Satake map for the mod$p$ derived Hecke algebra

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Abstract

In this paper we explore the structure of the derived Hecke algebra of a $p$-adic group, a graded associative algebra whose degree 0 subalgebra is the classical Hecke algebra. Working with $\mathbb{Z}/p^n$ coefficients, we will establish a Satake homomorphism relating the degree 1 component of this algebra, and the corresponding algebra for the torus.

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1 Introduction

Let $G$ be a reductive, connected, split group over a $p$-adic field $F$. Let $k_F$ be its residue field, and assume that $|k| \geq 5$. Let $K$ be a hyperspecial maximal compact subgroup of $G = G(F)$. Let $S$ be a ring of coefficients.

The classical Hecke algebra with $S$-coefficients is defined to be the algebra of compactly supported, bi-$K$-invariant functions on $G$ valued in $S$, under convolution. An equivalent description is given by the $G$-equivariant functions on $G/K \times G/K$ supported on finitely many orbits, again under convolution and valued in $S$. 
Our derived Hecke algebra \( \mathcal{H}_G \) is then defined (see definition 7 and propositions 7 and 12 for details) as the \( G \)-equivariant cohomology classes on \( G/K \times G/K \), supported on finitely many orbits: we require that \( F(xK, yK) \in H^* (\text{Stab}_G(xK, yK), S) \) and then \( G \)-equivariance means that \( \lambda_g^* F(gxK, gyK) = F(xK, yK) \), where \( \lambda_g^* \) is the isomorphism of cohomology groups induced by conjugation by \( g \). The derived Hecke algebra \( \mathcal{H}_G \) is naturally \( \mathbb{Z}_{\geq 0} \)-graded, and the degree \( n \)-submodule \( \mathcal{H}_G^n \) is a module for the degree 0 subalgebra \( \mathcal{H}_G^0 \) (which is the classical spherical Hecke algebra). We also denote \( \mathcal{H}_G^{\leq n} \), the direct sum of the submodules of degrees no larger than \( n \), with the obvious structure of \( \mathcal{H}_G^n \)-module.

A derived Hecke algebra was used by Schneider in [22] to describe the unbounded derived category of \( G \)-modules with \( k \)-coefficients (where \( k \) is any field of characteristic \( p \)); its properties have been further studied by Ollivier-Schneider in [20] where they explicitly compute the Hecke-module structure of some low degree cohomology groups related to the derived Hecke algebra for \( G = \text{SL}_2 \).

We will work instead with a different derived Hecke algebra that appears in [26]. Although these two definitions (the one mirroring Schneider’s and the one as ‘equivariant cohomology classes’ mentioned above) coincide in characteristic different from \( p \), we are not sure what the relationship is in characteristic \( p \): see appendix B.

**Theorem 1** (Main theorem). Let \( S = \mathbb{Z}/p^a \). Fix a maximal split \( F \)-torus \( T \) of \( G \), and denote \( T = T(F) \); let \( X_*(T) \) be the cocharacter group of the torus. There exists an injective homomorphism

\[
S_T^G : \mathcal{H}_G^{\leq 1} \longrightarrow \mathcal{H}_T^{\leq 1}
\]

whose degree zero component agrees with Herzig’s ([16]) and Henniart and Vigneras’ ([15]) Satake map, or rather with its generalization to \( S \)-coefficients as defined in section 3.

Moreover, with reference to the isomorphism \( \mathcal{H}_T \cong S[X_*(T)] \otimes_S H^*(T(0), S) \) (see proposition 8 for details) the image of \( S_T^G \) is supported on the following thickening of the antidominant cone:

\[
\{ \lambda \in X_*(T) | \langle \lambda, \alpha \rangle \cdot f \leq a \forall \alpha \in \Delta \}
\]

where the residue field \( k_F \) of \( F \) has cardinality \( p^f \geq 5 \) and \( \Delta \) is the basis of the root system \( \Phi(G, T) \) corresponding to the Borel subgroup chosen in the definition of the Satake homomorphism \( S_T^G \).

More precisely, if \( d = \langle \lambda, \alpha \rangle \cdot f \) for some simple root \( \alpha \), then \( S_T^G(F)(\lambda) \) is divisible by \( p^d \) for all \( F \in \mathcal{H}_G^1 \).

The description of a ‘lower bound’ for the image \( S_T^G (\mathcal{H}_G^{\leq 1}) \) - that is to say, a \( \mathcal{H}_G^0 \)-submodule of \( S_T^G (\mathcal{H}_G^{\leq 1}) \) - has been obtained in the PGL\(_2\) case (see also example 1 below), and is current work in progress for general \( G \). Roughly speaking, we expect this lower bound to be a ‘translate’ of the upper bound obtained in the theorem by a generic and ‘very antidominant’ cocharacter.

We remark once more that the classical Hecke algebras of \( G \) and \( T \), interpreted as degree 0 subalgebras of our derived Hecke algebras, are related by a ‘classical’ Satake homomorphism \( S : \mathcal{H}_G^0 \longrightarrow \mathcal{H}_T^0 \) due to Herzig (see [16]) and later generalized by Henniart and Vigneras (see [15]). The fact that our \( S_T^G \) is a map of modules under the classical Hecke algebras means exactly that for \( F_1 \in \mathcal{H}_G^0, F_2 \in \mathcal{H}_T^0 \), we have \( S_T^G(F_1 \circ F_2) = S(F_1) \circ S_T^G(F_2) \).

While an explicit definition of a Satake homomorphism in degrees higher than 1 is not available at the present time, for most global application a good understanding of the degree 1 part of the derived Hecke algebra is quite sufficient. Indeed as explained in [26], the derived Hecke algebra (and its global counterpart) acts on the cohomology of an arithmetic manifold and in many situations this action is generated in degree 1.

We describe explicitly our derived Satake homomorphism in a simple case.
Example 1. Let $G = \text{PGL}_2$ over $\mathbb{Q}_p$, where we suppose $p \geq 3$. Let $T$ be the diagonal torus, and $B$ be the Borel subgroup of upper unipotent matrices, so that the only positive root $\alpha$ has root space the unipotent upper triangular matrices. Let $K = \text{PGL}_2(\mathbb{Z}_p)$ be a fixed maximal compact: notice that by proposition 3.10 in [6], we have $\text{PGL}_2(\mathbb{Z}_p) = \text{GL}_2(\mathbb{Z}_p)/\mathbb{Z}_p^*$, and the matrix $(\frac{a}{c}, \frac{b}{d}) \in \text{PGL}_2(\mathbb{Z}_p)$ denotes in fact its scaling class. Let $S = \mathbb{Z}/p\mathbb{Z}$ be our ring of coefficients. We will write $\lambda$ for the function $\text{PGL}_2(\mathbb{Z}_p) \rightarrow \mathbb{Z}/p\mathbb{Z}$

$$\lambda : \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \rightarrow \log(ad^{-1}) \mod p,$$

which is the composition $\mathbb{Z}_p^* \rightarrow 1 + p\mathbb{Z}_p \xrightarrow{\log} \mathbb{Z}_p \rightarrow \mathbb{Z}_p/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$

Let $T_n \in \mathcal{H}_G^0$ be the indicator function of $K \left( \begin{smallmatrix} p^{-n} & 0 \\ 0 & 1 \end{smallmatrix} \right) K$ and $\tau_n \in \mathcal{H}_T^0$ be the indicator function of $\left( \begin{smallmatrix} p^{-n} & 0 \\ 0 & 1 \end{smallmatrix} \right) T(\mathcal{O})$. On this basis, the degree 0 Satake homomorphism is

$$S(T_n) = \begin{cases} \tau_n & \text{if } n = 0, 1 \\
\tau_n - \tau_{n-2} & \text{if } n \geq 2 \end{cases}$$

For each $n \geq 0$, let $K_n = \{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{PGL}_2(\mathbb{Z}_p) \mid c \in p^n\mathbb{Z}_p \}$ be the common stabilizer of the identity coset and $\left( \begin{smallmatrix} p^{-n} & 0 \\ 0 & 1 \end{smallmatrix} \right) K$. Define elements $f_n \in \mathcal{H}_G^1$ (for $n \geq 2$) and $c_n \in \mathcal{H}_T^1$ (for $n \in \mathbb{Z}$) so that

$$f_n : (K, \left( \begin{smallmatrix} p^{-n} & 0 \\ 0 & 1 \end{smallmatrix} \right) K) \mapsto \text{class of } \lambda \text{ in } H^1(K_n, \mathbb{Z}/p\mathbb{Z}),$$

$$c_n : (T(\mathcal{O}), \left( \begin{smallmatrix} p^{-n} & 0 \\ 0 & 1 \end{smallmatrix} \right) T(\mathcal{O})) \mapsto \text{class of } \lambda \text{ in } H^1(T(\mathcal{O}), \mathbb{Z}/p\mathbb{Z}),$$

and $f_n$, $c_n$ vanish off the $G$- and $T$-orbits of the left-hand elements. It is readily verified that the restriction of $\lambda$ to $K_n$ (for $n \geq 2$) and to $T(\mathcal{O})$ is indeed a homomorphism, and that the $f_n$ and $c_n$ give bases for $\mathcal{H}_G^1$ and $\mathcal{H}_T^1$.

With these notations, the degree 1 Satake homomorphism is given by

$$S_T^G(f_n) = c_n - c_{n-2} \quad \forall n \geq 2.$$

Given the multiplication rule on $\mathcal{H}_G^1$:

$$T_n \circ f_m = \begin{cases} f_{m+n} & \text{if } n = 0, 1 \\
f_{m+n} - f_{m+n-2} & \text{if } n \geq 2 \end{cases}$$

it is immediate to check that this is an extension of the degree 0 Satake homomorphism to a map of algebras $\mathcal{H}_G^1 \rightarrow \mathcal{H}_T^1$.

We notice that in this situation, both Herzig’s and Henniart and Vigneras’ degree 0 Satake homomorphism and our degree 1 extension have imaged supported on the antidominant cone, just as the theorem predicts. Indeed, since $\alpha(\left[ \begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix} \right]) = ts^{-1}$, the antidominant cone consists of characters $t \mapsto \left( \begin{smallmatrix} t^{-n} & 0 \\ 0 & 1 \end{smallmatrix} \right)$ having $n \geq 0$ so that they pair non-positively with $\alpha$.

Remark. A simple generalization of theorem 1 holds for more general coefficient rings of the form $S = \mathcal{O}_E/\varpi_E^m$, where $E/\mathbb{Q}_p$ is a Galois extension of degree $n$. The proof of theorem 1 goes through in the same way to show that if $d = \langle \lambda, \alpha \rangle \cdot f > 0$ for some simple root $\alpha$, then $S_T^G(F(\lambda))$ is divisible by $p^d$ for all $F \in \mathcal{H}_G^1$. Let now $k = \left[ \frac{m}{n} \right]$, so that $kn \leq m < (k+1)n$, then $\mathcal{O}_E \cong \mathbb{Z}_p^n$ as a free $\mathbb{Z}_p$-module and for an appropriate choice of basis we have $\varpi_E^m \mathcal{O}_E \cong (p^k \mathbb{Z}_p)^{n-(m-kn)} \oplus (p^{k+1} \mathbb{Z}_p)^{m-kn}$ so that

$$S = \mathcal{O}_E/\varpi_E^m \mathcal{O}_E \cong (\mathbb{Z}/p^k)^{n-(m-kn)} \oplus (\mathbb{Z}/p^{k+1})^{m-kn}.$$
1.1 Outline of the paper

We now explain the structure of the paper.

In section 2 we set up some notation that we will use in the course of the paper.

Section 3 is devoted to recalling the results of Herzig in [16] and of Henniart and Vigneras in [15] where they study the classical Hecke algebra with coefficients in a field of characteristic $p$. We extend their results to any coefficient ring of the form $S = \mathbb{Z}/p^n\mathbb{Z}$, the proofs go through almost verbatim and we get a Satake homomorphism in degree 0.

In section 4 we set up some machinery related to groupoids, which will greatly clarify and simplify some subsequent proofs. In particular, we are able to re-interpret the derived Hecke algebra of $G$ as compactly supported cohomology of the groupoid where $G$ acts diagonally on $G/K \times G/K$ - see propositions 7 and 12.

Section 5 is devoted to defining the Satake map via a push-pull diagram in groupoid cohomology. Using this setup, we also prove transitivity of the Satake map for inclusion among Levi subgroups: that is to say, if $M$ is a standard Levi subgroup with $G \supset M \supset T$, then $S^G_T = S^M_T \circ S^G_M$.

We give in section 6 another description of the Satake homomorphism by using the Universal Principal Series as a bi-module for two derived Hecke algebras. In particular, we prove here that the degree 1 Satake homomorphism is a map of modules over the respective degree 0 subalgebras.

Section 7 is devoted to completing the proof of our main theorem by studying the image of the Satake homomorphism $S^G_T : \mathcal{H}^1_G \rightarrow \mathcal{H}^1_T$. To study the support of the image, we use the transitivity result of section 6 to put us in the setting where $G$ has semisimple rank 1, where we compute the image as explicitly as needed.

In appendix A we collect the proofs of some results about groupoid cohomology, which we postponed until here to improve the flow of the reading.

Appendix B is devoted to describing the relationship between our definition of the derived Hecke algebra and a categorical definition, both in characteristic $p$ and different from $p$.

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2 Notation

Let $F$ be a $p$-adic field, with ring of integers $\mathcal{O}$, residue field $k$ of cardinality $q = p^f$ and fix a uniformizer $\varpi \in \mathcal{O}$. Assume $|k| \geq 5$.

Let $G$ be a reductive, connected, split group scheme over $\mathcal{O}$. In particular, $G_F$ is a reductive, connected, split group over $F$, and we denote $G = G_F(F)$ its group of $F$-points. We also fix $K = G(\mathcal{O})$, a hyperspecial maximal compact.

Thanks to the splitness condition we can fix a maximal split torus $\mathcal{O}$-scheme $T$, and we have $T(\mathcal{O})$ being the maximal compact subgroup of $T_F(F)$. We also fix a Borel $\mathcal{O}$-subgroup scheme $B$ of $G$, with Levi decomposition $B = T \times U$ defined over $\mathcal{O}$, so that $U$ is a unipotent $\mathcal{O}$-group scheme.

We will often suppress the base change notation and just denote the subgroups above by $B =$
B_{F}(F)$ and similarly for the torus and the unipotent subgroups. Other short-hand notations that we will use are $H_F$ or $H$ for $H_F(F)$, and $H_0$ for $H(0)$ by $H_0$ where $H$ is one of $G, B, T, \ldots$

The character lattice $X^*(T)$ is dual to the cocharacter lattice $X_*(T)$, and the latter is identified with $T(F)/T(0)$ (upon choosing the uniformizer $\varpi$) via

$$X_*(T) \longrightarrow T(F)/T(0) \quad \mu \mapsto \mu(\varpi)$$

Upon choosing the Borel $B$, we obtain a set of positive roots $\Phi^+(G, T) \subset \Phi(G, T)$ inside the root system of $(G, T)$, and a unique basis $\Delta \subset \Phi^+(G, T)$. We define the antidominant coweights

$$X_*(T)_- = \{ \mu \in X_*(T) \mid \langle \mu, \alpha \rangle \leq 0 \quad \forall \alpha \in \Phi^+(G, T) \} ;$$

generically these form a cone inside the lattice $X_*(T)$. The identification of the cocharacter lattice with $T(F)/T(0)$ carries the antidominant cone into $T^-/T(0)$, where

$$T^- = \{ t \in T(F) \mid \val_F(\alpha(t)) \leq 0 \quad \forall \alpha \in \Phi^+(G, T) \} .$$

Given two $p$-adic groups $H \supset L$ and a smooth representation $V$ of $L$, we denote by $\iota_{L}^{H}V$ the compactly induced representation from $L$ to $H$. This consists of the smooth part of the space of functions $f : H \longrightarrow V$ such that $f(lh) = l.f(h)$ for each $l \in L$, $h \in H$, with $H$ acting by right translation.

Our goal is to generalize Herzig’s Satake map from [16]. He defines a map between classical Hecke algebras with $\mathbb{F}_p$ coefficients

$$H_{\mathbb{F}_p}(G, K) \longrightarrow H_{\mathbb{F}_p}(T(F), T(0))$$

and we will extend it to the degree 1 piece of the derived Hecke algebras.

We set up our notation for the derived Hecke algebras (see also definition 7): whenever $H$ is a connected reductive $0$-group scheme, so that we have the $p$-adic group $H_F(F)$ and a maximal compact $H(0)$, we denote by $\mathcal{H}_H = \mathcal{H}(H(F), H(0))$ the derived Hecke algebra of $H(F)$. More generally, we denote by $\mathcal{H}_H^k$ (resp. $\mathcal{H}_H^{k \ast}$) the $k$-th graded submodule of the DHA (resp. the submodule supported on degrees at most $k$ with the induced subalgebra structure).

Each DHA is a graded algebra over its degree-$0$ part, hence we will extend the Satake homomorphism to a morphism $\mathcal{H}_G^1 \longrightarrow \mathcal{H}_T^1$ which is a map of modules compatible with the respective action of the degree-$0$ parts through the ‘classical’ Satake homomorphism defined by Herzig and later generalized by Henniart and Vigneras.

For each standard parabolic $P = M \ltimes V$ - where $M$ is the standard Levi factor containing the maximal torus $T$ - we consider the subgroup $P^0 = M(0) \ltimes V(F)$. We will sometimes use an object very similar to the derived Hecke algebra: that is, we denote by $\mathcal{H}_P$ the algebra of $P(F)$-invariant cohomology classes on $P(F)/P^0 \times P(F)/P^0$ supported on finitely many orbits - with convolution as multiplication.

Remark. Notice that there is no possible ambiguity with the previous notation for the derived Hecke algebra, since we only defined DHA for a reductive group, hence whenever the subscript $P$ is a parabolic we mean indeed the algebra defined in the previous paragraph.

We will often consider stabilizers and isotropy groups. If $G$ is a groupoid and $x \in \text{Ob}(G)$, we use $G_x$ as a shorthand for $\text{Stab}_G(x) = \text{Hom}_G(x, x)$. In case we consider common stabilizers we use multiple subscripts, for instance if $x, y \in G/K$ then $G_{x,y} = \text{Stab}_G(xK, yK) = \text{Ad}(x)K \cap \text{Ad}(y)K$. In case one of the cosets is $K$, we write $K_y = G_{1,y} = K \cap \text{Ad}(y)K$. 

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We will often define sums over orbits, in the following sense: let $G$ be a group acting on a set $X$, the notation
\[ \sum_{G \backslash X} \]
means that we sum over a choice of representatives for the $G$-orbits in $X$. Each time we do that, we will check that the sum does not depend on the chosen representatives, and it has finitely many nonzero summands.

3 Satake homomorphism in characteristic $p$

In this section we recall Herzig’s construction of a Satake homomorphism with $\mathbb{F}_p$-coefficients from [16] and [17] and generalize it (in the case of the trivial representation) to the case of $p$-torsion coefficients $\mathbb{Z}/p^s\mathbb{Z}$. This generalization (see corollary 3 below) is quite simple, and its proof basically consists of rephrasing Herzig’s arguments in the special case of the trivial representation and keeping track of the powers of $p$ appearing as coefficients. We remark that Henniart and Vigneras in [15] work out a wide generalization of Herzig’s results as well, allowing for instance any field of characteristic $p$ (such as $\mathbb{Z}/p\mathbb{Z}$) but not general $p$-torsion coefficients, which is what most interest us.

