There is a very similar story for a general reductive group \( G \). The main issues to take into account are:

1. The possibility of the existence of some nontrivial \( \mathbb{R} \)-split central torus that is not \( \mathbb{Q} \)-split.

2. The non-connectedness of the conjugacy class \( X \).

**Example 3** (Modular curves). Let \( \Gamma \subset \text{PGL}_2(\mathbb{Q}) \) contain the image of a congruence subgroup of \( \text{SL}_2(\mathbb{Q}) \). Then the setup above yields that \( \Gamma \backslash \mathbb{H} \) is an elliptic modular curve. The Shimura variety associated to the family of (torsion-free) subgroups \( \Gamma \) thus obtained is a projective system of modular curves, and the original object of study of Shimura.

Let’s develop this example in some more detail, by taking instead \( G = \text{GL}_2/\mathbb{Q} \) (the two viewpoints are more or less equivalent, since for a reductive group the axioms for a Shimura datum - which are not written on this note - are basically telling you to get rid of the maximal \( \mathbb{Q} \)-split central torus).

Let \( h : z = x + iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \) be defining our Hodge structure, and let then \( X \) be the \( \text{PGL}_2(\mathbb{R}) \) conjugacy class of such homomorphisms.

Consider now an elliptic curve \( E/\mathbb{C} \). Its adelic Tate module is the inverse limit of all torsion points

\[
V_f(E) = \varprojlim N \quad E[N] \otimes A_f = H_1(E, \mathbb{Q}) \otimes A_f.
\]

**Theorem 4.** The ‘full’ Shimura variety \( \text{Sh}(G, X) \) classifies isogeny classes of pairs \((E, \eta)\) where \( E/\mathbb{C} \) is an elliptic curve and \( \eta : A_f \times A_f \longrightarrow V_f(E) \) is an \( A_f \)-linear isomorphism which trivializes the adelic Tate module.

**Remark.**

1. Two pairs \((E, \eta), (E', \eta')\) are isogenous if there exists an isogeny carrying \( \eta \) to \( \eta' \).

2. The map \( \eta \) should be thought of as a ‘trivialization’ of the adelic Tate module, in the sense that we are picking a basis. This basis correspond to compatible choices of basis at each finite step \( E[N] \).
**Proof.** Exercise. □

**Example 4** (Siegel modular variety). Consider a $2n$-dimensional vector space $V$ over $\mathbb{Q}$ with a nondegenerate skew-symmetric form $\psi$ - for example let

$$\psi(x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}) = \sum_{i=1}^{n} x_i y_{n+i} - \sum_{i=1}^{n} x_{n+i} y_i.$$  

Notice that this is given by the matrix $J = \left( \begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix} \right)$, in the sense that $\psi(v, w) = v^T J w$.

Consider the algebraic group $G$ defined by ‘preserving $\psi$ up to homothety’: more explicitly, its functor of points is

$$G(A) = \{ g \in GL_{2n}(A) \mid \text{there exists } a = a(g) \in A^* \text{ with } \psi(gv, gw) = a \psi(v, w) \forall v, w \in V \}$$

on any $\mathbb{Q}$-algebra $A$. This is just a cover of the classical symplectic group, and in particular it is not semisimple as the center of $GL_{2n}$ is obviously contained in $G$.

Let $S^\pm$ be the set of Hodge structures of type $((-1,0),(0,1))$ on $V$ for which $\pm 2\pi i \psi$ is a polarization. This means: decompositions

$$V \otimes_{\mathbb{Q}} \mathbb{C} \cong V^{-1,0} \oplus V^{0,-1} \text{ with } V^{-1,0} = V^{0,-1}$$

such that the real-valued form

$$V(\mathbb{R})^2 \longrightarrow \mathbb{R} \quad (x, y) \mapsto (2\pi i) \psi(x, h(i)z)$$

is positive definite, where $h : S \longrightarrow GL(V)$ is the homomorphism canonically associated to the Hodge structure.