Recall the main theorem 1.2 from [16]:

**Theorem 2.** Let $H_{\bar{\mathbb{F}}}(G, K)$ be the Hecke algebra of $\bar{\mathbb{F}}$-valued, bi-$K$-invariant functions on $G$, and similarly by $H_{\bar{\mathbb{F}}}(T(F), T(\mathcal{O}))$ the Hecke algebra of $\bar{\mathbb{F}}$-valued, $T(\mathcal{O})$-invariant functions on $T(F)$. The Satake map
\[ S : H_{\bar{\mathbb{F}}}(G, K) \longrightarrow H_{\bar{\mathbb{F}}}(T(F), T(\mathcal{O})) \]
defined by
\[ f \mapsto \left( t \mapsto \sum_{u \in U(F)/U(\mathcal{O})} f(tu) \right) \]
is an injective $\bar{\mathbb{F}}$-algebra homomorphisms with image the subalgebra $H^{-}(T(F), T(\mathcal{O}))$ of functions supported on antidominant coweights.

**Remark.** In fact, Herzig’s statement is much more general (as in, he considers Hecke algebras not just for the trivial representations but for more general representations), but for our purpose the above result is sufficient.

Following Herzig, we remark that the formula for $S$ is extremely similar to the classical one, for Hecke algebras with $\mathbb{C}$-coefficients. The main difference is that the classical formula has a twist by $\delta_B(t)^{1/2}$ (where $\delta_B$ is the modulus character of $B$), whose purpose is mainly to make the Satake homomorphism land in the Weyl-invariant subalgebra of $H(T(F), T(\mathcal{O}))$.

As Gross remarks in [11], one can get rid of this twist and obtain a homomorphism defined over $\mathbb{Z}$, at the cost of losing the remarkable property of Weyl-invariance of the image (or having to twist the natural Weyl-invariant action on the image). This formula is then compatible with Herzig’s Satake map $S$ under reduction mod $p$ for the coefficients.

Herzig’s result has an immediate extension to our setup with general finite $p$-torsion coefficients. More precisely, we have the following corollary:
Corollary 3. Let $S = \mathbb{Z}/p^a\mathbb{Z}$ be the ring of coefficients for the Hecke algebras $H_S(G,K)$ and $H_S(T(F), T(\mathcal{O}))$. Consider the Satake map

$$S : H_S(G,K) \longrightarrow H_S(T(F), T(\mathcal{O}))$$

defined by

$$F \mapsto \left( t \mapsto \sum_{u \in U(F)/U(\mathcal{O})} F(tu) \right).$$

This is an injective $S$-algebra homomorphism with image supported on the thickening of the antidiagonal cone:

$$\{ \lambda \in X_*(T) \mid \langle \lambda, \alpha \rangle f < a \forall \alpha \in \Delta \}. $$

In fact, we have that if $0 < h = \langle \lambda, \alpha \rangle f < a$ for some simple root $\alpha$, then for each $F \in H_S(G,K)$ we have

$$(SF)(\lambda(\varpi)) \in p^h \mathbb{Z}/p^a\mathbb{Z} \subset \mathbb{Z}/p^a\mathbb{Z}.$$ 

Proof. This is a simple adaption of the proof in [16]: one starts with the following adaptation of lemma 2.7(iii):

Claim 4. Let $\lambda \in X_*(T)$ and $\alpha \in \Phi$. Let $t = \lambda(\varpi)$. Suppose that $A$ is an abelian group of exponent $p^a$, and that $\psi : U_\alpha(F)/tU_\alpha(\mathcal{O})t^{-1}$ is a function with finite support such that $\psi$ is left-invariant by $U_\alpha(\mathcal{O})$. Then

$$\sum_{u_\alpha \in U_\alpha(F)/tU_\alpha(\mathcal{O})t^{-1}} \psi(u_\alpha) \equiv 0 \mod p^{f(\lambda,\alpha)}.$$ 

In particular, if $\langle \lambda, \alpha \rangle \cdot f \geq a$, then

$$\sum_{u_\alpha \in U_\alpha(F)/tU_\alpha(\mathcal{O})t^{-1}} \psi(u_\alpha) = 0.$$ 

The proof of this claim is immediate since we are in the split case, so the argument given by Herzig adapts as it is: $tU_\alpha(\mathcal{O})t^{-1}$ is a subgroup of $U_\alpha(\mathcal{O})$ of index $[\mathcal{O} : \varpi^{\langle \lambda, \alpha \rangle} \mathcal{O}] = p^{f(\lambda,\alpha)}$. Therefore if $u_\alpha \in U_\alpha(F)/tU_\alpha(\mathcal{O})t^{-1}$ is in the support of $\psi$, we have $\psi(g_iu_\alpha) = \psi(u_\alpha)$ for each of the $p^{f(\lambda,\alpha)}$ coset representatives $g_i$ of $U_\alpha(\mathcal{O})/tU_\alpha(\mathcal{O})t^{-1}$, and hence the contribution of those summands is a multiple of $p^{f(\lambda,\alpha)}$.

Following through with Herzig’s proof of his theorem 1.2, step 1,2 and 3 go through word by word, while step 4 - whose proof relies entirely on lemma 2.7, in particular lemma 2.7(iii) - is replaced by

Claim 5 (Step 4'). Suppose $0 < h = \langle \mu, \alpha \rangle f$ for some simple root $\alpha$. Then $(SF)(\mu(\varpi)) \equiv 0 \mod p^h$.

In particular, if $f(\lambda, \alpha) \geq a$, then $(SF)(\mu(\varpi)) = 0$ in $\mathbb{Z}/p^a\mathbb{Z}$.

The proof of this claim follows the same argument used by Herzig, with the new input from our modified version of lemma 2.7(iii). \qed
4 Interlude with groupoids

In this section we introduce some notions related to groupoids that allow for a more formal and conceptual understanding of the derived Hecke algebra and the Satake map that we will define later. These notions include groupoid cohomology, pullback and pushforward maps and their properties, and we give an interpretation of the derived Hecke algebra in this setup. Some proofs of formal results, e.g. push-pull squares in groupoid cohomology, are postponed to appendix A, to not disrupt the flow of the paper.

Definition 1 (Topological groupoid). A topological groupoid $X$ is a groupoid together with the data of a topology on the morphism sets $\text{Hom}_X(x, y)$ for all $x, y \in \text{Ob}(X)$. We require that the composition maps $\text{Hom}_X(x, y) \times \text{Hom}_X(y, z) \to \text{Hom}_X(x, z)$ are continuous for all $x, y, z \in \text{Ob}(X)$.

A morphism of topological groupoids $f : X \to Y$ is a morphism of groupoids such that the induced maps on morphism sets $\text{Hom}_X(x_1, x_2) \to \text{Hom}_Y(f(x_1), f(x_2))$ are continuous.

Example 2. Let $G$ be a topological group acting on a set $X$. Consider the groupoid they generate: $X$ is the set of objects, and $\text{Hom}_p(x_1, x_2) = \{ g \in G \mid gx = y \}$. Then each morphism set has a natural topology as a subspace of $G$, and the composition map are clearly continuous, so this is a topological groupoid.

Every groupoid in this paper will be of the type discussed in this example, thus from now on we will often simply say ‘groupoid’, dropping the adjective ‘topological’.

Proposition 6. Given a $G$-set $X$ and a $H$-set $Y$, the pair of a group homomorphism $\rho : G \to H$ and a map of sets $f : X \to Y$ which is $\rho$-equivariant induces a morphism of groupoids. Explicitly, the morphism of groupoids $(f, \rho) : (G \to X) \to (H \to Y)$ is $f$ on the objects and $\rho$ on the morphisms.

Moreover, if $G$ and $H$ are topological groups and $\rho$ is continuous, then $(f, \rho)$ is a morphism of topological groupoids.

Proof. The first part of the proposition is clear. The only thing to check is that the map induced by $\rho$ on each morphism set is continuous. It suffices to check this for $\rho_{x, x} : \text{Hom}(x, x) \to \text{Hom}(y, y)$ where $f(x) = y$. But then $\rho_{x, x}$ is the composition of $\rho_{|G_x} : G_x \to H_y \cap \rho(G)$ and the inclusion of subgroups $H_y \cap \rho(G) \hookrightarrow H_y$. The former is continuous because $\rho$ is, and the latter is continuous since $H_y \cap \rho(G)$ has the subspace topology of $H$, and so does $H_y$. 

Definition 2 (Pullback and homotopy pullback). Let

$$
\begin{array}{ccc}
\text{E} & \to & \text{C} \\
\downarrow^p & & \downarrow \\
\text{A} & \to & \text{C}
\end{array}
$$

be a diagram of groupoids. The pullback is defined to be the subgroupoid $D$ of $A \times E$ whose objects are $\{(a, e) \in A \times E \mid f(a) = p(e)\}$. The morphisms are

$$
\text{Hom}_D((a_1, e_1), (a_2, e_2)) =
\{(a, \varepsilon) \in \text{Hom}_A(a_1, a_2) \times \text{Hom}_E(e_1, e_2) \mid f(a) = p(\varepsilon) \in \text{Hom}_C(f(a_1) = p(e_1), f(a_2) = p(e_2))\}
$$

This has projection maps onto $A$ and $E$ and satisfies the evident universal property

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The homotopy pullback is defined to be the groupoid $Z$ whose objects are triples $(a, \gamma, e) \in A \times \text{Hom}_C(f(a), p(e)) \times E$ and where the morphisms are

$$\text{Hom}_Z((a_1, \gamma_1, e_1), (a_2, \gamma_2, e_2)) = \{(\alpha, \varepsilon) \in \text{Hom}_A(a_1, a_2) \times \text{Hom}_E(e_1, e_2) \mid \gamma_2 \circ f(\alpha) = p(\varepsilon) \circ \gamma_1 \in \text{Hom}_C(f(a_1), p(e_2))\}$$

Again, this has obvious projection maps onto $A$ and $E$ and satisfies a ‘homotopy’ universal property.

If $E, A$ and $C$ are topological groupoids and $f, p$ are morphisms of topological groupoids, then the pullback and the homotopy pullback are also topological groupoids, where the topology on each morphism sets is the subspace topology of $\text{Hom}_A(a_1, a_2) \times \text{Hom}_E(e_1, e_2)$. Moreover, the projection maps to $A$ and $E$ are continuous.

**Remark.** These definitions above are clearly inspired by analogies with homotopy-theoretic constructions - like the following one.

**Definition 3** (Covering morphism). Let $q : X' \rightarrow X$ be a morphism of groupoids. We say that $q$ is a covering morphism if for each object $x' \in X'$ and each morphism $m : q(x') \rightarrow x$ in $X$, there exists a *unique* morphism $\tilde{m} : x' \rightarrow \tilde{x}$ in $X'$ such that $q(\tilde{m}) = m$.

Notice that by example 3.7(i) and remark 3.9(ii) of [3], we obtain that if in a diagram

$$\begin{array}{ccc}
E & \rightarrow & Z \\
p \downarrow & & \downarrow \\
A & \rightarrow & C
\end{array}$$

one of the map is a covering morphism of groupoids, then the homotopy pullback and the pullback are homotopy equivalent. Since the homotopy equivalence (defined before proposition 3.5 in [3]) is the identity on morphism sets, it is immediate that in the setup of topological groupoids the homotopy equivalence preserves the topological structure as well.

**Definition 4** (Connected components). Given a groupoid $X$, we denote by $\pi_0 X$ the set of its connected components - that is to say, the equivalence classes for the relation $x \sim y \iff \text{Hom}_X(x, y) \neq \emptyset$ for any $x, y \in \text{Ob}(X)$.

**Definition 5** (Homotopy equivalence). Let $f : X \rightarrow Y$ be a morphism of (topological) groupoids. This is called a homotopy equivalence if it induces a bijection $\pi_0 f : \pi_0 X \rightarrow \pi_0 Y$ between connected components and for each $a \in \text{Ob}(X)$ we have that $f_a : \text{Stab}_X(a) \rightarrow \text{Stab}_Y(f(a))$ is an isomorphism of (topological) groups.

### 4.1 Cohomology of topological groupoids

We give an ad-hoc definition for the (compactly supported) cohomology of a topological groupoid, which will correspond to our derived Hecke algebra as a module, and then define pullback and pushforward maps on groupoid cohomology.

We will consider cohomology groups for a general locally profinite topological group $L$ with trivial coefficient in the ring $S$: $H^*(L, S)$. By this, we mean the cohomology in the category of discrete $S$-modules with a continuous action of $L$, i.e. the derived functors of $\text{Hom}_{S[L]}(S, \cdot)$.
computed at $S$. In all our applications, $L$ will be a $p$-adic analytic group and $S$ a finite $p$-torsion ring, so section 2 of [9] provide details for the definition of these cohomology groups. See also [10] for a more general perspective on topological groups (and in particular section 9.2 for totally disconnected groups) and [25] for the special case of profinite groups (e.g. compact $p$-adic analytic groups).

**Definition 6** (Groupoid cohomology). Let $X$ be a topological groupoid and $S$ be a coefficient ring. We define the cohomology of $X$ to be the $S$-module $\mathbb{H}^*(X)$ of maps

$$F : \text{Ob}(X) \longrightarrow \bigoplus_{x \in \text{Ob}(X)} H^* (\text{Stab}_X(x), S)$$

such that

1. $F(x) \in H^* (\text{Stab}_X(x), S)$ for each object $x$ of $X$.

2. Each $\phi \in \text{Hom}_X(x,y)$ induces an isomorphism of groups $\text{Stab}_X(x) \longrightarrow \text{Stab}_X(y)$ given by $g \mapsto \phi \circ g \circ \phi^{-1}$, and hence a map in cohomology $H^* (\text{Stab}_X(y)) \xrightarrow{\phi^*} H^* (\text{Stab}_X(x))$. We require that $F(x) = \phi^*(F(y))$.

When $x = y$ this latter condition means that $F(x) \in H^* (\text{Stab}_X(x))$ is invariant under the action of any element $\phi \in \text{Stab}_X(x)$ - a well-known fact from group cohomology. Thus we can think of this second condition as a generalization of the above invariance and we will informally call it 'the $X$-invariance condition'.

We also define the finitely supported cohomology $\mathbb{H}_c^*(X)$ to be the subspace of $\mathbb{H}^*(X)$ satisfying the additional condition

3. $F$ is supported on finitely many connected components of $X$.

**Remark.** We notice that the second condition in the definition (the 'invariance' condition) means that on each connected component of $X$ either $F$ is zero everywhere or nonzero everywhere - thus making sense of condition 3.

Finally, we have a natural 'cup product' operation on $\mathbb{H}^*(X)$ defined as pointwise cup product:

$$(F_1 \circ F_2)(x) = F_1(x) \cup F_2(x) \in H^* (\text{Stab}_X(x)) \quad \forall F_1, F_2 \in \mathbb{H}(X). \quad (1)$$

It is immediate that this operation is associative (since cup product in group cohomology is) and it has a unit element, namely the map

$$1 : \text{Ob}(X) \longrightarrow \bigoplus_{x \in \text{Ob}(X)} H^* (\text{Stab}_X(x), S) \quad \text{defined as} \quad 1(x) = 1 \in H^0 (\text{Stab}_X(x), S)$$

for all objects $x \in \text{Ob}(X)$. Since obviously $\text{Supp} (F_1 \circ F_2) \subseteq \text{Supp} F_1 \cap \text{Supp} F_2$, we obtain that the compactly supported cohomology $\mathbb{H}_c^*(X)$ is an ideal of $\mathbb{H}^*(X)$ under this operation, and in particular cup product preserves $\mathbb{H}_c^*(X)$.

Let’s set up some notation. We denote $[G] = G(F)/G(\emptyset) = G/K$, and for any standard parabolic $P = M \times \mathbb{V}$ with standard Levi $M$, we similarly denote $[M] = M(F)/M(\emptyset) = P(F)/P^\circ$. We have the left-multiplication action of $G(F)$ on $[G]^2$, and we interpret this action as giving us a groupoid $G = (G_F \rightrightarrows [G]^2)$. Similarly we have the groupoid $M$. Explicitly, $G$ has underlying set $[G]^2$ and

$$\text{Hom} ((g_1, g_2), (g'_1, g'_2)) = \{ g \in G(F) \mid gg_1 = g'_1 \text{ and } gg_2 = g'_2 \text{ in } [G] \}.$$
Definition 7 (derived Hecke algebra). Let $G = (G_F \leadsto [G]^2)$ be our favorite groupoid. We define the derived Hecke algebra to be the compactly supported groupoid cohomology $\mathbb{H}^*_c(G)$. The multiplication operation is described in proposition 7. We will also denote the derived Hecke algebra by $\mathcal{H}_G$.

To describe what the convolution operation looks like on $\mathbb{H}^*_c(G)$, we need the notions of pullback and pushforward in cohomology.

Definition 8 (Pullback in cohomology). Let $i : X \longrightarrow Y$ be a continuous morphism of topological groupoids, and let $F \in \mathbb{H}^*(X)$. Since $i$ is a natural transformation, for each object $x \in \text{Ob}(X)$ we have an induced continuous map

$$\text{Stab}_X(x) = \text{Hom}_X(x,x) \xrightarrow{i_*} \text{Hom}_Y(i(x),i(x)) = \text{Stab}_Y(i(x))$$

and hence a map in cohomology in the other direction: $i^* : H^*(\text{Stab}_Y(i(x)),S) \longrightarrow H^*(\text{Stab}_X(x),S)$. We define

$$(i^*F)(x) := i^*F(i(x))$$

Remark. In general, pullback does not preserve the ‘compact support’ condition 3 of definition 6. But it does if we require the additional assumption that $i$ induces a map on connected components $\pi_0 i : \pi_0 X \longrightarrow \pi_0 Y$ which has finite fibers.

Notice in particular that an homotopy equivalence $i : X \longrightarrow Y$ as in definition 5 induces via pullback an isomorphism of (topological) groupoid cohomology, preserving the compactly supported cohomology.

In the appendix we check that the definition is well-posed (proposition 30).

We now define pushforward maps in cohomology for a special class of covering morphisms $i : X \longrightarrow Y$; notice that for each covering morphism $i : X \longrightarrow Y$ we have a natural injection between isotropy groups $X_x \hookrightarrow Y_{i(x)}$ for each $x \in \text{Ob}(X)$.

Definition 9 (Finite covering morphism). Let $i : X \longrightarrow Y$ be a continuous morphism of topological groupoids. We call it a finite covering morphism if it is a covering morphism such that for each $x \in \text{Ob}(X)$ the inclusion $X_x \hookrightarrow Y_{i(x)}$ is finite index and open.

Remark. By a theorem of Nikolov and Segal (see [19]) in case we have a finite index inclusion of profinite groups $G \hookrightarrow H$ with $H$ topologically finitely generated, then $G$ is automatically open. Moreover, a compact $p$-adic analytic group is necessarily finitely generated (this follows from theorem 8.1 in [7]), so the ‘open’ condition is redundant in many cases of interest for us, for instance whenever $H \subset K = G(\mathbb{O})$.

Definition 10 (Pushforward map in cohomology). Let $i : X \longrightarrow Y$ be a finite covering morphism. We define the pushforward as follows: let $F \in \mathbb{H}^*_c(X)$, and $y \in \text{Ob}(Y)$, we set

$$(i_*F)(y) = \sum_{Y_y \setminus x \in i^{-1}(y)} \text{cores}_{\text{Stab}_X(x)} \text{Stab}_Y(y) F(x).$$

Then $i_*F \in \mathbb{H}^*_c(Y)$.