**Remark.** More generally, a Hodge structure of type $((-1,0),(0,-1))$ on a real vector space $W$ corresponds to putting a complex structure on $W$. Given the complex structure, define $h : S \longrightarrow GL(V)$ as $h(z) = '\text{multiplication by } z'$. Given the Hodge structure, define the complex structure via the isomorphism $V \longrightarrow V_C / V^{0,-1}$.

In particular, for each such Hodge structure where $\pm 2\pi i \psi$ is a polarization, we get an associated homomorphism $h : S \longrightarrow G_\mathbb{R}$. The space of this homomorphisms $S^\pm$ is a $G(\mathbb{R})$-conjugacy class, and the pair $(G, S^\pm)$ satisfies the definition of Shimura datum (exercise!). The associated Shimura variety $Sh(G, S^\pm)$ is called Siegel modular variety.

The space $S^\pm$ is a disjoint union of two Hermitian symmetric domain and is called the **Siegel double space**. It is a generalization of the upper half plane.

**Example 5** (Hilbert modular variety: a non-example). Let $L/\mathbb{Q}$ be totally real, with degree $g$ and let $\{\sigma_1, \ldots, \sigma_g\}$ be the embeddings. Consider the base change $G = Res_{L/\mathbb{Q}} GL_2$. Notice in particular that $G(\mathbb{Q}) = GL_2(L)$ and that $G(\mathbb{R}) = \prod_{i=1}^{g} GL_2(R)$ with $GL_2(L) \ni \gamma \mapsto (\sigma_i(\gamma))_{i=1}^{g}$.

Let $h : S \longrightarrow G$ be defined on real points as $x + iy \mapsto \left( \begin{smallmatrix} x & -y \\ y & x \end{smallmatrix} \right)$ for $i = 1$. The stabilizer of $h$ under conjugation by $G^{ad}(\mathbb{R}) = \prod PGL_2(\mathbb{R})$ is

$$K_\times = \{ \left( \begin{smallmatrix} x_\sigma & -y_\sigma \\ y_\sigma & x_\sigma \end{smallmatrix} \right) \mid x_\sigma^2 + y_\sigma^2 \neq 0 \forall \sigma \}.$$ 

Notice that this is compact as we are in $PGL_2$-world!

On the other hand, $h(i) = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$ does not yield a Cartan involution on $G/Z_0(G)$, because $Z_0(G)$ is too small (this is the $\mathbb{R}$-split central torus).
2.1 The Borel embedding

Let \((G, X)\) be a Shimura datum. We now give a canonical embedding of \(X\) into a projective complex variety (in fact, a flag variety).

Recall that \(x \in X\) is a homomorphism \(S \to G_\mathbb{R}\) determining a Hodge decomposition and thus a Hodge filtration (which in this case is the same data, because of the specific type of the Hodge decomposition). Recall that we gave a description of \(x \in X\) as cocharacters \(\mu_x : \mathbb{G}_m \to G_\mathbb{C}\) inducing precisely that Hodge filtration, using some Tannakian formalism. Fix a faithful representation \((V, \xi)\) of \(G_3\) and define \(\hat{X}\) to be the \(G(\mathbb{C})\)-conjugacy class of the filtration induced by \(\mu_x\) on \(V\).

Remark. This conjugacy class only depends on \(X\), and not on our choice of \(x \in X\).

Fixing a point \(o \in X\) and letting \(P_o\) be the subgroup of \(G_\mathbb{C}\) stabilizing the filtration induced by \(\mu_o\) allow us to identify \(G_\mathbb{C}\{P_o - \hat{X}\). It is not hard to see that \(P_o\) is a parabolic subgroup (e.g. block-upper-triangular matrices in \(GL_n\)).

Definition 4. With the induced structure of complex projective variety, \(\hat{X}\) is called the compact dual symmetric Hermitian space of \(X\).