The action of the stabilizer $Y_y = \text{Stab}_Y(y) = \text{Hom}_Y(y,y)$ on the fiber $i^{-1}(y)$ is given as follows: let $h \in Y_y$ and $x \in i^{-1}(y)$, then the definition of covering morphism says that there exists a unique morphism $\tilde{h}$ in $X$ lifting $h : i(x) = y \leftrightarrow y$. So $\tilde{h} : x \leftrightarrow x'$, and since it lifts $h$ we must have $i(x') = i(\tilde{h}(x)) = h(i(x)) = h(y) = y$, so that $x' \in i^{-1}(y)$. The action is then defined as $h.x = x'$.  

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In the appendix, we check that this definition is well-posed (proposition 31).

Remark. In fact, the definition of pushforward via the same formula applies to a subspace of \( \mathbb{H}^*(X) \) slightly larger than \( \mathbb{H}^*_c(X) \).

For the formula \((i_*F)(y) = \sum_{y \in Y} \text{cores}_{\text{Stab}_Y(y)} F(x)\) to be well-defined, we will only use that \( F \) is compactly supported when checking that the summation has only finitely many summands - so in fact it is enough to require that for each \( y \in \text{Ob}(Y) \), \( F \) is supported on finitely many connected components of \( X \) above \( y \), a condition which we can sum up as \( i \)-fiberwise compactly supported.

If \( F \in \mathbb{H}^*(X) \) is \( i \)-fiberwise compactly supported, \( i_*F \in \mathbb{H}^*(Y) \) may be not compactly supported. Notice however that the submodule of \( i \)-fiberwise compactly supported cohomology classes is an ideal of \( \mathbb{H}^*(X) \) under pointwise cup product.

We describe now the convolution operation on the derived Hecke algebra.

**Proposition 7** (Convolution on the derived Hecke algebra). Consider the groupoid \( G = (G_F \simeq [G]^3) \) whose compactly supported cohomology \( \mathbb{H}^*_c(G) \) is by definition the derived Hecke algebra \( \mathbb{H}_G \). Consider the following diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{i_{1,2}} & G \\
\downarrow & & \downarrow \\
(G_F \simeq [G]^3) & \xrightarrow{i_{1,3}} & G
\end{array}
\]

where the map \( i_{1,2} : (G_F \simeq [G]^3) \to G \) is projection on the first and second factors on object, and the obvious inclusion of morphisms - similarly for \( i_{1,3} \) and \( i_{2,3} \).

Letting \( F_1, F_2 \in \mathbb{H}^*_c(G) \), we describe their convolution in the following way: we pullback them to \((G_F \simeq [G]^3)\) via the maps \( i_{1,2} \) and \( i_{2,3} \), we cup them as in equation 1, and then push them forward to \( G \) via \( i_{1,3} \).

This convolution operation puts an algebra structure on the derived Hecke algebra \( \mathbb{H}_G \).

Proof. This is a particular instance of our general setup described in the appendix A, fact 33. The well-definedness of this operation follows from the fact that \( K \) is open and compact in \( G \), as explained in the remark following fact 33. We will record in proposition 12 that this operation is indeed associative, but we can already notice that this too follows from fact 33.

We record for the sake of future computation an explicit formula for convolution in this algebra: fix \( x, y \in [G] \), we have

\[
(i_{1,3})_* \left( i_{1,2}^* F_1 \cup i_{2,3}^* F_2 \right) (x, y) = \sum_{G_{x,y} \setminus \{(x,y)\} \in (i_{1,3})^{-1}(x,y)} \text{cores}_{G_{x,y}} \left( i_{1,2}^* F_1 \cup i_{2,3}^* F_2 \right) (x, z, y) =
\]

\[
= \sum_{G_{x,y} \setminus \{x\} \in \text{Ob}[G]} \text{cores}_{G_{x,y}} \left( i_{1,2}^* F_1(x, z, y) \cup i_{2,3}^* F_2(x, z, y) \right) =
\]

\[
= \sum_{G_{x,y} \setminus \{x\} \in \text{Ob}[G]} \text{cores}_{G_{x,y}} \left( \text{res}_{G_{x,y}} F_1(x, z) \cup \text{res}_{G_{x,y}} F_2(z, y) \right).
\]

Notice that this is exactly the convolution formula for the derived Hecke algebra in its interpretation as equivariant cohomology classes as described, for instance, in [26].
We made use of the newly computed formula for convolution to show that the derived Hecke algebra of the torus has the simple description mentioned in the introduction:

**Proposition 8.** The map

\[ \mathcal{H}_T \longrightarrow S[X_*(T)] \otimes_S H^*(T(\mathcal{O}), S) \quad F \mapsto \tilde{F} \]

where

\[ \tilde{F}(\mu) = F(T(\mathcal{O}), \mu(\varpi)T(\mathcal{O})) \in H^*(T(\mathcal{O}), S) \text{ for all } \mu \in X_*(T) \]

is an isomorphism of graded algebras. The target \( S[X_*(T)] \otimes_S H^*(T(\mathcal{O}), S) \) is the tensor product of \( S \)-algebras and the grading is on the cohomology factor of this tensor product.

**Proof.** Since \( F \) is compactly supported, \( \tilde{F} \) is well-defined as an element of the tensor product. Moreover, the map \( F \mapsto \tilde{F} \) is obviously linear. We notice that \( X_*(T) \) and \( T(F)/T(\mathcal{O}) \) are in bijection as mentioned in the 2 section, via \( \mu \mapsto \mu(\varpi) \), and thus the map \( F \mapsto \tilde{F} \) gives a bijection between the ‘canonical bases’ of \( \mathcal{H}_T \cong \bigoplus_{\mu \in X_*(T)} H^*(\text{Stab}_{T(\mathcal{O})}(tT(\mathcal{O})), S) = \bigoplus_{t \in T(F)/T(\mathcal{O})} H^*(T(\mathcal{O}), S) \text{ and of } S[X^*(T)] \otimes_S H^*(T(\mathcal{O}), S). \)

It remains to check that the map respects the algebra structure, i.e. that given \( F_1, F_2 \in \mathcal{H}_T \) we have \( \tilde{F}_1 \cdot \tilde{F}_2 = \tilde{F}_1 \circ \tilde{F}_2 \) in \( S[X^*(T)] \otimes_S H^*(T(\mathcal{O}), S) \). Notice that when we specialize equation 2 to the case \( G = T \), we obtain that for any \( x, y, z \in [T] \) the stabilizers become \( G_{x,z} = G_{x,y} = G_{y,z} = G_{x,y,z} = T(\mathcal{O}) \) and hence the formula becomes

\[ (F_1 \circ F_2)(xT(\mathcal{O}), yT(\mathcal{O})) = \sum_{T(\mathcal{O}) \backslash \{z \in [T] \}} F_1(x, z) \cup F_2(z, y). \]

Since \( T(F) \) is abelian, the \( T(\mathcal{O}) \)-action on \([T]\) is trivial and hence we are not summing across orbits, but simply across points \( z \in [T] \). In particular we get

\[ \tilde{F}_1 \circ \tilde{F}_2(\mu) = (F_1 \circ F_2)(T(\mathcal{O}), \mu(\varpi)T(\mathcal{O})) = \sum_{\mu \in X_*(T) \equiv [T]} F_1(T(\mathcal{O}), \lambda(\varpi)T(\mathcal{O})) \cup F_2(\lambda(\varpi)T(\mathcal{O}), \mu(\varpi)T(\mathcal{O})). \]

On the other hand, the algebra tensor product structure gives

\[ \tilde{F}_1 \cdot \tilde{F}_2(\mu) = \sum_{\lambda \in X_*(T)} \tilde{F}_1(\lambda) \cdot \tilde{F}_2(\lambda^{-1}(\mu) = \sum_{\lambda \in X_*(T)} F_1(T(\mathcal{O}), \lambda(\varpi)T(\mathcal{O})) \cup F_2(T(\mathcal{O}), \lambda(\varpi)^{-1} \mu(\varpi)T(\mathcal{O})), \]

so it remains to show that \( F_2(T(\mathcal{O}), \lambda(\varpi)^{-1} \mu(\varpi)T(\mathcal{O})) = F_2(\lambda(\varpi)T(\mathcal{O}), \mu(\varpi)T(\mathcal{O})) \) for every \( \lambda, \mu \in X_*(T) \). This follows immediately from \( T \)-invariance of \( F_2 \) and the fact that the conjugation action of \( \lambda(\varpi) \) on \( H^*(T(\mathcal{O}), S) \) is trivial. \( \square \)

In a way similar to proposition 7, we can re-interpret convolution on the algebra \( \mathcal{H}_P \) whose definition was outlined in section 2.

**Proposition 9.** Let \( [M] = P_F/P^\circ = M_F/M_0 \). Consider the groupoid \( (P_F \rightrightarrows [M]^2) \) whose compactly supported cohomology \( H^*_c((P_F \rightrightarrows [M]^2)) \) is the derived Hecke algebra \( \mathcal{H}_P \). Consider the following diagram:

\[
\begin{array}{ccc}
(P_F \rightrightarrows [M]^2) & \xrightarrow{i_{1,2}} & (P_F \rightrightarrows [M]^2) \\
(P_F \rightrightarrows [M]^2) & \xrightarrow{i_{2,3}} & (P_F \rightrightarrows [M]^2) \\
(P_F \rightrightarrows [M]^2) & \xrightarrow{i_{1,3}} & (P_F \rightrightarrows [M]^2)
\end{array}
\]
where the map \( i_{1,2} : (P_F \sim [M]^3) \to (P_F \sim [M]^2) \) is projection on the first and second factors on object, and the obvious inclusion of morphisms - similarly for \( i_{1,3} \) and \( i_{2,3} \).

Letting \( F_1, F_2 \in H_*^c(\{P_F \sim [M]^2\}) \), we describe their convolution in the following way: we pullback them to \( (P_F \sim [M]^3) \) via the maps \( i_{1,2} \) and \( i_{2,3} \), we cup them as in equation 1, and then push them forward to \( (P_F \sim [M]^2) \) via \( i_{1,3} \).

**Proof.** This follows from fact 33 once we check that the three conditions of the fact are satisfied. Obviously \( i_{1,3} \) is a covering morphism. Given \((m_1 P^o, m_2 P^o, m_3 P^o) \in [M]^3\) its stabilizer is \( M(F)_{m_1,m_2,m_3} V(F) \) while the stabilizer of \((m_1 P^o, m_2 P^o) \) is \( M(F)_{m_1,m_2} V(F) \). The index is then \([M(F)_{m_1,m_2} : M(F)_{m_1,m_2,m_3}]\) which is finite since \( M(\emptyset) \) is open compact in \( M(F) \). This proves condition 3.

Condition 1 says that \( i_{1,2} F_1 \cup i_{2,3} F_2 \) is \( i_{1,3} \)-fiberwise compactly supported for each \( F_1, F_2 \in H_F \) and as explained in the appendix, it suffices to check that for all \( m_1, m_2 \in [M] \) we have finite index of \( \text{Stab}_{P_F}(m_1, m_2) \subset \text{Stab}_{P_F}(m_1) \) which holds exactly as in the previous paragraph.

Finally, to check condition two we notice that since \( V_F \) is normal in \( P_F \) we have

\[
P^o m P^o n P^o = P^o m M_0 n P^o = V_F (M_0 m M_0 n M_0) V_F.
\]

Since \( M_0 \subset M_F \) is open and compact, we have a finite disjoint union \( M_0 m M_0 n M_0 = \bigcup_{i=1}^N M_0 m_i M_0 \) and thus

\[
P^o m P^o n P^o = V_F \left( \bigcup_{i=1}^N M_0 m_i M_0 \right) V_F = \bigcup_{i=1}^N P^o m_i P^o.
\]

We mention here a few relevant general results concerning pushforward and pullback of cohomology classes, whose proofs we delay until the appendix.

**Lemma 10.** Let

\[
\begin{array}{ccc}
X & \xrightarrow{k} & Z \\
\downarrow{i} & & \downarrow{j} \\
Y & & \\
\end{array}
\]

be a commutative triangle of topological groupoid morphisms. Suppose that \( i \) is a finite covering morphism and that \( j \) induces inclusions between isotropy groups.

Let then \( F \in \mathbb{H}^*(X) \) be \( i \)-fiberwise compactly supported and \( G \in \mathbb{H}^*(Z) \), we have

\[
i_* F \cup j^* G = i_*(F \cup k^* G) \quad \text{in} \quad \mathbb{H}^*(Y).
\]

Moreover, if \( j \) and \( k \) induce finite-fibers maps on connected components, then each operation on cohomology preserves the compact support and thus if \( F \in \mathbb{H}^*_c(X) \) and \( G \in \mathbb{H}^*_c(Z) \), the formula above holds in \( \mathbb{H}^*_c(Y) \).

**Lemma 11.** Let

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi} & X \\
\downarrow{\tilde{i}} & & \downarrow{i} \\
Y & \xrightarrow{\pi} & A \\
\end{array}
\]

be a pullback square of topological groupoids, where \( i \) is a finite covering morphism. Suppose also that \( \pi \) is continuous and induces open injections of isotropy groups at all objects (e.g. \( \pi \) is also a
finite covering morphism, but in fact it suffices that every morphisms in $A$ has at most a unique
lift under $\pi$).

Let $F \in \mathbb{H}^*(X)$ be $i$-fiberwise compactly supported, then $\tilde{\pi}^*F$ is $\tilde{i}$-fiberwise compactly supported and we have

$$\tilde{(i)}_*((\tilde{\pi})^*F) = \pi^*(i_*F).$$

If moreover $\pi$ induces a finite-fibers map on connected components, then given $F \in \mathbb{H}^*_c(X)$ the same formula above holds in $\mathbb{H}^*_c(Y)$.

As a first application of these lemmas, we can use fact 33 again which shows immediately that convolution on our derived Hecke algebra as defined in proposition 7 is associative.

**Proposition 12.** Let $A, B, C \in \mathbb{H}^*_c(G)$. Then $(A \circ B) \circ C = A \circ (B \circ C)$.

**Proof.** We already checked that fact 33 applies to this setup, and hence we obtain an algebra structure. \hfill \Box

# 5 Satake map via groupoids

In this section we construct our Satake map as a combination of pushforwards and pullbacks in groupoid cohomology. Recall that the classical Satake homomorphism, both over $\mathbb{C}$ (see for instance [11]) and over $\mathbb{F}_p$ (as in [16]), can be described as ‘integration over the unipotent radical’. More precisely, in [17], sections 2.2 and 2.3, Herzig defines the Satake transform between the mod $p$ classical Hecke algebra of a reductive $p$-adic group $G$ and that of a standard Levi subgroup $M$. Letting $\mathcal{P}_F = M_F \ltimes N_F$ be the Levi decomposition of the standard parabolic $P$ containing $M$, he defines an algebra homomorphism $S^M_G : \mathcal{H}_G^0 \to \mathcal{H}_M^0$ as

$$f \mapsto \left( m \mapsto \sum_{n \in N_{F}/N_{O}} f(mn) \right).$$

The notation above takes $f \in \mathcal{H}_G^0$ to be a bi-$K$-invariant function: if we want to re-interpret this map in terms of the description as $G$-invariant functions on $G/K \times G/K$, we obtain

$$S^M_G : f \mapsto \left( (M_0, mM_0) \mapsto \sum_{n \in N_{F}/N_{O}} f(K, mnK) \right).$$

This is more suited for a generalization to our groupoid cohomology language: noticing that pushforward as in definition 10 corresponds to ‘integration along the fiber’, we define our Satake map to be the result of the pushforward and pullbacks as follows. Recall that we denote $[M] = M_F/M_0 = P_F/P^0$. Consider the diagram:

$$\begin{array}{ccc}
(G_F \twoheadrightarrow [G] \times G_F/P_0) & \xrightarrow{i} & \mathcal{G} = (G_F \twoheadrightarrow [G]^2) \\
\downarrow \pi & & \\
(M_F \twoheadrightarrow [M]^2) & \xrightarrow{i_M^G} & (G_F \twoheadrightarrow [G] \times G_F/P^0)
\end{array}$$

The map $i_M^G$ is an inclusion, both at the level of groups and at the level of sets. $i$ is the identity at the group level, and the surjection $G_F/P_0 \twoheadrightarrow G_F/K$ at the level of sets. $\pi$ is a quotient map at
the level of sets, and it is the map encoding ‘integration over the unipotent radical’ as discussed above. It is clear that all the morphisms are continuous with respect to the topologies induced on each groupoid as in example 2.

**Definition 11** (Satake map in groupoid cohomology). We define the Satake map as the function $\mathbb{H}^*_c(G) \to \mathbb{H}^*_c(M)$ induced by the sequence of pullbacks and pushforwards in the diagram above.

**Remark.** To prove that this is well-defined, we need to check that $\pi$ is a finite covering morphism and that each inclusion map induces a map on connected components which has finite fibers.

The fact that $\pi$ is a covering is immediate, since in both groupoids the same group $G_F$ is acting. To check the finiteness assumption, it suffices to check finite index inclusion of isotropy groups on a representative for each connected component of the source groupoid: let then $(K, xP_0)$ be one such representative with $x \in P_F$ by the Iwasawa decomposition.

Then the isotropy group of $(K, xP_0)$ is $K \cap \text{Ad}(x)P_0$. We also have $\pi((K, xP_0)) = (K, xP^\circ)$ whose isotropy group is $K \cap \text{Ad}(x)P^\circ$.

Since $x \in P_F$, we have $\text{Ad}(x)P^\circ \subset P_F$ so that $K \cap \text{Ad}(x)P^\circ = K \cap \text{Ad}(x)P^\circ \cap P_F = P_0 \cap \text{Ad}(x)P^\circ$.

Similarly, $K \cap (xP_0) = P_0 \cap \text{Ad}(x)P_0$, which is finite index in $P_0$ since $P_0 \subset P_F$ is open and closed. A fortiori, $P_0 \cap \text{Ad}(x)P_0$ will be finite index in $P_0 \cap \text{Ad}(x)P^\circ$, and in particular open since $P_0 \cap \text{Ad}(x)P^\circ$ is compact. This shows that we can pushforward along $\pi$ while preserving compact support.

It remains to check that each inclusion map has finite fibers on connected components.

For $i^G_M$ we have $\pi_0M = M_0\backslash M_F/M_0$ but also $(G_F \hookrightarrow [G] \times G_F/P^\circ)$ has connected components indexed by $K\backslash G_F/P^\circ \equiv P_0 \backslash P_F/P^\circ \equiv M_0\backslash M_F/M_0$ where the first equivalence follows by the Iwasawa decomposition and the second by the Levi decomposition of $P_F$ and the fact that $P^\circ = M_0V(F)$. Hence in fact $i^G_M$ induces a bijection on connected components.

Finally, we need to show that $i$ induces a finite-fibers map on connected components, which is to say we need to show that each double coset $KgK$ decomposes into finitely many double cosets in $K\backslash G/P_0$. But by compactness of $K$ we already know that $KgK$ decomposes in finitely many left $K$-cosets, so a fortiori it will split into finitely many double $(K, P_0)$-cosets. This proves that the Satake homomorphism is well-defined as a function $\mathbb{H}^*_c(G) \to \mathbb{H}^*_c(M)$.