In fact, there is a more canonical way to view \(\hat{X}\), as the space representing the functor of filtrations on the trivial bundle with some additional data: given the faithful representation \((V, \xi)\) of \(G\) as above, fix a family of tensors \(t_\alpha\) in \(V\) such that \(G\) is exactly the stabilizer of those tensors inside \(GL(V)\). Notice that the \((t_\alpha)\) define a global tensor on the trivial bundle \(V_S = S \times V_\mathbb{C} \to S\) for each connected complex variety \(S\).

For our fixed \(o \in X\), let \(F_o^*\) be the induced Hodge filtration on \(V(\mathbb{C})\). Consider the functor on connected complex varieties

\[ S \to \mathcal{F}_o(S) = \{\text{Filtrations } F^\bullet\text{ on } V_S | (V_s, F^\bullet_s, (t_\alpha)) \cong (V, F^\bullet_o, (t_\alpha)) \forall s \in S\} \]

where \(V_s\) is the fiber at \(s\), and \(F^\bullet_s\) the induced filtration at that fiber.

Then \(\hat{X}\) represents exactly the functor \(\mathcal{F}_o\).

Definition 5 (Borel embedding). The map

\[ \beta : X \hookrightarrow \hat{X} \quad x \mapsto \text{Fil}(\mu_x) \]

associating to a homomorphism \(x : S \to G_\mathbb{R}\) the conjugacy class of the filtration induced by the associated \(\mu_x\) on the faithful representation \((V, \xi)\) is an open embedding of complex manifolds. Let \(K_o\) be the stabilizer under \(G(\mathbb{R})\) of our \(o \in X\), then the following diagram commutes

\[ \begin{array}{ccc}
G(\mathbb{R})/K_o & \to & G(\mathbb{C})/P_o \\
\uparrow \cong & \nearrow \cong \\
X^\mathbb{C} & \to & \hat{X}
\end{array} \]

Corollary 5. The Shimura variety \(\text{Sh}_K(G, X)\) is a complex manifold.

\(^3\)This is not necessary, one could instead think of \(G(\mathbb{C})\)-conjugacy classes of filtrations on the category of complex representations of \(G\).
3 Moduli interpretation in terms of classifying abelian varieties

We want to see how certain Shimura varieties are moduli spaces for abelian varieties, in a generalization of the fact that modular curves are moduli spaces for elliptic curves.

Let \( f : (G, X) \frown (\text{GSp}, S^\pm) \) be an embedding of Shimura datum, where the target is as in example 4. This just means an embedding \( f : G \frown \text{GSp} \) carrying the conjugacy class of homomorphism \( X \) into \( S^\pm \). By general constructions, this gives rise to a morphism of schemes

\[
\text{Sh}(G, X) \to \text{Sh}(\text{GSp}, S^\pm)
\]

which is equivariant for \( f(\mathbb{A}_f) : \text{G}(\mathbb{A}_f) \to \text{GSp}(\mathbb{A}_f) \) in the obvious way.

**Theorem 6** (Mumford 1965). For every faithful representation \( (V, \xi) \) of \( G \) and every \( x \in X \), the Hodge structure \( (V, \xi \circ h_x) \) is the rational Hodge structure attached to an abelian variety \( A_x \) over \( \mathbb{C} \). Moreover, \( A_x \) is well-determined up to isogeny.

**Remark.** Milne formulates this in the general setup of motives (replacing abelian varieties) but then it’s only an ‘hope’, not a theorem.

Let’s give a bit more details. Let \( A \) be the category of abelian varieties over \( \mathbb{C} \), and let \( A^0 \) be the isogeny category: morphisms are defined from those of \( A \) by tensoring with \( \mathbb{Q} \), so that isogenies in \( A \) (finite kernel and cokernel) become isomorphisms in \( A^0 \).

Recall that a Hodge structure of type \( \{(-1,0), (0,-1)\} \) on a real vector space \( V \) is the same as a complex structure on \( V \). More generally, for a free \( \mathbb{Z} \)-module \( L \), an integral Hodge structure is a complex structure on \( L \otimes_{\mathbb{Z}} \mathbb{R} \).