We check that this definition coincides with the one to be obtained in section 6 via an explicit use of the derived universal principal series. Let then $F \in \mathbb{H}^*_c(G)$, we have that

$$\pi_*(i^*F)(x, y) = \sum_{\text{cores}_{\text{Stab}_G(x,y)}^{\text{Stab}_G(x,y)}} \text{cores}_{\text{Stab}_G(x,y)}(i^*F)(x, y) = \sum_{\text{cores}_{\text{Stab}_G(x,y)}^{\text{Stab}_G(i(x,y))}} \text{res}_{\text{Stab}_G(x,y)}^{\text{Stab}_G(i(x,y))} F(i(x, y)).$$

To check it coincide with our formula below we can pick one representative for each orbit, so let’s fix $x = K$ and $y = mM_0V_F$. Then $\text{Stab}_G(x, y) = M(\mathcal{O})_mV(\mathcal{O})$, while for each $(K, mvP_0) \in \pi^{-1}(K, mM_0V_F)$ we have $\text{Stab}_G(K, mvP_0) = P(\mathcal{O})_{mv}$. The above sum becomes

$$\sum_{M(\mathcal{O})_mV(\mathcal{O})} \text{cores}_{P(\mathcal{O})_{mv}}^{M(\mathcal{O})_mV(\mathcal{O})} \text{res}_{P(\mathcal{O})_{mv}}^{K_{mv}} F(K, mvK)$$

which coincides with formula 5 for $(F.1)$ after the definition 13 of the Satake homomorphism in section 6 once we prove that $P(\mathcal{O})_{mv} = M(\mathcal{O})_mV(\mathcal{O}) \cap K_{mv}$.
forces \( \text{Ad}(m)\eta \in M(\mathcal{O}) \) and hence the Levi component of each element of \( P(\mathcal{O})_{mv} \) is in \( M(\mathcal{O})_m \), which proves that \( P(\mathcal{O}) \subset M(\mathcal{O})_m V(\mathcal{O}) \).

For the opposite inclusion, we notice that \( K_{mv} \cap M(\mathcal{O})_m V(\mathcal{O}) = K \cap \text{Ad}(mv)K \cap P(\mathcal{O}) \cap M(\mathcal{O})_m V(\mathcal{O}) = \text{Ad}(mv)P(\mathcal{O}) \cap P(\mathcal{O}) \cap M(\mathcal{O})_m V(\mathcal{O}) \subset P(\mathcal{O})_{mv} \).

The map \( i^G_M \) induces a pullback in cohomology which corresponds simply to restriction, using the canonical inclusion of isotropy groups. Hence the map \( \mathbb{H}^*_c(G) \xrightarrow{S} \mathbb{H}^*_c(M) \) results to be

\[
S(F)(M, m \mathcal{O}) = \text{res}_{M(\mathcal{O})_m V(\mathcal{O})}^{\mathcal{O}(\mathcal{O})_m V(\mathcal{O})} \sum_{\mathcal{M}(\mathcal{O})_m V(\mathcal{O}) \setminus \{mv \in P(\mathcal{O})_m \}} \text{cores}_{P(\mathcal{O})_m}^{\mathcal{M}(\mathcal{O})_m V(\mathcal{O})} F(K, mvK)
\]

which coincides with formula 6 in section 6.

### 5.1 Transitivity of the Satake map

In the subsection we show that the Satake map is transitive for inclusion among Levi factors, using the groupoid setup described above: given a standard parabolic \( P \) containing the fixed Borel subgroup \( B \), we obtained three groupoid diagrams for \( S^G_M, S^M_F \) and \( S^G_F \) and we show that the composition of the first two induces on cohomology the same map as the third one, via repeated applications of lemmas 10 and 11. Roughly speaking, the equivalence of the diagrams boils down to the fact that integrating along the unipotent radical \( U_F \) of \( B_F \) is equivalent to integrating along the unipotent radical \( V_F \) of \( P_F \) first, then along the unipotent radical \( (U_F \cap M_F) \) of \( (B_F \cap M_F) \), since \( U_F = (U_F \cap M_F) \times V_F \).

Let us then choose a standard parabolic \( P = M \times V \supset T \times U = B \) such that \( M \supset T \) and denoting \( B' = B \cap M \) a Borel of \( M \), we have the diagram

\[
\begin{array}{ccc}
(G_F \hookrightarrow [G] \times G_F/B_0) & \xrightarrow{i} & G = (G_F \hookrightarrow [G]^2) \\
\downarrow \pi & & \downarrow \pi \\
(T_F \hookrightarrow [T]^2) & \xrightarrow{i^G_F} & (G_F \hookrightarrow [G] \times G_F/B^\circ)
\end{array}
\]

which describes the Satake map \( S^G_F \). We want to prove that \( S^G_F = S^M_F \circ S^G_M \) which is to say that the same function between compactly supported cohomology algebras is obtained by composing the following two diagrams:

\[
\begin{array}{ccc}
(G_F \hookrightarrow [G] \times G_F/P_0) & \xrightarrow{i} & G = (G_F \hookrightarrow [G]^2) \\
\downarrow \pi & & \\
(M_F \hookrightarrow [M]^2) & \xrightarrow{i^G_M} & (G_F \hookrightarrow [G] \times G_F/P^\circ)
\end{array}
\]

and then

\[
\begin{array}{ccc}
(M_F \hookrightarrow [M] \times M_F/B_0) & \xrightarrow{i} & M = (M_F \hookrightarrow [M]^2) \\
\downarrow \pi & & \\
(T_F \hookrightarrow [T]^2) & \xrightarrow{i^G_T} & (M_F \hookrightarrow [M] \times M_F/B^\circ)
\end{array}
\]

To prove the statement we will repeatedly apply lemma 11, and replace the diagrams with ones that have the same effect on cohomology, until the two diagrams will coincide.

Let’s start with the second one.
Claim 13. The diagram

$$(M_F \rightrightarrows [M] \times P_F/P_0) \overset{i'}{\rightarrow} (G_F \rightrightarrows [G] \times G_F/P_0)$$
$$\pi' \downarrow \quad \pi'$$
$$M = (M_F \rightrightarrows [M]^2) \overset{i}{\rightarrow} (G_F \rightrightarrows [G] \times G_F/P^\circ)$$

is a pullback square of groupoids. Moreover, the maps involved satisfy the assumptions of lemma 11: that is to say, $\pi$ is a finite covering morphism and $i$ induces inclusion on isotropy groups, and a finite-fibers map on connected components.

Proof. While checking that the Satake map was well-defined we already checked all conditions to make sure that the pushforward $\pi_*$ and the pullback $i^*$ were well-defined on compactly supported groupoid cohomology: the only thing that remains to check is that $i$ induces inclusions of isotropy groups.

Letting then $(m_1 M_0, m_2 M_0) \in [M]^2$, we have that $i((m_1 M_0, m_2 M_0)) = (m_1 K, m_2 P^\circ)$. We have

$$\text{Stab}_{M_F}(m_1 M_0, m_2 M_0) = \text{Ad}(m_1) M_0 \cap \text{Ad}(m_2) M_0 \subset \text{Ad}(m_1) K \cap \text{Ad}(m_2) P^\circ = \text{Stab}_{G_F}(m_1 K, m_2 P^\circ)$$

and hence $i$ induces inclusions on isotropy groups.

In the case of groupoids consisting of groups acting on sets with compatible maps, the pullback square is obtained by taking the pullback of the groups acting on the pullback of the sets (proposition 4.4 (ii) in [3]). It’s clear that the pullback of the groups is $M_F$ and that the first factor of the pullback of the sets is $[M]$. For the second factor, by definition of pullback of sets we obtain

$$G_F/P_0 \times_{G_F/P^\circ} [M] = \{(g P_0, m M_0) \in G_F/P_0 \times [M] | \pi(g P_0) = i(m M_0)\} =$$

$$= \{(g P_0, m M_0) \in G_F/P_0 \times [M] | g P^\circ = m P^\circ\}.$$  

As $m \in M_F$, we have $m P^\circ \subset P_F$ and thus the above condition forces $g \in P_F$ too. Let then $g = n v$ be its Levi decomposition and notice than that $g P_0 = n v P_0 \overset{\pi^\circ}{\rightarrow} n v P^\circ = n P^\circ$. In particular, $m P^\circ = n P^\circ$ with $m, n \in M_F$ forces then $m M_0 = n M_0$, i.e. $m = n$ as cosets in $[M]$. We obtain

$$G_F/P_0 \times_{G_F/P^\circ} [M] \cong P_F/P_0$$

with maps

$$P_F/P_0 \overset{i'}{\rightarrow} G_F/P_0 \text{ and } P_F/P_0 \overset{\pi'}{\rightarrow} [M]$$

being respectively inclusion and projection onto the $M_F$-component of the Levi factorization $P_F = M_F \ltimes V_F$. \qed

Applying lemma 11 shows that for the purpose of moving around cohomology classes, we can replace the second diagram with the following one:

$$(M_F \rightrightarrows [M] \times P_F/P_0) \overset{i'}{\rightarrow} (G_F \rightrightarrows [G] \times G_F/P_0) \overset{i}{\rightarrow} G = (G_F \rightrightarrows [G]^2)$$
$$\pi' \downarrow \quad \pi'$$
$$M = (M_F \rightrightarrows [M]^2)$$

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Using again claim 13 but replacing $M$ with $T$, we obtain that

$$
(T_F \xrightarrow{\sim} [T] \times B_F/B_0) \xrightarrow{\pi'} (G_F \xrightarrow{\sim} [G] \times G_F/B_0) \xrightarrow{\pi} \overline{T} = (T_F \xrightarrow{\sim} [T]^2) \xrightarrow{i} (G_F \xrightarrow{\sim} [G] \times G_F/B^2)
$$

is also a pullback square of groupoids, with maps satisfying the additional assumptions of lemma 11. Hence we can replace the first diagram by

$$
(T_F \xrightarrow{\sim} [T] \times B_F/B_0) \xrightarrow{\pi'} (G_F \xrightarrow{\sim} [G] \times G_F/B_0) \xrightarrow{i} G = (G_F \xrightarrow{\sim} [G]^2)
$$

Finally claim 13 with $G$ replaced by $M$ and $M$ by $T$ yields the pullback square of groupoids

$$
(T_F \xrightarrow{\sim} [T] \times B_F'/B_0') \xrightarrow{\pi'} (M_F \xrightarrow{\sim} [M] \times M_F/B_0') \xrightarrow{i} M = (M_F \xrightarrow{\sim} [M]^2)
$$

with maps again satisfying the additional assumption of lemma 11. This allows us to replace the third diagram with

$$
(T_F \xrightarrow{\sim} [T] \times B_F'/B_0') \xrightarrow{\pi'} (M_F \xrightarrow{\sim} [M] \times M_F/B_0') \xrightarrow{i} M = (M_F \xrightarrow{\sim} [M]^2)
$$

Notice that the map $i : (G_F \xrightarrow{\sim} [G] \times G_F/B_0) \longrightarrow \overline{G} = (G_F \xrightarrow{\sim} [G]^2)$ involved in the first diagram factors through the similar map $i : (G_F \xrightarrow{\sim} [G] \times G_F/P_0) \longrightarrow \overline{G} = (G_F \xrightarrow{\sim} [G]^2)$ of the second diagram, hence it remains to show that the diagrams

$$
(T_F \xrightarrow{\sim} [T] \times B_F/B_0) \xrightarrow{\pi} (G_F \xrightarrow{\sim} [G] \times G_F/B_0) \xrightarrow{i} (G_F \xrightarrow{\sim} [G] \times G_F/P_0)
$$

and

$$
(M_F \xrightarrow{\sim} [M] \times P_F/P_0) \xrightarrow{i} (G_F \xrightarrow{\sim} [G] \times G_F/P_0)
$$

induce the same map in cohomology. This is a consequence of the following
Claim 14. The diagram

\[
\begin{array}{ccc}
(T_F \hookrightarrow [T] \times B_F/B_0) & \xrightarrow{\iota'} & (M_F \hookrightarrow [M] \times P_F/P_0) \\
\pi' \downarrow & & \pi \downarrow \\
(T_F \hookrightarrow [T] \times B'_F/B'_0) & \xrightarrow{i} & \bar{M} = (M_F \hookrightarrow [M]^2)
\end{array}
\]

is a pullback square of groupoids. Moreover, it satisfies the assumption of lemma 11.

Proof. First of all, \(\pi\) comes from the pullback square of claim 13, hence it is a finite covering of groupoids as shown in the proof of lemma 11.

Also, \(i\) is the composition of two maps \((T_F \hookrightarrow [T] \times B'_F/B'_0) \to (M_F \hookrightarrow [M] \times M_F/B'_0) \to \bar{M}\), with \(\iota'\) coming from the pullback square of claim 13 and thus inducing injections on isotropy groups and a finite-fibers map on connected components by the proof of lemma 11. On the other hand, \((M_F \hookrightarrow [M] \times M_F/B'_0) \to \bar{M}\) induces injections on isotropy groups and a finite-fibers map on connected components by the remark following definition 11.

It remains to check that \((T_F \hookrightarrow [T] \times B_F/B_0)\) with the given maps \(\pi'\) and \(\iota'\) is indeed the pullback square. Just as before, by proposition 4.4(ii) in [3] we can compute the pullback of the groups and the sets separately. It is clear that the pullback of the groups is \(T_F\), and that the pullback of the sets - computed separately on each factor - has first factor equal to \([T]\). Finally, For the second factor by definition of pullback of sets we obtain

\[
P_F/P_0 \times_{M_F/M_0} B'_F/B'_0 = \{(pP_0, b'B'_0) \in P_F/P_0 \times B'_F/B'_0 \mid \pi(pP_0) = i(b'B'_0)\} = \\
= \{(pP_0, b'B'_0) \in P_F/P_0 \times B'_F/B'_0 \mid mM_0 = b'M_0\text{ using the Levi decomposition }p = mv\}.
\]

In other words, there exists \(m_0 \in M_0\) such that \(m = b'm_0\). Notice that we have then \(pP_0 = mvP_0 = b'm_0vm_0^{-1}P_0 = b'vP_0\), since \(M_F\) normalizes \(V_F\). We obtain then

\[
\{(pP_0, b'B'_0) \in P_F/P_0 \times B'_F/B'_0 \mid mM_0 = b'M_0\text{ with }p = mv\} = \\
= \{(b'vP_0, b'B'_0) \in P_F/P_0 \times B'_F/B'_0\} \equiv B_F/B_0.
\]

with maps

\[
B_F/B_0 \xrightarrow{\iota'} P_F/P_0 \text{ and } B_F/B_0 \xrightarrow{\pi'} B'_F/B'_0
\]

the first one being inclusion and the second one induced by the semidirect product decomposition 
\(B_F = B'_F \ltimes V_F\), which is just the restriction to \(B_F\) of the Levi decomposition \(P_F = M_F \ltimes V_F\). \(\square\)

6 Satake via the Universal Principal Series

In this section we prove that the map defined in the previous one is a morphism of graded algebras in degrees 0 and 1: notice indeed that the construction of section 5 holds in any degree, and provides us with a Satake map in all degrees - but at the present time we cannot show that it is an algebra homomorphism past degree 1.

To show that our Satake map is a morphism, we re-interpret it in a different way, via a generalization to the derived setting of the classical universal principal series. We start by recalling the classical definition of the Satake homomorphism via the universal principal series (see for instance [12], section 4). Consider the space of compactly supported, locally constant functions on
Consider the groupoid $P = (P_F \hookrightarrow [M]^2)$ (where the use that $[M] = M_F/M_0 = P_F/P_0$ and the continuous morphism $i : M \rightarrow P$ defined as the identity on the spaces, and the inclusion $M_F \hookrightarrow P_F$ at the level of groups.

This induces a pullback map in compactly supported cohomology $i^* : \mathbb{H}^*_c(P) \rightarrow \mathbb{H}^*_c(M)$, and each compactly supported cohomology is a derived Hecke algebra (respectively $\mathcal{H}_P$ and $\mathcal{H}_M$) as per propositions 9 and 7.

The pullback $i^* : \mathcal{H}_P \rightarrow \mathcal{H}_M$ is an isomorphism of graded algebras.

Proof. We start by remarking that the pullback along $i$ indeed preserves compact support, because $\pi_0 P = P \setminus P_F/P_0 = M_0 \setminus M_F/M_0 = \pi_0 M$ by the Levi decomposition. The map induced by $i$ on connected components is thus the identity, and in particular it has finite fibers.

As usual, we interpret the algebra $\mathcal{H}_P$ as $P(F)$-equivariant cohomology classes

$$A : P(F)/P_0 \times P(F)/P_0 \rightarrow \bigoplus_{x,y \in P(F)/P_0} H^*(P(F)_{x,y})$$

with support on finitely many orbits and such that $A(mP_0, nP_0) \in H^*(P(F)_{m,n})$.

Denoting by $P(F)_{m,n} = \text{Stab}_{P(F)}(mP_0, nP_0)$ and $M(F)_{m,n} = \text{Stab}_{M(F)}(mM_0, nM_0)$, we have an explicit formula for the pullback map $i^* : \mathcal{H}_P \rightarrow \mathcal{H}_M$, $A \mapsto \tilde{A}$ as

$$\tilde{A}(mM_0, nM_0) = \text{res}_{M(F)_{m,n}}^{P(F)_{m,n}} A(mP_0, nP_0)$$

(3)

To check that this is an algebra homomorphism, we need to show that given $A, B \in \mathcal{H}_P$ we have $i^* A \circ i^* B = i^*(A \circ B)$ as elements of $\mathbb{H}^*_c(M)$. We interpret both sides in terms of the usual groupoids diagrams:

\[
\begin{array}{ccc}
A \in P & \xrightarrow{i_{1,2}} & P \ni B \\
(P_F \hookrightarrow [M]^3) & & \\
\downarrow i_{1,3} & & \downarrow i \\
M & & \end{array}
\]
is the diagram representing the right hand side, while

\[
\begin{array}{ccc}
A \in P & \overset{i}{\to} & P \ni B \\
\uparrow & & \uparrow \\
\overset{i_1,2}{i^*A} \in M & \overset{i_{1,3}}{\to} & M \ni \overset{i_{2,3}}{i^*B} \\
\downarrow & & \downarrow \\
(M_F \rightsquigarrow [M]^3) & \overset{i_{1,3}}{\to} & M \\
\end{array}
\]

is the diagram representing the left hand side. By composing the arrows on top, this latter diagram simplifies to

\[
\begin{array}{ccc}
A \in P & \overset{i_{1,3}}{\to} & P \ni B \\
\uparrow & & \uparrow \\
\overset{i_{1,2}}{i^*A} \in M & \overset{i_{2,3}}{\to} & M \ni \overset{i_{3}}{i^*B} \\
\downarrow & & \downarrow \\
(M_F \rightsquigarrow [M]^3) & \overset{i_{1,3}}{\to} & M \\
\end{array}
\]

As for the other diagram, we use the following fact:

**Claim 16.** The diagram

\[
\begin{array}{ccc}
(M_F \rightsquigarrow [M]^3) & \overset{i_{1,3}}{\to} & M \\
\uparrow & & \uparrow \\
(P_F \rightsquigarrow [M]^3) & \overset{i_{1,3}}{\to} & P \\
\end{array}
\]

is a pullback square of groupoids. Moreover, it satisfies the assumption of lemma 11: that is to say,\(i_{1,3}\) is a finite covering morphism and \(i\) induces inclusion on isotropy groups, and a finite-fibers map on connected components.

**Proof of claim.** Notice that \(i_{1,3}\) is clearly a continuous covering morphism, since we have the same group \(P_F\) acting on both the source and the target groupoid. Moreover, given \((xP^\circ, yP^\circ, zP^\circ) \in \text{Ob}([M]^3)\) we have that

\[
\text{Stab}_{P}((xP^\circ, yP^\circ, zP^\circ)) = \text{Ad}(x)P^\circ \cap \text{Ad}(y)P^\circ \cap \text{Ad}(z)P^\circ = M(F)_{x,y,z}V(F)
\]

while

\[
\text{Stab}_{P}((xP^\circ, zP^\circ)) = \text{Ad}(x)P^\circ \cap \text{Ad}(z)P^\circ = M(F)_{x,z}V(F)
\]

so that the index of the inclusion of isotropy groups is the index \([M(F)_{x,z} : M(F)_{x,y,z}]\), which is finite since \(M(\emptyset)\) is open and compact in \(M(F)\). This proves that \(i_{1,3}\) is a finite covering morphism, since it also implies that \(M(F)_{x,y,z}\) is open in \(M(F)_{x,z}\). That \(i\) induces an inclusion of isotropy groups is immediate, and as for the connected components we have that \(\pi_0 M = M_0 \backslash M/F/M_0 \equiv P^\circ \backslash P/F/P^\circ = \pi_0 P\), so that the map induced by \(i\) is in fact a bijection.
It remains to show that the pullback square is indeed \((M_F \sim [M]^3)\), with the maps as given in the diagram. As mentioned before, the pullback square of a diagram of groupoids given by groups acting on sets with compatible maps is the pullback of the groups acting on the pullback of the sets. The pullback groups is obviously \(M_F\), while for the sets we get \([M]^3\). The maps are the ones given.

The usual application of lemma 11 shows then that the diagram for the right hand side is replaced by

\[
\begin{array}{ccc}
A \in P & \xleftarrow{i_{1,2}} & P \ni B \\
& \xrightarrow{i_{2,3}} & \\
(P_F \sim [M]^3) & \xrightarrow{i} & (M_F \sim [M]^3) \\
& \xleftarrow{i_{1,3}} & \\
& & M
\end{array}
\]

and after composing the upper arrows the latter diagram becomes identical to the one for \(i^*A \circ i^*B\).

This proves that the pullback is an algebra homomorphism. It remains to show that it is bijective: this amounts to showing that the two spaces of compactly supported cohomology classes are identified as \(S\)-modules under pullback.

Given that we have an equality of double coset spaces \(P^\circ \backslash P_F/P^\circ = M_0 \backslash M_F/M_0\) and in light of the explicit formula 3, it suffices to check that

\[
H^i \left( \text{Stab}_{P(F)}(P^\circ, xP^\circ), S \right) \xrightarrow{\text{res}} H^i \left( \text{Stab}_{M(F)}(M_0, xM_0), S \right)
\]

is an isomorphism for all \(i \geq 0\). We find

\[
\text{Stab}_{P(F)}(P^\circ, xP^\circ) = P^\circ \cap xP^\circ x^{-1} = (M_0 \cap xM_0 x^{-1}) V_F = M(O)_x V(F)
\]

and

\[
\text{Stab}_{M(F)}(M_0, xM_0) = M_0 \cap xM_0 x^{-1} = M(O)_x
\]

Hence we need to show that the restriction map

\[
\text{res} : H^i (M(O)_x V(F), S) \rightarrow H^i (M(O)_x, S)
\]

is an isomorphism in all degrees 0 and 1. In degree 0 this is obvious, as both cohomology groups are copies of \(S\) and the map is the identity map.

In degree 1, both cohomology groups are Hom-groups, so it suffices to show that \(V(F) \subseteq [M(O)_x V(F), M(O)_x V(F)]\). The equality \(\pi_0 P = \pi_0 M\) implies that \(x\) can be chosen among a set of coset representatives for \(M_0 \backslash M_F/M_0\), and the Cartan decomposition for \(M\) says that we can then choose \(x \in T_F\). Therefore, \(T(O) \subseteq M(O)_x\) and hence we can show that \(V(F)\) is in the commutator subgroup of \(T(O)V(F) \subseteq M(O)_x V(F)\) by working on each root space \(U_\alpha(F) \subseteq V(F)\) - here we use crucially that the cardinality of the residue field of \(F\) is at least 5.

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The following argument that finishes the proof for all degrees \( i > 1 \) (and in fact works for \( i = 1 \) as well) was suggested to us by Florian Herzig, who we are very thankful to.

Firstly, by identifying the quotient with the Levi factor, consider the short exact sequence \( 1 \rightarrow V(F) \rightarrow M(\mathbb{O})_x V(F) \rightarrow M(\mathbb{O})_x \rightarrow 1 \). The Hochschild-Serre spectral sequence for continuous cohomology with trivial \( S \)-coefficients gives

\[
E_2^{p,q} = H^p(M(\mathbb{O})_x, H^q(V(F), S)) \Rightarrow H^{p+q}(M(\mathbb{O})_x V(F), S).
\]

It suffices to prove that \( H^q(V(F), S) = 0 \) for all \( q \geq 1 \). Indeed, it will follow that the spectral sequence collapses on the second page, and the inflation map \( H^p(M(\mathbb{O})_x, S) \xrightarrow{\text{infl}} H^p(M(\mathbb{O})_x V(F), S) \) is an isomorphism in all degrees \( p \geq 0 \). A consideration on the cochain complex shows easily that inflation is a right inverse to restriction, and thus we will obtain that the restriction map \( H^p(M(\mathbb{O})_x V(F), S) \rightarrow H^p(M(\mathbb{O})_x, S) \) is an isomorphism as well.

We prove that \( H^q(V(F), S) = 0 \) for any \( q \geq 1 \) by induction on \( \dim V \). The inductive step is immediate: as \( V \) is nilpotent, if \( \dim V \geq 2 \) we can choose a short exact sequence \( 1 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 1 \) having \( \dim V > \dim V' \), \( \dim V \geq 1 \) and then Hochschild-Serre yields

\[
E_2^{p,q} = H^p(V''(F), H^q(V'(F), S)) \Rightarrow H^{p+q}(V(F), S).
\]

By induction, \( H^q(V'(F), S) = 0 \) for all \( q \geq 1 \) and thus the Hochschild-Serre spectral sequence collapses on the second page, to yield that the inflation map \( H^p(V''(F), S) \rightarrow H^p(V(F), S) \) is an isomorphism in all degrees \( p \geq 0 \). By repeating the argument replacing \( V \) with \( V'' \), we can assume that \( \dim V'' = 1 \), and then the claim will follow from the base case of the induction.

It remains thus to show that \( H^q(F, S) = 0 \) for all \( q \geq 1 \). For \( F = \mathbb{Q}_p \) this has been proved by Emerton in the course of the proof of lemma 4.3.7 in [9]. By induction on \( d = \dim_{\mathbb{Q}_p} F \) (the base case being Emerton’s result), the general case follows by using again the Hochschild-Serre spectral sequence for \( 1 \rightarrow \mathbb{Q}_p^{\otimes d-1} \rightarrow F \approx \mathbb{Q}_p^{\otimes d} \rightarrow \mathbb{Q}_p \rightarrow 1 \) - or alternatively the Künneth formula for continuous cohomology.

We now give a brief description of the content of the rest of this section. Consider the topological groupoid \((P_F \rightharpoonup [P] \times [M])\) whose compactly supported cohomology \( \mathbb{H}^*_c(P_F \rightharpoonup [P] \times [M]) \) can be identified with the space \( \mathcal{H}_P(P_0, P^c) \) of \( P(F) \)-invariant cohomology classes \( A \) on \( P_F/P_0 \times P_F/P^c \), supported on finitely many orbits and such that

\[
A(xP_0, yP^c) \in H^* \left( \text{Stab}_{P(F)}(xP_0, yP^c), S \right).
\]

We have a continuous morphism of topological groupoids

\[
(P_F \rightharpoonup [P] \times [M]) \xrightarrow{i} (G_F \rightharpoonup [G] \times G/P^c)
\]

given by inclusion \( P_F \hookrightarrow G_F \) at the level of groups, ‘identity’ on the first component \([P] = P_F/P_0 \equiv G/K = [G]\) and injection on the second component \( P_F/P^c \hookrightarrow G_F/P^c \). This turns out to be a homotopy equivalence (see fact 18) and thus gives an isomorphism on cohomology via pullback.

We also have a convolution action of \( \mathcal{H}_G \) on the left of \( \mathbb{H}^*_c(G_F \rightharpoonup [G] \times G/P^c) \) (denoted \( \mathcal{H}_G(K, P^c) \) from now on) as well as a convolution action of \( \mathcal{H}_P \) on the right of \( \mathcal{H}_P(P_0, P^c) \).

The diagram

\[
\begin{array}{ccc}
\mathcal{H}_G = \mathcal{H}_G(K, K) & \mathcal{H}_G(K, P^c) & \\
\mathcal{H}_P(P_0, P^c) & \mathcal{H}_P(P^c, P^c) = \mathcal{H}_P & \\
& \cong & \\
& i^* & \\
\end{array}
\]

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where the curved arrows denote the aforementioned actions, allows us to view the middle column \( \mathcal{H}_G(K, P^c) \cong \mathcal{H}_P(P_0, P^o) \) as a \((\mathcal{H}_G, \mathcal{H}_P)\)-bimodule: this is proved in fact 19.

It turns out (see proposition 17) that by choosing a specific \(^1\) element \( p \in \mathcal{H}_P(P_0, P^o) \), the convolution action \( \mathcal{H}_P(P_0, P^o) \rightharpoonup \mathcal{H}_P \) provides a morphism of (right) \( \mathcal{H}_P \)-modules via \( A \mapsto 1.A \) which is an isomorphism in degrees 0 and 1.

This allows us to define an algebra morphism \( S : \mathcal{H}_G^{\leq 1} \longrightarrow \mathcal{H}_P^{\leq 1} \) intrinsically via the rule \( F.1 = 1.S(F) \), as in definition 12. Composing with the isomorphism given by lemma 15 yields an algebra morphism \( \mathcal{H}_G^{\leq 1} \longrightarrow \mathcal{H}_M^{\leq 1} \) as in definition 13 and the final part of this section proves - by comparing explicit formulas - that this morphism coincides with the Satake map defined in section 5.

**Proposition 17.** Consider the following push-pull diagram of groupoids

\[
\begin{array}{ccc}
(P_F \rightharpoonup [P] \times P_F/P^o) & \overset{i_{1,2}}{\longrightarrow} & (P_F \rightharpoonup [P] \times P_F/P^o \times P_F/P^o) \\
\downarrow i_{1,3} & & \downarrow i_{1,3} \\
(P_F \rightharpoonup [P] \times P_F/P^o) & & (P_F \rightharpoonup P_F/P^o \times P_F/P^o) = P
\end{array}
\]

The above diagram yields a well-posed action by convolution of \( \mathcal{H}_P \) on \( \mathcal{H}_P(P_0, P^o) \), giving the latter the structure of a graded right \( \mathcal{H}_P \)-module.

In particular, let 1 be the element of \( \mathcal{H}_P(P_0, P^o) \) supported on the \( P_F \)-orbit of \( (P_0, P^o) \) and such that \( 1 (P_0, P^o) \in H^0(P_0, S) \) is the element 1 of \( S \). Then the map

\[ \mathcal{H}_P \longrightarrow \mathcal{H}_P(P_0, P^o) \quad F \mapsto 1.F \]

is a degree-preserving morphism of graded \( \mathcal{H}_P \)-modules, and when restricted to \( \mathcal{H}_P^{\leq 1} \) it is an isomorphism onto its image \( \mathcal{H}_P^{\leq 1}(P_0, P^o) \).

**Proof.** The first part of the proposition is an application of fact 35, so we need to check the three conditions in the statement of that result to obtain a well-posed algebra action of \( \mathcal{H}_P \) on \( \mathcal{H}_P(P_0, P^o) \).

To show condition 3, i.e. that \( i_{1,3} \) is a finite covering morphism, it suffices to check that the injection of isotropy groups is an open inclusion of finite index for one object in each connected component of the source groupoid. Let then \( (P_0, mP^o, nP^o) \in [P] \times [M]^2 \), whose stabilizer is \( P(0) \cap \text{M}(F)_{m,n} \text{V}(F) = \mathcal{M}(0)_{m,n} \text{V}(0) \). We have \( i_{1,3} \left( (P_0, mP^o, nP^o) \right) = (P_0, nP^o) \) whose stabilizer is \( P(0) \cap \text{M}(F)_n \text{V}(F) = \mathcal{M}(0)_n \text{V}(0) \) and hence the index is \([\mathcal{M}(0)_n : \mathcal{M}(0)_{m,n}]\) which is finite as \( \mathcal{M}(0) \subset \text{M}(F) \) is open and compact; moreover \( \mathcal{M}(0)_n \text{V}(0) \) is compact, so the inclusion holds.

Next we check condition 1: given \( A \in \mathbb{H}^*_c(P_F \rightharpoonup [P] \times [M]) \) and \( B \in \mathcal{H}_P \), we show that the cohomology class \( i^*_{1,2} A \cup i^*_{2,3} B \) is \( i_{1,3} \)-fiberwise compactly supported. This boils down to the index \([\text{Stab}_{p_F}(xP_0) : \text{Stab}_{p_F}(xP_0, mP^o)]\) being finite. Conjugating by \( x^{-1} \) yields that this index is the same as \([\text{Stab}_{p_F}(P_0) : \text{Stab}_{p_F}(P_0, x^{-1}mP^o)]\), and since \( x^{-1}mP^o = m'P^o \) for some \( m' \in M_F \), we obtain that the index is \([\text{P}(0) : \text{P}(0) \cap \text{M}(F)_{m'} \text{V}(F)] = [\text{P}(0) : \text{M}(0)_{m'} \text{V}(0)] = [\text{M}(0) : \text{M}(0)_{m'}]\) which is finite.

\(^1\)This is the spherical vector.
Finally, we need to check condition 2: since $V_F$ is normal in $P_F$, we have

$$P_0 m P^0 n P^0 = P_0 m M_0 n P^0 = V_0 (M_0 m M_0 n M_0) V_F.$$  

Since $M_0 \subset M_F$ is open and compact, we have a finite disjoint union $M_0 m M_0 n M_0 = \bigcup_{i=1}^{N} M_0 m_i M_0$ and thus

$$P_0 m P^0 n P^0 = V_0 \left( \bigcup_{i=1}^{N} M_0 m_i M_0 \right) V_F = \bigcup_{i=1}^{N} P_0 m_i P^0.$$  

This completes the proof of the first part of the proposition. It is then an immediate consequence of the structure of right $\mathcal{H}_P$-module on $\mathcal{H}_P(P_0, P^0)$ that the map $F \mapsto 1.F$ is a morphism of right $\mathcal{H}_P$-modules. It is also clearly degree preserving, since $1 \in \mathcal{H}_P(P_0, P^0)$ is supported in degree 0. It remains to show that $F \mapsto 1.F$ is bijective in degree at most 1. Both $\mathcal{H}_P$ and $\mathcal{H}_P(P_0, P^0)$ can be broken down into direct sums of cohomology algebras of groups indexed by the same double coset set $M_0 \setminus M_F / M_0$: more precisely, $\mathcal{H}_P \cong \bigoplus_{m \in M_0 \setminus M_F / M_0} H^*(M(0)_m V(F), S)$ and $\mathcal{H}_P(P_0, P^0) \cong \bigoplus_{m \in M_0 \setminus M_F / M_0} H^*(M(0)_m V(0), S)$.

By definition of the convolution action, we have

$$(1.F)(P_0, m P^0) = \sum_{x P^0 \in P_F / P^0} 1(P_0, x P^0) \cup F(x P^0, m P^0),$$

but since 1 is supported on the $P_F$-orbit of $(P_0, P^0)$, for a summand to be nonzero we need $x \in P_0 \subset P^0$ and hence we can pick $x P^0 = P^0$ and get

$$(1.F)(P_0, m P^0) = 1(P_0, P^0) \cup F(P^0, m P^0).$$

More precisely, since $\text{Stab}(P_0, P^0) = P_0$, $\text{Stab}(P^0, m P^0) = (M_0 \cap m M_0 m^{-1}) V_F$ and $\text{Stab}(P_0, m P^0) = (M_0 \cap m M_0 m^{-1}) V_0$ we obtain

$$(1.F)(P_0, m P^0) = \text{cores}_{M(0)_m V(0)}^{M(0)_m V(F)} \left( \text{res}_{M(0)_m V(0)}^{P(0)} 1(P_0, P^0) \cup \text{res}_{M(0)_m V(0)}^{M(0)_m V(F)} F(P^0, m P^0) \right)$$

and since cupping with 1 does not do anything, even after restriction,

$$(1.F)(P_0, m P^0) = \text{res}_{M(0)_m V(0)}^{M(0)_m V(F)} F(P^0, m P^0). \quad (4)$$

This explicit formula shows that bijectivity amounts to the restriction map

$$\text{res}_{M(0)_m V(0)}^{M(0)_m V(F)} : H^i(M(0)_m V(F), S) \rightarrow H^i(M(0)_m V(0), S)$$

being an isomorphism for all $i = 0, 1$ and for all $m$ in a choice of representatives for $P_0 \setminus P_F / P^0 \equiv M_0 \setminus M_F / M_0$. Then choosing $m \in X^m(T)_-$, the antidominant cocharacters for $M$, allows us to show that $T(0) \subset M(0)_m$ and then run the same argument as in lemma 15, using again that $|k| \geq 5$. \hfill \Box

Remark. As in the lemma 15, it is immediate from the proof that the map $F \mapsto 1.F$ is an isomorphism in degree less than $k$, where $k$ is the greatest integer such that the restriction map $H^i(M(0)_m V(F), S) \rightarrow H^i(M(0)_m V(0), S)$ is an isomorphism for all $m \in M_0 \setminus M_F / M_0$ and for all $i \leq k$. 

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Recall that the compactly supported cohomology of the groupoid \((G_F \curvearrowright [G] \times G/P^\circ)\) is the space \(\mathcal{H}_G(K, P^\circ)\) of \(G\)-invariant cohomology classes
\[
f : G/K \times G/P^\circ \longrightarrow \bigoplus H^* (\text{Stab}_G(xK, yP^\circ), S)
\]
supported on finitely many orbits and such that \(f(xK, yP^\circ) \in H^* (\text{Stab}_G(xK, yP^\circ), S)\).

**Fact 18.** We have a degree-preserving isomorphism \(\mathcal{H}_G(K, P^\circ) \longrightarrow \mathcal{H}_P(P_0, P^\circ)\) induced as pullback on cohomology by the homotopy equivalence of topological groupoids
\[
(P_F \curvearrowright [P] \times [M]) \overset{i}{\longrightarrow} (G_F \curvearrowright [G] \times G/P^\circ)
\]
given by inclusion \(P_F \hookrightarrow G_F\) at the level of groups, ‘identity’ on the first component \([P] = P_F/P_0 \equiv G/K = [G]\) and injection on the second component \(P_F/P^\circ \hookrightarrow G_F/P^\circ\).

Explicitly, the map is realized as
\[
\mathcal{H}_G(K, P^\circ) \longrightarrow \mathcal{H}_P(P_0, P^\circ) \quad F \mapsto \tilde{F} \text{ with } \tilde{F}(P_0, mP^\circ) = F(K, mP^\circ)
\]
for all \(m \in M_0 \setminus M_F/M_0\), which are representatives for the \(G\)-orbits on \(G/K \times G/P^\circ\) as well as for the \(P_F\)-orbits on \(P_F/P_0 \times P_F/P^\circ\).