**Definition 6.** An integral Hodge structure of type \( \{(-1,0), (0,-1)\} \) is polarizable if there exists a non-degenerate alternating map

\[
\psi : L \times L \to \mathbb{Z}
\]

such that \( \psi_c(L^{-1,0}, L^{-1,0}) = 0 \) and similarly for \( L^{0,-1} \).

**Theorem 7.**

1. The functor \( A \mapsto H_1(A, \mathbb{Z}) \) is an equivalence of categories from \( A \) to polarizable integral Hodge structures of type \( \{(-1,0), (0,-1)\} \).

2. The functor \( A \mapsto H_1(A, \mathbb{Q}) \) is an equivalence of categories from \( A^0 \) to the category of polarizable rational Hodge structures of type \( \{(-1,0), (0,-1)\} \).

**Proof.** Exercise.

The proof of the theorem (in particular, the construction of an inverse to the given bijection) will show that the polarization on the Hodge structure produces a polarization on the abelian variety \( = \) an isogeny between \( A \) and its dual which behaves well with respect to double duality.

Let now \( A \) be an abelian variety over \( \mathbb{C} \). Consider its adelic Tate module \( V_f(A) = \lim_N A[N] \otimes \mathbb{A}_f \): this is a free \( \mathbb{A}_f \)-module of rank 2, and it is isomorphic to \( H_1(A, \mathbb{A}_f) = H_1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_f \). We call \( \lambda \) a polarization on \( A \), as well as the induced one on \( H_1(A, \mathbb{Z}) \) and thus on \( V_f(A) \).

**Theorem 8.** Let’s go back to our symplectic Shimura datum: \( (V, \psi) \) is a symplectic space over \( \mathbb{Q} \), and \( G = \text{GSp}(V) \). \( X \) is the set of complex structures on \( V \) polarized by \( \pm \psi \), and for a compact open \( K \subset G(\mathbb{A}_f) \) we defined

\[
\text{Sh}_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K.
\]
Sh\(_K\)(G, X) classifies isomorphism classes of triples \((A, \lambda, \eta_K)\) where \(A\) is an isogeny class of abelian varieties (= an object of \(A^0\)), \(\pm \lambda\) is a polarization of \(A\) and \(\eta_K\) is the \(K\)-orbit of an \(A_f\)-linear isomorphism
\[
\eta : V \otimes \mathbb{A}_f \rightarrow V_f(A)
\]
carrying \(\psi\) into an \(\mathbb{A}^*_f\)-multiple of \(\lambda\).

**Remark.** An isomorphism between triples is an isomorphism in \(A^0\) between the two abelian varieties, identifying the two polarizations up to \(\mathbb{Q}^*_f\)-multiples, and the two \(K\)-orbits of \(A_f\)-linear isomorphisms.

**Proof.** Exercise. \(\square\)

## 4 Definition and construction of automorphic vector bundles

We want to generalize the notion of automorphic form to higher dimensional groups. The main twist is about the weight: while for classical modular forms the weight is simply a positive number (usually \(k \geq 2\)), it turns out that in fact that was meant to be a representation of the maximal compact subgroup: but since this is SO\(_2(\mathbb{R})\), its representations are just indexed by integers.

**Definition 7** (G-vector bundle). Let \(S/k\) be an algebraic variety with an action \(G \times S \rightarrow S\) of an algebraic group \(G/k\). A G-vector bundle on \(S\) is a vector bundle \(V \rightarrow S\) together with an action \(G \times V \rightarrow V\) which makes \(\rho\) \(G\)-equivariant in the obvious way, and such that the maps \(g : V_s \rightarrow V_{g.s}\) are linear for all \(s \in S\).

We are interested in this when \(S = \check{X}\) is the compact dual Hermitian symmetric space to some \(X\) appearing in the Shimura datum \((G, X)\). Let \(J\) be a \(G_C\)-vector bundle on \(\check{X}\). Using the commutative diagram 1, we can pull this back via the Borel embedding to get \(\beta^{-1}(J)\), a \(G(\mathbb{R})\)-vector bundle on \(X\).

Now, Milne is assuming the condition that the \(G_C\)-action on \(J\) 'factors through \(G_{C'}^c\). I was not able to find in his notes what the superscript \(c\) stands for. Obviously this quotient can’t be something too small or the resulting action would be almost trivial. Maybe he means that the restriction of the action to the connected component completely determines it? Or is he quotienting by some central subtorus (he uses the superscript \(ad\) for the adjoint quotient, so he cannot be quotienting by the whole center)? At any case, I checked a few other sources and none seem to mention this point, so we will safely sweep it under the rug.

If we also pick a congruence subgroup \(K\) small enough (so that it acts without fixed points on \(\beta^{-1}(J)\)), we can pass to the quotient and obtain a vector bundle
\[
\mathcal{V}_K(J) = G(\mathbb{Q}) \backslash \beta^{-1}(J) \times \mathcal{G}(\mathbb{A}_f)/K.
\]
For every \(L \supset K\) with finite index there are obvious maps \(\mathcal{V}_K(J) \rightarrow \mathcal{V}_L(J)\), and then for every \(g \in G(\mathbb{A}_f)\) and open compact subgroups \(L \supset g^{-1}Kg\) we also have maps
\[
\mathcal{V}_K(J) \rightarrow \mathcal{V}_L(J) \quad [x, a] \mapsto [x, ag].
\]

**Theorem 9.** 1. The vector bundles \(\mathcal{V}_K(J)\) and the ‘change of groups’ maps above are algebraic.
2. If $X$ has no factor isomorphic to the Poincare disk\(^4\) then every analytic section of $\mathcal{V}_K(J)$ is algebraic, and $H^0(X, \mathcal{V}_K(J))$ has finite dimension over $\mathbb{C}$.

3. The family $(\mathcal{V}_K(J))_K$ is a scheme with a right action of $G(A_f)$.

Obviously the proof of this theorem (in particular the first two points) is everything but trivial: it requires GAGA-type results due to Serre, Grothendieck...

**Definition 8.** A vector bundle of the form $\mathcal{V}_K(J)$ is an **automorphic vector bundle**. A section $f \in H^0(G(\mathbb{Q}) \backslash X \times G(A_f)/K, \mathcal{V}_K(J))$ is an automorphic form of type $J$ and level $K$.

**Example 6.** Here’s an easy way of producing an automorphic vector bundle (at least abstractly). When you stare at the definitions, you see that the initial data is this $G_\mathbb{C}$-vector bundle on $\breve{\mathcal{X}}$.

Notice in particular that for any $G_\mathbb{C}$-vector bundle $I_o$ on $\breve{\mathcal{X}}$, $P_o$ acts on the fiber $I_o$. On the other hand, given an algebraic representation of $P_o$ we can identify its space with an abstract fiber, then use the $G_\mathbb{C}$-action to spread it out to a vector bundle on $G_\mathbb{C}/P_o \cong \breve{\mathcal{X}}$.

It turns out that we have thus an equivalence of categories between $G_\mathbb{C}$-vector bundles on $\breve{\mathcal{X}}$ and representations of $P_o$: so it’s enough to understand $P_o$.

Let’s try to make even more explicit the connection between this setup and the standard notion of ‘automorphic form’. Let $\Gamma \subset \text{Aut}(X)$ be a discrete subgroup - or if you wanna be more comfortable, notice that $G(\mathbb{R}) \longrightarrow \text{Aut}(X)$, and choice a discrete subgroup in there. An **automorphy factor** is a mapping

$$j : \Gamma \times X \longrightarrow \text{GL}(V)$$

into the automorphisms of a complex vector space $V$ such that

1. $x \rightarrow j(\gamma, x)$ is holomorphic for each $\gamma \in \Gamma$.
2. cocycle condition: for each $\gamma, \gamma' \in \Gamma$ and each $x \in X$ we have

$$j(\gamma\gamma', x) = j(\gamma, \gamma'x) \cdot j(\gamma', x).$$

**Definition 9** (Automorphic form). An automorphic form of type $J$ is a holomorphic function $f : X \longrightarrow V$ such that $f(\gamma x) = j(\gamma, x)f(x)$ and $f$ is ‘holomorphic at the cusps’.