Notice that the map \(F \mapsto \tilde{F}\) indeed coincide with the pullback \(i^* : \mathbb{H}^*_c (G_F \curvearrowright [G] \times G/P^\circ) \longrightarrow \mathbb{H}^*_c (P_F \curvearrowright [P] \times P_F/P^\circ)\). In particular, \(i\) induces injections on isotropy groups which are in fact bijections, so that no restriction map is needed: we have
\[
\text{Stab}_G(K, mP^\circ) = K \cap mP^\circ m^{-1} = K \cap P_F \cap mP^\circ m^{-1} = P_0 \cap mP^\circ m^{-1} = \text{Stab}_{P(F)}(P_0, mP^\circ),
\]
showing that \(i\) is indeed a homotopy equivalence.

**Fact 19.** The above isomorphism allows us then to view \(\mathcal{H}_G(K, P^\circ) \cong \mathcal{H}_P(P_0, P^\circ)\) as a bimodule for the algebras \(\mathcal{H}_G\) and \(\mathcal{H}_P\), the former acting by convolution on the left factor of \(\mathcal{H}_G(K, P^\circ)\) and the latter acting by convolution on the right factor of \(\mathcal{H}_P(P_0, P^\circ)\).

We show that we have indeed a bi-module structure - that is to say, the two convolution actions commute with each other. This resembles strongly fact 37 from appendix B, but is not quite the same since we have the map \(i\) inducing an isomorphism in cohomology ‘in the middle’. We give then all details, even though the proof boils down to a diagram chasing very similar to that of fact 37.

**Proof.** Let \(A \in \mathcal{H}_G\), \(B \in \mathcal{H}_P\) and \(F \in \mathbb{H}^*_c (G_F \curvearrowright [G] \times G/P^\circ)\). The convolution \((A.F).B\) is represented by the following diagram:

```
A \in \mathcal{H}_G \quad (G_F \curvearrowright [G] \times G_F/P^\circ) \ni F
\downarrow \quad \downarrow
(G_F \curvearrowright [G]^2 \times G_F/P^\circ) \quad (P_F \curvearrowright [P] \times [M]) \ni B
\downarrow \quad \downarrow
(G_F \curvearrowright [G] \times G_F/P^\circ) \ni i (P_F \curvearrowright [P] \times [M]^2)
\downarrow \quad \downarrow
((A.F).B) \in (P_F \curvearrowright [P] \times [M])
```

![Diagram](image-url)
Claim 20. The diagram

\[
\begin{array}{ccc}
(G_F \xrightarrow{\sim} [G]^2 \times G_F/P^\circ) & \xrightarrow{\text{iso}_{1,3}} & (P_F \xrightarrow{\sim} [P]^2 \times [M]^2) \\
\downarrow i_{1,3} & & \downarrow i_{1,3} \\
(G_F \xrightarrow{\sim} [G] \times G_F/P^\circ) & \xleftarrow{\text{iso}_{1,2}} & (P_F \xrightarrow{\sim} [P] \times [M]^2)
\end{array}
\]

is a pullback square of groupoids. Moreover, it satisfies the assumption of lemma 11: that is to say, \(i_{1,3}\) is a finite covering morphism and \(i \circ i_{1,2}\) induces inclusion on isotropy groups.

Proof of claim. We have shown above that \(i\) induces a bijection between isotropy groups, and since \(i_{1,2} : (P_F \xrightarrow{\sim} [P] \times [M]^2) \longrightarrow (P_F \xrightarrow{\sim} [P] \times [M])\) is clearly a covering morphism, the composition \(i \circ i_{1,2}\) also induces inclusion of isotropy groups.

Clearly \(i_{1,3}\) is a covering morphism, since \(G_F\) acts on both groupoids. To prove that the inclusion of isotropy groups is open and finite index, we take a representative \((p_1K, p_2K, P^\circ)\) for a fixed connected component of \((G_F \xrightarrow{\sim} [G]^2 \times G_F/P^\circ)\). Then \(\text{Stab}_{p_F}(p_1K, p_2K, P^\circ) = \text{Ad}(p_1)K \cap \text{Ad}(p_2)K \cap P^\circ = \text{Ad}(p_1)P_0 \cap \text{Ad}(p_2)P_0 \cap P^\circ\), while \(\text{Stab}_{p_F}(i_{1,3}(p_1K, p_2K, P^\circ)) = \text{Stab}_{p_F}(p_1K, P^\circ) = \text{Ad}(p_1)K \cap P^\circ = \text{Ad}(p_1)P_0 \cap P^\circ\). Since \(P^\circ\) is closed inside \(P_F\), we obtain that \(\text{Ad}(p_1)P_0 \cap P^\circ\) is closed inside the compact \(\text{Ad}(p_1)P_0\), hence compact. But \(\text{Ad}(p_2)P_0\) is open in \(P_F\), thus the intersection \(\text{Ad}(p_2)P_0 \cap (\text{Ad}(p_1)P_0 \cap P^\circ)\) is open in the compact \(\text{Ad}(p_1)P_0 \cap P^\circ\), and thus finite index.

It remains to show that the pullback is the given one. The group acting is obviously \(P_F\), as for the set we have that

\[
\left(\left([G]^2 \times G_F/P^\circ\right) \times [G] \times G_F/P^\circ \left([P] \times [M]^2\right)\right) = \{(p_1K, p_2K, p_3P^\circ), (p_1K, p_2K, p_3P^\circ) : (p_1K, p_3P^\circ) = (p_1K, p_3P^\circ)\}
\]

which forces \(g_3 \in P_F\), and completes the proof. \(\square\)

Thus lemma 11 allows us to replace the original diagram with the following:

\[
\begin{array}{ccc}
G & \xrightarrow{i_{1,2}} & (G_F \xrightarrow{\sim} [G] \times G/P^\circ) \\
\downarrow i_{2,3} & & \downarrow i_{2,3} \\
(G_F \xrightarrow{\sim} [G]^2 \times G/P^\circ) & \xleftarrow{\text{iso}_{1,2}} & (P_F \xrightarrow{\sim} [P]^2 \times [M]^2) \\
\downarrow i_{1,2,4} & & \downarrow i_{2,3} \\
(P_F \xrightarrow{\sim} [P] \times [M]^2) & \xrightarrow{i_{1,3}} & (P_F \xrightarrow{\sim} [P] \times [M])
\end{array}
\]

Applying lemma 10 to the subdiagram

\[
\begin{array}{ccc}
(P_F \xrightarrow{\sim} [P]^2 \times [M]^2) & \xrightarrow{i_{1,2,4}} & (P_F \xrightarrow{\sim} [P] \times [M]^2) \\
\downarrow i_{1,3} & & \downarrow i_{1,3} \\
(P_F \xrightarrow{\sim} [P]^2 \times [M]^2) & \xrightarrow{i_{2,3}} & (P_F \xrightarrow{\sim} [P] \times [M])
\end{array}
\]
makes it equivalent to

\[(P_F \rightsquigarrow [P]^2 \times [M]^2) \xrightarrow{i_{3,4}} (P_F \rightsquigarrow [M]^2) \]

\[\xrightarrow{i_{1,2,4}} (P_F \rightsquigarrow [P] \times [M]^2) \]

\[\xrightarrow{i_{1,3}} (P_F \rightsquigarrow [P] \times [M]) \]

and replacing this into the last diagram and composing pullbacks we obtain that \((A.F).B\) is given by

\[A \in G \quad F \in (G_F \rightsquigarrow [G] \times G/P^\circ) \quad (P_F \rightsquigarrow [M]^2) \ni B \]

\[\xrightarrow{i_{1,2}} (P_F \rightsquigarrow [P] \times [M]) \quad (P_F \rightsquigarrow [P] \times [M]^2) \ni B \]

\[\xrightarrow{i_{1,3}} (P_F \rightsquigarrow [P] \times [M]) \]

Consider now \(A.(F.B)\): the relevant diagram is

\[F \in (G_F \rightsquigarrow [G] \times G/P^\circ) \xleftarrow{i} (P_F \rightsquigarrow [P] \times [M]) \]

\[\xrightarrow{i_{1,2}} (P_F \rightsquigarrow [P] \times [M]) \quad (P_F \rightsquigarrow [M]^2) \ni B \]

\[\xrightarrow{i_{1,3}} (P_F \rightsquigarrow [P] \times [M]) \]

\[\xrightarrow{i_{2,3}} (P_F \rightsquigarrow [P] \times [M]) \]

\[A \in G \quad (G_F \rightsquigarrow [G] \times G/P^\circ) \xrightarrow{i} (P_F \rightsquigarrow [P] \times [M]) \]

\[\xrightarrow{i_{1,2}} (P_F \rightsquigarrow [P] \times [M]) \quad (P_F \rightsquigarrow [G]^2 \times G/P^\circ) \]

\[\xrightarrow{i_{2,3}} (P_F \rightsquigarrow [P] \times [M]) \]

\[\xrightarrow{i_{1,3}} (P_F \rightsquigarrow [G] \times G/P^\circ) \]

\[\xrightarrow{i_{1,3}} (P_F \rightsquigarrow [P] \times [M]) \]

where the symbol ◆ is to remember that although the pullback in cohomology goes as \(i^* : H^* (G_F \rightsquigarrow [G] \times G/P^\circ) \to H^* (P_F \rightsquigarrow [P] \times [M])\), we use the fact that it is an isomorphism and instead move the cohomology class \(F.B\) in the opposite direction.

**Claim 21.** The diagram

\[(G_F \rightsquigarrow [G]^2 \times G/P^\circ) \xleftarrow{i} (P_F \rightsquigarrow [P]^2 \times [M]) \]

\[\xrightarrow{i_{1,3}} (P_F \rightsquigarrow [P]^2 \times [M]) \]

\[(G_F \rightsquigarrow [G] \times G/P^\circ) \xleftarrow{i} (P_F \rightsquigarrow [P] \times [M]) \]

\[\xrightarrow{i_{1,3}} (P_F \rightsquigarrow [P] \times [M]) \]

is a pullback square of groupoids. Moreover, it satisfies the assumption of lemma 11: that is to say, \(i_{1,3}\) is a finite covering morphism and \(i\) induces inclusion on isotropy groups.
Proof of claim. We have already shown that $i_{1,3}$ is a finite covering morphism, and that $i$ induces bijection on isotropy groups. It remains to show that the pullback is the given one: it is clear that the group acting is $P_F$, and as for the sets we have

$$([G]^2 \times G_{F/\mathcal{P}^c}) \times_{[G] \times G_{F/\mathcal{P}^c}} ([P] \times [M]) =$$

$$= \{(p_1 K, p_2 K, gP^c), (p'_1 P_0, p'_2 P^c) \mid (p_1 K, gP^c) = (p'_1 P_0, p'_2 P^c)\}$$

which forces $g \in P_F$ and concludes the proof.

We apply lemma 11 and then distribute and compose pullbacks, to obtain that the diagram for $A.(F.B)$ is equivalent to

$$\begin{array}{ccc}
G & \xrightarrow{\sim} & G \times \mathcal{G}_{F/\mathcal{P}^c} \\
\downarrow{i_{1,2}} & \downarrow{i_{2,3}} & \downarrow{i_{1,3}} \\
(P_F \sim [P] \times [M]) & \xrightarrow{\sim} & (P_F \sim [P] \times [M]) \\
\downarrow{i_{1,3}} & \downarrow{i_{2,3}} & \downarrow{i_{1,3}} \\
(P_F \sim [P] \times [M]) & \xrightarrow{\sim} & (P_F \sim [P] \times [M]) \\
\end{array}$$

The two occurences of $i$ cancel out since $i^\ast$ is an isomorphism, and we obtain then

$$\begin{array}{ccc}
G & \xrightarrow{\sim} & G \times \mathcal{G}_{F/\mathcal{P}^c} \\
\downarrow{i_{1,2}} & \downarrow{i_{2,3}} & \downarrow{i_{1,3}} \\
(P_F \sim [P] \times [M]) & \xrightarrow{\sim} & (P_F \sim [P] \times [M]) \\
\downarrow{i_{1,3}} & \downarrow{i_{2,3}} & \downarrow{i_{1,3}} \\
(P_F \sim [P] \times [M]) & \xrightarrow{\sim} & (P_F \sim [P] \times [M]) \\
\end{array}$$

Claim 22. The diagram

$$\begin{array}{ccc}
(P_F \sim [P] \times [M]) & \xrightarrow{i_{2,3,4}} & (P_F \sim [P] \times [M]) \\
\downarrow{i_{1,2,4}} & \downarrow{i_{1,3}} & \downarrow{i_{1,3}} \\
(P_F \sim [P] \times [M]) & \xrightarrow{i_{2,3}} & (P_F \sim [P] \times [M]) \\
\end{array}$$

is a pullback square of groupoids. Moreover, it satisfies the assumption of lemma 11: that is to say, $i_{1,3}$ is a finite covering morphism and $i_{2,3}$ induces inclusion on isotropy groups.
Proof of claim. Both \( i_{1,3} \) and \( i_{2,3} \) are covering morphisms, so in particular \( i_{2,3} \) induces inclusion on isotropy groups. We have already seen that \( i_{1,3} \) is a finite covering morphism, so it remains to prove that the pullback is the given one. It is obvious that the group acting is \( P_F \), and that the set is the given one.

Applying lemma 11 yields then

\[
\begin{align*}
(G_F \sim [G] \times G/P^o) & \xrightarrow{i} \ (P_F \sim [P] \times [M]) & \ (P_F \sim [M]^2) \\
(G) & \xrightarrow{i_{1,3}} (P_F \sim [P]^2 \times [M]) & \ (P_F \sim [P] \times [M]^2) \\
& \xrightarrow{i_{2,3}} (P_F \sim [P] \times [M]) & \ (P_F \sim [P] \times [M])
\end{align*}
\]

Applying now lemma 10 to the subdiagram

\[
\begin{align*}
(G) & \xrightarrow{i_{1,3}} (P_F \sim [P]^2 \times [M]) & \ (P_F \sim [P] \times [M]^2) \\
& \xrightarrow{i_{2,3}} (P_F \sim [P] \times [M]) & \ (P_F \sim [P] \times [M])
\end{align*}
\]

and composing pullbacks yields the same diagram that \((A.F).B\) was equivalent to, and completes the proof.

**Definition 12.** We define the homomorphism: \( S^G_P : \mathcal{H}^\leq_G \rightarrow \mathcal{H}^\leq_P \) implicitly by the following formula:

\[
F.1 = 1. S^G_P(F) \quad \forall F \in \mathcal{H}^\leq_G
\]

where recall that \( 1 \in \mathcal{H}_G(K, P^o) \cong \mathcal{H}_P(P_0, P^o) \) is supported on the orbit of \((K, P^o)\) and takes value \( 1 \in H^0(P_0, S) \) there.

This is a degree-preserving morphism of algebras: indeed letting \( F_1, F_2 \in \mathcal{H}_G \) be such that \( \deg(F_1 \circ F_2) \leq 1 \) we have

\[
(F_1 \circ F_2).1 = 1. S^G_P(F_1 \circ F_2)
\]

but also

\[
(F_1 \circ F_2).1 = F_1.(F_2.1) = F_1. (1. S^G_P(F_2)) = (F_1.1) . S^G_P(F_2) = (1. S^G_P(F_1)) . S^G_P(F_2) = 1. (S^G_P(F_1) \circ S^G_P(F_2))
\]

and finally since by proposition 17 the map \( G \rightarrow 1.G \) is an isomorphism of \( \mathcal{H}_P \)-modules between \( \mathcal{H}^\leq_P \) and \( \mathcal{H}^\leq_P (P_0, P^o) \), we must have

\[
S^G_P(F_1 \circ F_2) = S^G_P(F_1) \circ S^G_P(F_2)
\]
Definition 13 (Satake homomorphism). We define the Satake homomorphism: $S^G_M : \mathcal{H}_G^{<1} \to \mathcal{H}_M^{<1}$ by composing the previous homomorphism and the isomorphism $\mathcal{H}_P^{<1} \cong \mathcal{H}_M^{<1}$ from lemma 15. Explicitly,

$$S^G_M F (mM_0, nM_0) = \text{res}_{M(F),m,n}^{(F)} S^G_P F (mP^0, nP^0)$$

Remark. Following the remarks after lemma 15 and proposition 17, it follows that the same maps $S^G_P$ and $S^G_M$ are degree-preserving algebra homomorphisms on $\mathcal{H}_G^{<k}$, where $k$ is the largest integer such that the restriction maps

$$\text{res} : H^i (M(0)_m V(F), S) \to H^i (M(0)_m V(0), S) \quad \text{and} \quad \text{res} : H^i (M(0)_m V(F), S) \to H^i (M(0)_m, S)$$

are isomorphisms for all $m \in M_0 \setminus M_F / M_0$ and for all $i \leq k$.

We want then to get a very explicit formula for $S^G_M$. In formula 4 we obtained that $(1.G)(P_0, mP^0) = \text{res}_{M(0)_m V(0)}^{M(0)_m V(F)} G(P^0, mP^0)$ for all $G \in \mathcal{H}_P$.

Let then $F \in \mathcal{H}_G^{<1}$ and consider

$$(F.1)(K, mP^0) = \sum_{x \in G / K \equiv P_F / P_0} F(K, xK) \cup 1(xK, mP^0)$$

We now restrict the choice of possible nonzero summands. We can write $x = mv \in M_F \ltimes V_F$, and since 1 is supported on the $P_F$-orbit of $(P_0, P^0)$, we need $(nvP_0, mP^0) \sim (P_0, P^0)$. The left pair is in the same orbit as $(P_0, v^{-1}n^{-1}mP^0) = (P_0, n^{-1}mP^0)$ where the last equality holds since $P^0 \supset V_F$. For the summand to be nonzero we need then $n^{-1}m \in P^0$ which amounts to saying that $nM_0 = mM_0$. We conclude that

$$(F.1)(K, mP^0) = \sum_{mv \in P_F / P_0} F(K, mvK) \cup 1(mvK, mP^0)$$

with the usual understanding about taking $\text{Stab}_G(K, mP^0) = M(0)_m V(0)$-orbits in the sum, and using the appropriate restriction and corestriction maps: explicitly

$$(F.1)(K, mP^0) = \sum_{M(0)_m V(0) / \{mv \in P_F / P_0\}} \text{cores}_{M(0)_m V(0)}^{M(0)_m V(0) \cap K_{mv}} \text{res}_{M(0)_m V(0) \cap K_{mv}}^{K_{mv}} F(K, mvK)$$

where as usual we have noted that restricting the element 1 and then cupping with it has no effect.