Now the connection: pick a $G_\mathbb{C}$-vector bundle $I$ on $\breve{\mathcal{X}}$ and let $V = I_{\beta(o)}$ be the fiber at a fixed $o \in X$. Notice that since $X$ is simply connected, when we pull back along the Borel embedding the isomorphism $V \cong \beta^{-1}(I)_o$ extends to an isomorphism $X \times V \cong \beta^{-1}(I)$, so that by transport of structure we have a $G(\mathbb{R})^0$-action on $X \times V \cong \beta^{-1}(I)$. We are ready to implicitly define the automorphy factor $j$ associated to $I$ by putting

$$\gamma.(x, v) = (\gamma x, j(\gamma, x)v) \quad \forall \gamma \in G(\mathbb{R})^0, x \in X, v \in V. \quad (2)$$

**Exercise.** Check that the restriction of $j$ to $(\Gamma \cap K) \times V$ is an automorphy factor.

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\(^4\)This condition is technical: avoiding dimension 2 issues allows us to work with actual sections, rather than having to restrict to sections with only logarithmic poles and zeros.
Example 7. For piece of mind, here’s our classical modular forms re-interpreted in this setup. Let $G = \text{SL}_2$, and $X = \mathbb{H}$ the upper half plane, which we identify with the open disk in $\mathbb{C}$ by the isomorphism $z \mapsto \frac{z-i}{z+i}$.

The compactification $\tilde{X}$ is (surprise surprise) the Riemann sphere, and the Borel embedding $\beta : X \hookrightarrow \tilde{X}$ embeds $X$ as the upper hemisphere.

Take $o = i \in \mathbb{H}$, so that $P_o = \text{SO}_2(\mathbb{R}) = \{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \}$. Let $\chi_k$ be the $(2k)$-th power of the character $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mapsto e^{2\pi i \theta}$ of $P_0$. Let $V_k$ be the corresponding automorphic vector bundle via example 6. Then sections of $V_k$ holomorphic at infinity are elliptic modular forms of weight $k$.

Exercise. Prove the last statement.
5 Problems

Exercise 1. Find a point $z \in \mathbb{H}$ whose stabilizer under the $\text{SL}_2(\mathbb{R})$-action is the ‘standard’ maximal compact $\text{SO}_2(\mathbb{R})$ of $\text{SL}_2(\mathbb{R})$.


Suppose given a pair $(E, \eta)$ where $E/\mathbb{C}$ is an elliptic curve and $\eta : \mathbb{A}_f \times \mathbb{A}_f \rightarrow V_f(E)$ is an $\mathbb{A}_f$-linear isomorphism that trivializes the adelic Tate module.

Denote $W = H_1(E, \mathbb{Q})$. Fix an isomorphism $\mathbb{Q}^2 = V \xrightarrow{\alpha} W$. As $E$ puts a complex structure on $W$, we can transport it via $\alpha$ to a complex structure on $V$. But a complex structure on $\mathbb{Q}^2$ is exactly a point on $X = \mathbb{C}/\mathbb{R}$, the conjugacy class of $h : x + iy \mapsto (\frac{x}{y}, \frac{y}{x})$, so we get a point $x \in X$.

The adelic Tate module $V_f(E)$ is (basically by definition) isomorphic to $W \otimes_{\mathbb{Q}} \mathbb{A}_f$, so that the level structure $\eta$ becomes an isomorphism

$$\eta : \mathbb{A}_f \times \mathbb{A}_f \rightarrow W \otimes_{\mathbb{Q}} \mathbb{A}_f$$

and on the other hand we also have

$$\mathbb{A}_f \times \mathbb{A}_f \cong \mathbb{Q}^2 \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\alpha \otimes 1} W \otimes \mathbb{A}_f$$

so that the composition

$$\mathbb{A}_f \times \mathbb{A}_f \xrightarrow{\eta} W \otimes \mathbb{A}_f \xrightarrow{(\alpha \otimes 1)^{-1}} \mathbb{A}_f \times \mathbb{A}_f$$

is an invertible $\mathbb{A}_f$-linear map on $\mathbb{A}_f \times \mathbb{A}_f$ - that is: an element $g \in \text{GL}_2(\mathbb{A}_f)$.

We then assign $(E, \eta) \mapsto (x, g) \in \text{Sh}(G, X)$ - the full level Shimura variety.

Now check that the above construction is well-defined. Changing $a$ to some $a'$ is equivalent of multiplying on the left by $\gamma \in \text{GL}_2(\mathbb{Q})$, which translates both the complex structure $h$ and the resulting $g$, so that yields the same point in $\text{Sh}(G, X)$. Given an isogeny $f \in \text{Hom}(E, E') \otimes \mathbb{Q}$ between two pairs $(E, \eta)$ and $(E', \eta')$, we obtain isomorphism $H_1(E, \mathbb{Q}) \rightarrow H_1(E', \mathbb{Q})$ and $V_f(E) \rightarrow V_f(E')$. This change can be absorbed into the construction by replacing $a$ with its post-composition with $H_1(E, \mathbb{Q}) \rightarrow H_1(E', \mathbb{Q})$, and the above argument shows that $(x, g) \in \text{Sh}(G, X)$ is unchanged.

The opposite direction...

Exercise 3. Go back to example 4 and check that the given construction satisfies all the conditions for the Shimura datum. In particular, compute the Hodge structure and check that the given $h$ yields indeed a Cartan involution.


We start by giving the opposite construction. Suppose given a polarizable integral Hodge structure of type $\{(−1, 0), (0, −1)\}$, that is we have a free $\mathbb{Z}$-module $V_\mathbb{Z}$ with a nondegenerate alternating morphism

$$\psi : V_\mathbb{Z} \times V_\mathbb{Z} \rightarrow \mathbb{Z}$$

for which both $V^\vee_{(-1,0)}$ and $V^{(0,-1)}$ are isotropic.

To this, we associate the abelian variety $A = V^{-1,0}/V_\mathbb{Z}$, so that $V_\mathbb{Z} = H_1(A, \mathbb{Z})$ and the Lie algebra is $\text{Lie}(A) = V^{-1,0} = V_{\mathbb{C}}/\text{Fil}^0V_{\mathbb{C}} = V_{\mathbb{C}}/V^{0,-1}$. 

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We give some more details for the second part of the statement, the one about isogeny classes. Recall that in this case we want an equivalence of categories from isogeny classes of abelian varieties over \( \mathbb{C} \) to polarizable rational Hodge structures of type \( ((-1,0),(0,-1)) \). It is clear that the functor is well-defined, so we construct the inverse.

Let \( V_\mathbb{Q} \) be a polarizable rational Hodge structure of the wanted type, so that \( \psi : V_\mathbb{Q} \to V_\mathbb{Q} \to \mathbb{Q} \) is a polarization (with properties similar to the one for \( V_{\mathbb{Z}} \) above). Take a lattice \( V_{\mathbb{Z}} \subset V_{\mathbb{Q}} \) small enough so that \( \psi \) is integral on \( V_{\mathbb{Z}} \). Then on \( V_{\mathbb{Z}} \), \( \psi \) restricts to an integral polarizable Hodge structure of the same type, so by part a we get an abelian variety \( A \) over \( \mathbb{C} \).

It remains to show that the isogeny class of \( A \) is well-defined. If we had taken another lattice \( V'_{\mathbb{Z}} \), they would both be commensurable to their intersection \( V''_{\mathbb{Z}} = V_{\mathbb{Z}} \cap V'_{\mathbb{Z}} \) which is to say that the corresponding abelian variety \( A'' \) receives isogenies from both \( A \) and \( A' \).