The last term is then equal to

$$(1.S^G_P)(P_0, mP^0) = \text{res}_{M(0)_m V(0)}^{M(0)_m V(F)} S^G_P F (P^0, mP^0)$$

In particular,

$$S^G_M F (M_0, mM_0) = \text{res}_{M(0)_m V(F)}^{M(0)_m V(0)} S^G_P F (P^0, mP^0) = \text{res}_{M(0)_m V(0)}^{M(0)_m V(F)} \circ \text{res}_{M(0)_m V(0)}^{M(0)_m V(F)} S^G_F (P^0, mP^0) =$$

$$= \text{res}_{M(0)_m V(0)}^{M(0)_m V(F)} (1.S^G_P)(P_0, mP^0) = \text{res}_{M(0)_m V(0)}^{M(0)_m V(F)} (F.1)(K, mP^0) =$$

$$= \text{res}_{M(0)_m V(0)}^{M(0)_m V(0)} \sum_{mv \in P_F / P_0} \text{cores}_{M(0)_m V(0) \cap K_{mv}}^{M(0)_m V(0) \cap K_{mv}} \text{res}_{M(0)_m V(0) \cap K_{mv}}^{K_{mv}} F(K, mvK)$$

We want to apply the double coset formula: notice that for each $x \in M(0)_m \setminus M(0)_m V(0) / M(0)_m V(0) \cap K_{mv}$ we have that $x \cdot (M(0)_m V(0) \cap K_{mv}) x^{-1} = M(0)_m V(0) \cap K_{xmv}$. We obtain that

$$\text{res}_{M(0)_m V(0)}^{M(0)_m V(0)} \text{cores}_{M(0)_m V(0) \cap K_{mv}}^{M(0)_m V(0) \cap K_{mv}} = \sum_{x \in M(0)_m \setminus M(0)_m V(0) / M(0)_m V(0) \cap K_{mv}} \text{cores}_{M(0)_m \cap K_{xmv}}^{M(0)_m \cap K_{xmv}} \text{res}_{M(0)_m \cap K_{xmv}}^{M(0)_m \cap K_{xmv}} c_x$$

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Now clearly each \( x \) can be taken in \( V_\Omega \), in particular \( xm = m\hat{x} \) for another \( \hat{x} \in V_F \), then we also have \( \hat{x}v = \hat{v} \): this is saying that \( xmvK = m\hat{v}K \) for a change of variables \( v \mapsto \hat{v} \) in \( V_F/V_\Omega \).

Also, notice that in the formula for \( S^G_M F \) we are summing over \( M(\Omega)_mV(\Omega) \)-orbits on classes \( mv \in P_F/P_\Omega \): the new inner sum \( \sum_{x \in M(\Omega)_m \backslash M(\Omega)_mV(\Omega)/M(\Omega)_mV(\Omega) \cap K_{mv}} \) correspond to splitting each \( M(\Omega)_mV(\Omega) \)-orbit into \( M(\Omega)_m \)-orbits, and then considering one contribution for each of the latter. It is clear that this amounts to considering \( M(\Omega)_m \)-orbits on \( mv \in P_F/P_\Omega \) to start with, so we get

\[
S^G_M F(M_\Omega, mM_\Omega) = \sum_{M(\Omega)_m \backslash \{mv \in P_F/P_\Omega\}} \text{cores}_{M(\Omega)_m} F(K, m\hat{v}K)
\]

This will be our formula for the derived Satake homomorphism.

### 7 The image of the Satake homomorphism

We finally prove the main theorem 1: we first show injectivity of the Satake homomorphism by an adaptation of the standard ‘unipotence’ argument, and then we describe the upper bound on the image. The description of the image is where our work on the Satake homomorphism in the previous sections comes in handy: recall that we want to show that a certain power of \( p \) divides \( (S^G_T F)(T_\Omega, \lambda(\varpi)T_\Omega) \) if \( \lambda \in X_*(T) \) pairs positively with some simple root \( \alpha \). We briefly summarize the argument: first of all, we restrict ourselves to the semisimple rank 1 case by factoring \( S^G_T \) through the standard Levi factor having root system \( \Phi(M, T) = \{ \pm \alpha \} \). Then, a careful analysis of the possible summands involved in the explicit formula for \( (S^G_T F)(T_\Omega, \lambda T_\Omega) \) and some SL2-computations yield that each such summand is divisible by the required power of \( p \).

From now on, we will sometimes use the notation \( S^G_T F(\mu) = S^G_T F(T_\Omega, \mu(\varpi)T_\Omega) \) whenever \( \mu \in X_*(T) \) is a cocharacter.

**Proposition 23.** Let \( \lambda \in X_*(T)_- \) be an antidominant cocharacter and \( f_\lambda \in \mathcal{H}_G \) be supported on \( H^*(K_\lambda, \mathcal{S}) \). Then \( S^G_T (f_\lambda)(\mu) = 0 \) unless \( \mu \geq \lambda \), and \( S^G_T (f_\lambda)(\lambda) = \text{res}^{K_\lambda}_{T_\Omega} f_\lambda \).

**Proof.** By formula 7, in order for \( S^G_T (f_\lambda)(\mu) \) to be nonzero we must have \( \varpi^\mu u \in K \varpi^\lambda K \) for some \( u \in U(F) \). Then lemma 3.6(i) in [16] shows that we must have \( \mu \geq \lambda \).

Similarly, lemma 3.6(ii) in [16] says that \( K \varpi^\lambda K \cap \varpi^\lambda U(F) = \varpi^\lambda U(\Omega) \), so that in particular there is only one summand contributing to \( S^G_T (f_\lambda)(\lambda) \), and we can take \( u = \hat{u} \) to obtain

\[
S^G_T (f_\lambda)(\lambda) = \text{cores}^{T_\Omega(\lambda)}_{K_\lambda} \text{res}^{K_\lambda}_{T_\Omega(\lambda)} f(K, \varpi^\lambda K) = \text{res}^{K_\lambda}_{T_\Omega(\lambda)} f_\lambda
\]

since \( K_\lambda \supset T(\Omega) \).

**Corollary 24.** The derived Satake homomorphism is injective, as a map from \( \mathcal{H}^*_G \) to \( \mathcal{H}^*_T \).

**Proof.** The previous proposition is the main ingredient of the usual ‘unipotence’ argument for the injectivity of the Satake homomorphism. Since \( \text{res}^{B(\Omega)}_{T(\Omega)} : H^1(B(\Omega), S) \longrightarrow H^1(T(\Omega), S) \) is an isomorphism, what remains to check is that \( \text{res}^{K_\lambda}_{B(\Omega)} \) is injective: we will prove this by showing that \( B(\Omega) [K_\lambda, K_\lambda] = K_\lambda \). Notice that since \( [K_\lambda, K_\lambda] \) is normal in \( K_\lambda \), this is equivalent to showing that the subgroup \( \langle B(\Omega), [K_\lambda, K_\lambda] \rangle \) of \( K_\lambda \) generated by \( B(\Omega) \) and \( [K_\lambda, K_\lambda] \) is the whole \( K_\lambda \).

\footnote{Here \( [K_\lambda, K_\lambda] \) is the commutator subgroup of \( K_\lambda \).}
Bruhat-Tits theory provides a factorization of $K_{\lambda}$: we follow [21] (section 1.1) in explaining it. Let $x_0 \in A(T, F) \subset B(G, F)$ be the hyperspecial vertex corresponding to the maximal compact $K$: we turn this into the origin of the apartment for $T$ (so that $x_0 = 1 \in X_*(T)$ is the trivial cocharacter) and set $\Omega = \{x_0, \lambda(\varpi)x_0\} \subset X_*(T) \subset X_*(T) \otimes \mathbb{R} = A(T, F) \subset B(G, F)$, so that we have

$$K_{\lambda} = K \cap \lambda(\varpi)K\lambda(\varpi)^{-1} = \text{Stab}_G(x_0) \cap \lambda(\varpi)\text{Stab}_G(x_0)\lambda(\varpi)^{-1} = \text{Stab}_G(x_0) \cap \text{Stab}_G(\lambda(\varpi)x_0) = \text{Stab}_G(\Omega).$$

Define now $f_\Omega : \Phi(G, T) \to \mathbb{R}$ as

$$f(\alpha) = -\inf_{x \in \Omega} \alpha(x),$$

where $\alpha(x)$ is the action of the root $\alpha$ as an affine function on $A(T, F)$. In the setup of [21], the translation action of $T(F)$ on $A(T, F)$ is defined as $g.x = x + \nu(g)$, where $\nu : T(F) \to X_*(T) \otimes \mathbb{R}$ is defined implicitly via the perfect pairing $\langle \cdot, \cdot \rangle : X_*(T)_{\mathbb{R}} \times X^*(T)_{\mathbb{R}} \to \mathbb{R}$ as

$$\langle \nu(g), \chi \rangle = -\text{val}_F(\chi(g)) \quad \forall \chi \in X^*(T).$$

In particular, we obtain that $\nu(\lambda(\varpi)) = -\lambda \in X_*(T)_+$ is a dominant cocharacter. We also have that $\lambda(\varpi)x_0 = x_0 - \lambda$, and hence $\alpha(\lambda(\varpi)x_0) = \alpha(x_0 - \lambda) = \langle -\lambda, \alpha \rangle$, since $\alpha(x_0) = 0$ for all $\alpha \in \Phi(G, T)$. By dominance of $-\lambda$, this yields $\alpha(\lambda(\varpi),x_0) \geq 0$ for all $\alpha \in \Phi^+$ and $\alpha(\lambda(\varpi),x_0) \leq 0$ for all $\alpha \in \Phi^-$. We conclude

$$f(\alpha) = -\inf_{x \in \Omega} \alpha(x) = -\inf \{0, \langle -\lambda, \alpha \rangle \} = \begin{cases} 0 & \text{if } \alpha \in \Phi^+ \\ \langle -\lambda, \alpha \rangle & \text{if } \alpha \in \Phi^- \end{cases} = \begin{cases} 0 & \text{if } \alpha \in \Phi^+ \\ \langle \lambda, \alpha \rangle & \text{if } \alpha \in \Phi^- \end{cases}$$

Following through with section 1.1 in [21], we obtain that

$$U_{\alpha,f_\Omega(\alpha)} = \begin{cases} U_\alpha(0) & \text{if } \alpha \in \Phi^+ \\ U_\alpha(\varpi^{\langle \lambda, \alpha \rangle}0) & \text{if } \alpha \in \Phi^- \end{cases}$$

and denote by $U_\Omega$ the subgroup of $G$ generated by the $U_{\alpha,f_\Omega(\alpha)}$’s as $\alpha$ varies across the roots $\Phi(G, T)$.

The subgroup $N_\Omega = \{n \in N_G(T) \mid nx = x \forall x \in \Omega\}$ normalizes $U_\Omega$, and so we can consider the subgroup $P_\Omega = N_\Omega U_\Omega$, which has $U_\Omega$ as a normal subgroup. A crucial fact is that $P_\Omega = \text{Stab}_G(\Omega)$ - see [21] page 104.

In particular, each $U_{\alpha,f_\Omega(\alpha)}$ is contained in $\text{Stab}_G(\Omega) = K_{\lambda}$, so using the $T(\mathcal{O})$-action by conjugation (and the fact that clearly $T(\mathcal{O}) \subset K_{\lambda}$) we obtain that $U_{\alpha,f_\Omega(\alpha)} = [K_{\lambda}, K_{\lambda}]$ for all $\alpha \in \Phi(G, T)$, so that $U_\Omega \subset [K_{\lambda}, K_{\lambda}]$.

It remains to show that $N_\Omega = B(\mathcal{O})[K_{\lambda}, K_{\lambda}]$. Notice that this is immediate whenever $\lambda$ is regular, because then $N_\Omega = \{n \in K \cap N_G(T) \mid n.\lambda = \lambda\}$ is simply $T(\mathcal{O})$, so this concludes the proof of corollary 24 in the case of regular cocharacter.

Recall that the action of $N_G(T)/T_0$ on the apartment $A(T, F)$ via affine transformations is defined by matching up the translation action of $X_*(T)$ and the Weyl-action of $N_G(T)/T_F$ on $A(T, F)$ under the short exact sequence

$$1 \to T_F/T_0 = X_*(T) \to N_G(T)/T_0 \xrightarrow{pr} W = N_G(T)/T_F \to 1,$$

as explained for example in [18], section 1 of chapter 1.

Since every $n \in N_\Omega \subset N_G(T)$ fixes $x_0 = 1$ and $\lambda$, we obtain that $pr(n) = w_n \in W$ must fix $\lambda$ under the Weyl group action, that is to say that $w_n \in W(M) = N_G(T) \cap M(F)$ where $M = Z_G(\lambda)$.
is a standard Levi subgroup (see for instance lemma 2.17 of [24] for this well-known fact).
This implies that $N_\Omega \subset M(F)$: now lemma 2.14(a) and (b) in [24] say that since $M = Z_G(\lambda)$ is connected, it is generated by $T$ and the $U_\alpha$'s such that $\langle \lambda, \alpha \rangle = 0$. Upon intersecting its $F$-points with $K$, we get that

$$N_\Omega \subset K \cap M_F \cap N_G(T) \subset \left< T(\mathcal{O}), U_\alpha(\mathcal{O}) \mid \langle \lambda, \alpha \rangle = 0 \right>.$$  

We have $T(\mathcal{O}) \subset B(\mathcal{O})$ and for each root $\alpha$ such that $\langle \lambda, \alpha \rangle = 0$, we have already shown that $U_\alpha(\mathcal{O}) \subset [K_\lambda, K_\lambda]$, so this concludes the proof.

We are finally ready to prove the main theorem, regarding the image of the Satake homomorphism. The rest of the section will be devoted to proving the following result.

**Theorem 25.** Suppose that the residue field of $F$ has size $|k_F| = p^f \geq 5$. Let $\alpha$ be a simple root, and suppose that $\mu \in X_*(T)$ is such that $h = \langle \mu, \alpha \rangle \cdot f \geq 1$. Then $S_F^G F(\mu) \equiv 0 \mod p^h$ for all $F \in \mathcal{H}_G^1$.

In particular, if our ring of coefficients $S$ is $p^n$-torsion and $\langle \mu, \alpha \rangle \cdot f \geq a$, then $S_F^G F(\mu) = 0$ for all $F \in \mathcal{H}_G^1$.

**Proof.** Let $M = Z_G((\ker \alpha)^0)$ be a standard Levi for $G$, with maximal torus $T$ and root system $\Phi_M = \{\alpha, -\alpha\}$. In particular $M$ has semisimple rank 1 and we want to adapt the PGL2-computations outlined in the introduction to this setup. Notice that $M = M^0 = Z_G((\ker \alpha)^0) = Z_G(\ker \alpha)^0$ where the last equality holds since both groups are smooth, connected, and have the same Lie algebra - and one is contained in the other. From now on, we assume $M = Z_G(\ker \alpha)^0$, so that $\ker \alpha$ is central in $M$.

By the transitivity results in subsection 5.1, we have that

$$S_F^G f = S_T^M \circ S_M^G f,$$

hence it suffices to show that $S_T^M F(\mu) \equiv 0 \mod p^h$ for all $F \in \mathcal{H}_M^1$.

For ease of notation, for the rest of the proof we put ourselves in the setting where $G = M$, with maximal torus $T$ and Borel subgroup $B \cap M$ which we denote $B$. The unipotent radical of $B$ is $U \cap M = U_\alpha$. Similar adjustment to the notation are made for $F$- and $\mathcal{O}$-points, so that for instance $K = M(\mathcal{O})$ in the following.

Notice then that $B = T \times U_\alpha$ as $\mathcal{O}$-group schemes. We also fix a pinning: isomorphism of $\mathcal{O}$-group schemes $i_\alpha : G_a \to U_\alpha$ and $i_{-\alpha} : G_a \to U_{-\alpha}$ which forms an SL2-triple with the coroot $\alpha^\vee : G_m \to T$, in the sense of [5], section 3.2.1.

By linearity of the Satake map, it suffices to show that $S_T^M F(\mu) \equiv 0 \mod p^h$ for $F$ in a basis of $\mathcal{H}_G^1$, assume then that $F$ is supported on the $G$-orbit of $(K, \lambda K)$ for $\lambda$ an antidominant cocharacter for $G$.

By lemma 15, we have that $\mathcal{H}^{\leq 1}_T \cong \mathcal{H}^{\leq 1}_B$, hence it suffices to show that $S_T^B F(\mu) \equiv 0 \mod p^h$. As $\res_{B(\mathcal{O})}^{B^\circ} : H^1(B^\circ, S) \to H^1(B(\mathcal{O}), S)$ is an isomorphism, by formula 5 it suffices to show that $\res_{B(\mathcal{O})}^{B^\circ} S_B^G F(\mu) = 0$. We obtain

$$\res_{B(\mathcal{O})}^{B^\circ} S_B^G F(B^\circ, \mu(\varpi) B^\circ) = \sum_{B(\mathcal{O}) \setminus \{\varpi^\mu u \in B_F / B_0\}} \text{cores}_{B(\mathcal{O}) \cap K_{\varpi^\mu u}} S_{B(\mathcal{O}) \cap K_{\varpi^\mu u}}^F(K, \varpi^\mu u K), \quad (8)$$

so it suffices to show that the right hand side is zero mod $p^h$. Notice that $B(\mathcal{O}) \cap K_{\varpi^\mu u} = B(\mathcal{O}) \cap \Ad(\varpi^\mu u)K = B(\mathcal{O}) \cap \Ad(\varpi^\mu u)B(\mathcal{O})$, and from now on we will denote this by $B(\mathcal{O})_{\varpi^\mu u}$. 

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We describe more explicitly the possible \( u \in U_F = U_\alpha(F) \) appearing in the sum: since \( \varpi^u u \) is a coset representative for \( B_F/B_\alpha \), multiplying by \( U_\alpha(\mathfrak{O}) \) on the right shows that we can assume that \( u \in U_\alpha(F)/U_\alpha(\mathfrak{O}) \). On the other hand, we consider \( B(\mathfrak{O}) \)-orbits (under left multiplication) on the cosets \( \varpi^u \mathcal{B}(\mathfrak{O}) \), hence multiplying on the left by \( U_\alpha(\mathfrak{O}) \), and using that \( U_\alpha(\mathfrak{O}) \varpi^u = \varpi^u U_\alpha(\varpi^{-(\mu,\alpha)}), \) we can assume that \( u \in U_\alpha(\varpi^{-(\mu,\alpha)}) \mid U_\alpha(F) \).

By assumption \( \langle \mu, \alpha \rangle \geq 1 \), and since \( U_\alpha \) is abelian, this second condition means we can forget the first one we found, and we end up with \( u \in U_\alpha(F)/U_\alpha(\varpi^{-(\mu,\alpha)}) \).

Fix now one \( u \in U_\alpha(F) \) appearing in the convolution sum in formula 8, we aim to show that its contribution to the sum is divisible by \( p^h \).

We want to obtain some necessary conditions for an element \( y = tv \in T(\mathfrak{O}) \ltimes U_\alpha(\mathfrak{O}) = \mathcal{B}(\mathfrak{O}) \) to belong to \( \mathcal{B}(\mathfrak{O})_{\varpi^u} \). We have that \( y \in \mathcal{B}(\mathfrak{O})_{\varpi^u} \) if and only if \( \text{Ad}(\varpi^u)^{-1} y \in \mathcal{B}(\mathfrak{O}) \): explicitly

\[
 u^{-1} \varpi^u tv \varpi^u u = t \left( (t^{-1} u^{-1}) \left( \varpi^{-\mu} v \varpi^{\mu} \right) \right) u \\
 \frac{U_\alpha(F)}{U_\alpha(F)}
\]

As conjugation by \( T \) preserves \( U_\alpha \), the above element is in \( \mathcal{B}(\mathfrak{O}) = T(\mathfrak{O}) \ltimes U_\alpha(\mathfrak{O}) \) if and only if

\[
 (t^{-1} u^{-1}) \left( \varpi^{-\mu} v \varpi^{\mu} \right) u \in U_\alpha(\mathfrak{O}).
\]

Using implicitly the isomorphism \( i_\alpha(\mathfrak{O}) : \mathfrak{O} \cong U_\alpha(\mathfrak{O}) \) and passing to additive notation, the above is equivalent to

\[
 u \left( 1 - \alpha(t^{-1}) \right) + \varpi^{-(\mu,\alpha)} v \in \mathfrak{O}.
\]

which is equivalent to

\[
 \varpi^{(\mu,\alpha)} u \left( 1 - \alpha(t^{-1}) \right) + v \in \varpi^{(\mu,\alpha)} \mathfrak{O}.
\]

Since \( v \in \mathfrak{O} \cong U_\alpha(\mathfrak{O}) \), we must have \( \varpi^{(\mu,\alpha)} u \left( 1 - \alpha(t^{-1}) \right) \in \mathfrak{O} \) as well.