**Exercise 5.** Prove theorem 8.

We have the symplectic Shimura variety \( \text{Sh}(\text{GSp}(V),X) \), and we want to show that at level \( K \), the variety \( \text{Sh}_K(\text{GSp}(V),X) \) classifies triples \( (A,\lambda,\eta_K) \) where \( A \) is an isogeny class of abelian varieties, \( \pm \lambda \) is a polarization of \( A \) and \( \nu K \) is a \( K \)-orbit of \( \text{A}_f \)-linear trivializations \( V \otimes \text{A}_f \to V_\mathbb{A}(A) \) into the adelic Tate module of \( A \).

Suppose given a triple \( (A,\lambda,\eta_K) \). By the theorem 7, \( A \) corresponds to a polarizable rational Hodge structure \( W = H_1(A,\mathbb{Q}) \), and the polarization \( \lambda \) corresponds to \( \pm \) a polarization on \( W \) (still denoted \( \lambda \)). The isomorphism \( \eta : V \otimes \text{A}_f \to V_\mathbb{A}(A) \cong W \otimes \text{A}_f \) shows that \( V,W \) are rational symplectic modules of the same dimension.

**Fact 10.** There is only one isomorphism class of such things!

Let’s pick then an isomorphism \( a : V \to W \) as symplectic modules, so that \( \lambda(a(v_1),a(v_2)) = \psi(v_1,v_2) \) for all \( v_1,v_2 \in V \).

By transport of structure, the Hodge structure on \( W \) pulls back to a Hodge structure \( h \) on \( V \), and this is now polarized by \( \pm \psi \) and hence yields an element of \( X \) (which recall had among the many possible interpretations that of complex structure on our symplectic space \( V \) for which \( \pm \psi \) was a polarization).

Play the same game as in the previous theorem, to get an element \( g \in \text{GL}(V \otimes \text{A}_f) \) defined as \( g(v) = (a \otimes 1)^{-1} \eta(v) \) where \( a \otimes 1 : V \otimes \text{A}_f \to W \otimes \text{A}_f \). The assumption that \( \eta \) carries \( \psi \) into \( \lambda \) up to \( \mathbb{Q}^* \)-multiples guarantees that in fact \( g \in \text{GSp}(V) \).

We send then \( (A,\lambda,\eta_K) \) to the class of \( (h,g) \in \text{Sh}_K(\text{GSp}(V),X) \). We check it is well-defined.

Replacing \( a \) by another isomorphism \( a' : V \to W \) yields a left translation by some element \( \lambda \in \text{GSp}(V)(\mathbb{Q}) \), and so the same element in the Shimura variety at level \( K' \).

In the opposite direction: if \( (h,g) \in \text{Sh}_K(\text{GSp}(V),X) \), then by definition \( h \) corresponds to a Hodge structure on \( V \) for which \( \pm \psi \) is a polarization. Like in the previous theorem, find a lattice \( V_{\mathbb{Z}} \) where the polarization is integral, and define \( A = V/V_{\mathbb{Z}} \), with now \( \lambda \) being the induced \( \pm \) polarization. Then \( V = H_1(A,\mathbb{Q}) \) and \( V \otimes \text{A}_f \cong V_\mathbb{A}(A) \). Defining \( \eta \) to be the composition \( V \otimes \text{A}_f \overset{g}{\to} V \otimes \text{A}_f \cong V_\mathbb{A}(A) \) guarantees that \( \eta \) carries \( \psi \) onto a \( \mathbb{Q}^* \)-multiple of \( \lambda \), as \( g \) was a symplectic similitude.

**Exercise 6.** Check that the map \( j : G(\mathbb{R})^0 \times X \to \text{GL}(V) \) defined as in equation 2 defines upon restriction to \( (\Gamma \cap K) \times V \) an automorphy factor.

**Exercise 7.** Go back to the example 7 and check that the sections of the line bundle \( V_k \) that are holomorphic at infinity are indeed elliptic modular forms of weight \( k \) in the classical sense.