Denote now \( B_{\alpha,\mu} = T(\mathfrak{O}) \ltimes U_\alpha(\varpi^{(\mu,\alpha)} \mathfrak{O}) \).

**Lemma 26.** \( \text{cores}_{B_{\alpha,\mu}^B(\mathfrak{O})} : H^1(B_{\alpha,\mu}, S) \longrightarrow H^1(B(\mathfrak{O}), S) \) is multiplication by \( p^h \).

**Proof.** The index is

\[
 [B(\mathfrak{O}) : B_{\alpha,\mu}] = [U(\mathfrak{O}) : U(\varpi^{(\mu,\alpha)} \mathfrak{O})] = [\mathfrak{O} : \varpi^{(\mu,\alpha)} \mathfrak{O}] = |k_F|^{(\mu,\alpha)} = p f^{(\mu,\alpha)} = p^h
\]

and on the other hand the restriction map \( \text{res} : H^1(B(\mathfrak{O}), S) \longrightarrow H^1(B_{\alpha,\mu}, S) \) is an isomorphism in degree 1, since the derived subgroup of \( B_{\alpha,\mu} \) contains the unipotent part \( U_\alpha(\varpi^{(\mu,\alpha)} \mathfrak{O}) \), by using the action of the torus as usual.

Therefore, the corestriction map \( \text{cores}_{B_{\alpha,\mu}^B(\mathfrak{O})} : H^1(B_{\alpha,\mu}, S) \longrightarrow H^1(B(\mathfrak{O}), S) \) is simply multiplication by the index \( p^h \). \( \square \)

We now consider a special case for the possible summand \( \varpi^u u \): suppose first that \( u \in U(F) \) is in fact in \( U(\varpi^{-(\mu,\alpha)}) \mathfrak{O} \). Then as a representative for the \( B(\mathfrak{O}) \)-orbit of \( \varpi^u \mathcal{B}(\mathfrak{O}) \) we can take \( u = \text{id} \), which corresponds via \( i_\alpha \) to \( 0 \).

Equation 10 yields then \( v \in U_\alpha(\varpi^{(\mu,\alpha)} \mathfrak{O}) \), and since this holds for all \( y \in B(\mathfrak{O})_{\varpi^u} \), we have \( B(\mathfrak{O})_{\varpi^u} = T(\mathfrak{O}) \ltimes U_\alpha(\varpi^{(\mu,\alpha)} \mathfrak{O}) = B_{\alpha,\mu} \).

The lemma above showed that \( \text{cores}_{B_{\alpha,\mu}^B(\mathfrak{O})} \) is multiplication by \( p^h \) - which proves that this summand yields a contribution to \( S_T^0 F(\mu) \) that is zero mod \( p^h \).

From now on, we assume that \( u = u_\alpha \notin U_\alpha(\varpi^{-(\mu,\alpha)} \mathfrak{O}) \) - or equivalently in additive notation that \( \varpi^{(\mu,\alpha)} u_\alpha \notin \mathfrak{O} \). Fix then \( -n = \text{val}(i_\alpha^{-1}(u)) < -\langle \mu, \alpha \rangle \) for some \( n > \langle \mu, \alpha \rangle \geq 1 \), i.e. \( n \geq 2 \).
Claim 27. Suppose that \(-n = \text{val}(i_{\alpha}^{-1}(u)) < -\langle \mu, \alpha \rangle\), then for each \(v = i_{\alpha}(z) \in U_{\alpha}(O)\) we can find \(t \in T(O)\) such that \(tv = y\) satisfies equation 10 and hence \(tv \in B(O)_{\omega^\mu u}\).

Proof of claim. Let \(u = i_{\alpha}(x)\). It suffices to find a solution \(t \in T(O)\) to the equation

\[\omega^{\langle \mu, \alpha \rangle} x (1 - \alpha(t^{-1})) + z = 0\]

which is equivalent to

\[\alpha(t^{-1}) = 1 + zx^{-1} \omega^{-\langle \mu, \alpha \rangle}.\]

As \(z \in O\) and \(\text{val}(x^{-1}) = n > \langle \mu, \alpha \rangle\) by assumption, it suffices to show that \(\alpha : T(O) \rightarrow O^*\) surjects onto \(1 + \omega O\).

For the coroot \(\alpha^\vee : G_m \rightarrow T\) we have \(\alpha^\vee(O) : O^* \rightarrow T(O)\) and moreover the composition \(O^* \xrightarrow{\alpha^\vee} T(O) \xrightarrow{\alpha} O^*\) is just squaring, so it suffices to show that the squaring homomorphism \(O^* \rightarrow O^*, \alpha \mapsto \alpha^2\) surjects onto \(1 + \omega O\).

Notice that the squaring morphism respects the canonical filtration \(1 + \omega^k O\) of \(O^*\), and that for \(k \geq 1\) on each quotient \((1 + \omega^k O) / (1 + \omega^{k+1} O)\) it induces an isomorphism, because each such quotient is a finite \(p\)-group with \(p \neq 2\). Then lemma 2 in chapter V of [23] proves that the squaring morphism is an isomorphism from \(1 + \omega O\) to itself, and hence finishes the proof of this claim. \(\square\)

We can then assume that for our fixed \(u = i_{\alpha}(x)\), the mapping \(B(O)_{\omega^\mu u} \rightarrow U_{\alpha}(O)\) sending \(tu \mapsto u\) is surjective. In particular, we have that \(T(O)B(O)_{\omega^\mu u} = B(O)\), hence applying the isomorphism \(\text{res}_{T(O)}^{B(O)} : H^1(B(O), S) \rightarrow H^1(T(O), S)\) followed by the restriction/corestriction formula we obtain that

\[\text{res}_{T(O)}^{B(O)} \text{cores}_{B(O)_{\omega^\mu u}} \text{res}_{B(O)_{\omega^\mu u}} \text{Fres}_{T(O)}^{B(O)} \text{res}_{T(O) \cap B(O)_{\omega^\mu u}} \text{res}_{T(O) \cap B(O)_{\omega^\mu u}} \text{F}(K, \omega^\mu u K) = \text{cores}_{T(O)}^{T(O) \cap B(O)_{\omega^\mu u}} \text{res}_{T(O) \cap B(O)_{\omega^\mu u}} \text{F}(K, \omega^\mu u K)\]

We notice that

\[T(O) \cap B(O)_{\omega^\mu u} = \{ t \cdot 1 | u(1 - \alpha(t^{-1})) + \omega^{-\langle \mu, \alpha \rangle} \cdot 0 \in O \} = \{ t \in T(O) | \alpha(t) - 1 \in u^{-1}O = \omega^n O \}\]

Denote this last subgroup by \(T_{\alpha,n}\) and notice that \(T(O) \supset T_{\alpha,n} \supset Z(G) \subseteq \ker \alpha\).

Recall that \(F\) is supported on the double coset \(K\lambda(\omega)K\). Now lemma 3.6i) in [16] says that \(F(K, \omega^\mu u K)\) is only nonzero if \(\mu \geq \lambda\), which means that \((\mu - \lambda) \in \mathbb{R}_{\geq 0} \alpha^\vee\) is a non-negative real linear combination of simple coroots.

Claim 28. Recall that \(u = i_{\alpha}(x)\) so that \(n = -\text{val}(x)\). Letting

\[k = \text{Ad}(\omega^\mu) i_{-\alpha}(x^{-1}) = \mu(\omega) i_{-\alpha}(x^{-1}) \mu(\omega)^{-1},\]

we have \(k \in U_{-\alpha}(O)\) and \(\omega^\mu u \in k \omega^\lambda K\). In particular we also obtain \(\mu - \lambda = n \alpha^\vee\).

Proof. First of all, notice that by assumption \(\text{val}(x^{-1}) = -\text{val}(x) = n > \langle \mu, \alpha \rangle\) so that \(\text{val}(x^{-1}) + \langle \mu, -\alpha \rangle > 0\), and hence

\[k = \text{Ad}(\omega^\mu) i_{-\alpha}(x^{-1}) = i_{-\alpha}(\omega^{\langle \mu, -\alpha \rangle} x^{-1}) \in i_{-\alpha}(O) = U_{-\alpha}(O) \subset K.\]

To show the rest of the claim, we prove that \(\omega^\mu u \in k(\mu - n\alpha^\vee)(\omega)K\). Notice that \(\langle \mu - n\alpha^\vee, \alpha \rangle = \langle \mu, \alpha \rangle - 2n < n - 2n = -n \leq -2\); in particular \(\mu - n\alpha^\vee\) is an antidominant weight, and hence by uniqueness of the Cartan decomposition for \(\omega^\mu u\) it must coincide with \(\lambda\), proving the last statement of the claim.

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We only need to prove that \((\varpi^\mu u)^{-1} \cdot k(\mu - n\alpha^\gamma)(\varpi) \in K\). That is the same as
\[
u^{-1}i_\alpha(x^{-1})\varpi^{-\mu}(\mu - n\alpha^\gamma)(\varpi) = i_\alpha(-x)i_\alpha(x^{-1})(n\alpha^\gamma)(\varpi^{-1}) = i_\alpha(-x)i_\alpha(x^{-1})\alpha^\gamma(\varpi^{-n}).
\]

Recall that the coroot \(\alpha^\gamma\) is defined ‘in the same way’ both for \(\text{SL}_2\) and for \(\text{PGL}_2\) as \(t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)\).

We can compute the above element (using the previously fixed pinning) inside our \(\text{SL}_2\)-triple, and we obtain
\[
i_\alpha(-x)i_\alpha(x^{-1})\alpha^\gamma(\varpi^{-n}) = \left(\frac{1}{0} - x \right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & x^{-1} \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -x \\ 0 & x^{-1} \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x^{-1} & -x \varpi^n \\ 0 & \varpi^n \end{smallmatrix}\right)
\]

and as \(n = \text{val}(x^{-1}) \geq 2\), the last element is indeed in \(\text{SL}_2(\mathcal{O})\) or \(\text{PGL}_2(\mathcal{O})\) depending on the case.

We have then \(F(K, \varpi^\mu uK) = F(K, k\varpi^\lambda K) = (c_k^*)^{-1}F(K, \lambda K)\). We now analyze the effect of conjugation by \(k\) and restriction to \(T_{\alpha,n}\) to finish the proof of the theorem.

Recall that we are studying the following summand:
\[
\text{cores}^T_{T_{\alpha,n}}\text{res}^K_{T_{\alpha,n}}F(K, \varpi^\mu uK) = \text{cores}^T_{T_{\alpha,n}}\text{res}^K_{T_{\alpha,n}}F(K, k\varpi^\lambda K) = \text{cores}^T_{T_{\alpha,n}}\text{res}^K_{T_{\alpha,n}}c_kF(K, \varpi^\lambda K).
\]

We focus on \(\text{res}^K_{T_{\alpha,n}}c_kF(K, \varpi^\lambda K) \in H^1(T_{\alpha,n}, S)\). Denote \(f = F(K, \lambda K) \in H^1(K, S)\).

**Claim 29.** The subgroup \(\alpha^\gamma(\mathcal{O}^*) \cdot (\ker \alpha)(\mathcal{O})\) of \(T(\mathcal{O})\) has index coprime to \(p\).

**Proof of claim.** Consider the exact sequence
\[
1 \longrightarrow (\ker \alpha)(\mathcal{O}) \longrightarrow T(\mathcal{O}) \longrightarrow \mathcal{O}^* \longrightarrow 1.
\]

Then we have for the quotient
\[
T(\mathcal{O})/\alpha^\gamma(\mathcal{O}) \cdot (\ker \alpha)(\mathcal{O}) \cong \text{Im}(\alpha)/\alpha(\alpha^\gamma(\mathcal{O})) \subset \mathcal{O}^*/(\mathcal{O}^*)^2.
\]

So it suffices to show that the cokernel of the squaring morphism \(\mathcal{O}^* \longrightarrow \mathcal{O}^*, a \mapsto a^2\) has size coprime to \(p\).

In the proof of claim 27 we have shown that the squaring morphism is an isomorphism when restricted to the pro-\(p\) Sylow \(1 + \varpi\mathcal{O}\) of \(\mathcal{O}^*\), so the size of the cokernel is coprime to \(p\). \(\square\)

We show how this claim finishes the proof of theorem 25. As the index is coprime to \(p\), the behavior of a morphism \(f : T(\mathcal{O}) \longrightarrow S\) is completely determined by its restriction to \(\alpha^\gamma(\mathcal{O}^*) \cdot (\ker \alpha)(\mathcal{O})\). Let then \(t = \alpha^\gamma(z)t_0 \in \alpha^\gamma(\mathcal{O}^*) \cdot (\ker \alpha)(\mathcal{O})\). Since \((\ker \alpha)\) is central in \(G\) by construction, conjugation by \(k^{-1}\) acts trivially on it while on the other hand
\[
\text{Ad}(k^{-1}) \left(\begin{smallmatrix} z \\ \varpi^{-\langle \mu, \alpha \rangle}x^{-1}z^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} z \\ \varpi^{-\langle \mu, \alpha \rangle}x^{-1}z^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \varpi^{-\langle \mu, \alpha \rangle}x^{-1}(z^{-1} - 1) \\ \varpi^{-\langle \mu, \alpha \rangle}x^{-1}(z^{-1} - 1) \end{smallmatrix}\right)
\]

so that we conclude
\[
\text{Ad}(k^{-1})t = \text{Ad}(k^{-1})(\alpha^\gamma(z)t_0) = \text{Ad}(k^{-1})(\alpha^\gamma(z)) \cdot t_0 = \left(\begin{smallmatrix} \varpi^{-\langle \mu, \alpha \rangle}x^{-1}(z^{-1} - 1) \\ \varpi^{-\langle \mu, \alpha \rangle}x^{-1}(z^{-1} - 1) \end{smallmatrix}\right)
\]

We notice that this latter element is not just on \(K_{\lambda}\), but also in \(K_{\lambda} \cap B^{-}(\mathcal{O}) = U_{-\alpha}(\varpi^{-\langle \lambda, \alpha \rangle}\mathcal{O}) \times T(\mathcal{O})\). Therefore, when computing
\[
\text{res}^K_{T_{\alpha,n}}c_kF(K, \varpi^\lambda K)(t) = \text{res}^K_{T_{\alpha,n}}(c_kF)((c_kf)|_{T_{\alpha,n}}(t) = f(k^{-1}tk)
\]

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we can replace $f$ by its image under the restriction map \( \text{res}^{K_\lambda}_{K_\lambda \cap B^{-}(0)} : H^1(K_\lambda, S) \to H^1(K_\lambda \cap B^{-}(0), S) \). Now since \( \text{res}^{U_\alpha(\omega(\lambda,\alpha)\otimes T(\mathbb{O}))}_{T(\mathbb{O})} : H^1(U_\alpha(\omega(\lambda,\alpha)\otimes T(\mathbb{O}), S) \to H^1(T(\mathbb{O}), S) \) is also an isomorphism, we can assume $f$ is just the unique extension to $U_\alpha(\omega(\lambda,\alpha)\otimes T(\mathbb{O}))$ of a homomorphism $T(\mathbb{O}) \to S$. This unique extension is obtained via the quotient map $U_\alpha(\omega(\lambda,\alpha)\otimes T(\mathbb{O})) \to T(\mathbb{O})$ - in other words, $f(ut) = f(t)$ if $ut \in U_\alpha(\omega(\lambda,\alpha)\otimes T(\mathbb{O}))$.

Notice that \( \text{U}_p \) is equal to \( \text{U}_p \) of a homomorphism $T(\mathbb{O}) \to S$. This yields a map in cohomology in the opposite direction:

\[
\text{cores}_{T_{\alpha,n}}(\text{res}^{T(\mathbb{O})}_{T_{\alpha,n}} f ) (t) = \text{cores}_{T_{\alpha,n}}(f) (t)
\]

since $\alpha^\vee(z) \cdot t_0 = t \in T_{\alpha,n}$.

Applying the corestriction map \( \text{cores}_{T_{\alpha,n}} \) to \( \text{res}^{T(\mathbb{O})}_{T_{\alpha,n}} f \) yields then \( [T(\mathbb{O}) : T_{\alpha,n}] \cdot f \). Recall that \( T_{\alpha,n} = \{ t \in T(\mathbb{O}) \mid \alpha(t) \in 1 + \mathbb{O}^n \} \) is the kernel of the composition $\alpha_n : T(\mathbb{O}) \to \mathbb{O}^*/(1 + \mathbb{O}^n)$, hence the index \( [T(\mathbb{O}) : T_{\alpha,n}] \) is equal to \( |\text{Im}(\alpha_n)| \). We proved before that $\alpha$ surjects onto $1 + \mathbb{O}$, hence the size of the image is divisible by \( |(1 + \mathbb{O}^n)/(1 + \mathbb{O}^n)| = |k_F|^{|n-1|} = p^{f(n-1)} \).

Our standing assumption that $n > \langle \mu, \alpha \rangle$ means that $f \cdot (n-1) \geq f \cdot \langle \mu, \alpha \rangle = h$, where the last equality is the assumption on $\mu$ in the statement of the theorem. Therefore, multiplication by \( [T(\mathbb{O}) : T_{\alpha,n}] \) lands into the $p^b$-divisible subgroup of the cohomology with $S$-coefficients and this completes the proof of the theorem.

\[\square\]

### A Appendix

We collect here the proofs of several facts on groupoids and groupoid cohomology used in the main body of the paper.

**Proposition 30** (Well-posedness of pullback). Let \( i : G \to H \) be a continuous morphism of topological groupoids. Then $i^*F$ satisfies conditions 1 and 2 of definition 6. Moreover, if \( \pi_0 i : \pi_0 G \to \pi_0 H \) has finite fibers and $F \in \mathbb{H}_c^*(H)$, then $i^*F \in \mathbb{H}_c^*(G)$.

**Proof.** That $i^*F$ satisfies condition 1 of definition 6 is obvious, so we focus on condition 2. Fix then $x,y \in \text{Ob}(G)$ and $\phi \in \text{Hom}_G(x,y)$. As $i$ is a natural transformation, this induces $i(\phi) \in \text{Hom}_H(i(x),i(y))$ and thus an isomorphism $\text{Stab}_H(i(x)) \overset{i(\phi)_*}{\longrightarrow} \text{Stab}_H(i(y))$ mapping $h \mapsto i(\phi) \circ h \circ i(\phi)^{-1}$. This yields a map in cohomology in the opposite direction:

\[
(i(\phi)^*)_* : H^*(\text{Stab}_H(i(y)), S) \to H^*(\text{Stab}_H(i(x)), S)
\]

and since $F \in \mathbb{H}_c^*(H)$ we have

\[
F(i(x)) = (i(\phi))^* F(i(y)).
\]
Moreover, the functorial properties of $i$ give group homomorphisms

$$\text{Stab}_G(x) = \text{Hom}_G(x, x) \xrightarrow{i^x} \text{Hom}_H(i(x), i(x)) = \text{Stab}_H(i(x))$$

and

$$\text{Stab}_G(y) = \text{Hom}_G(y, y) \xrightarrow{i^y} \text{Hom}_H(i(y), i(y)) = \text{Stab}_H(i(y))$$

and via the maps in cohomology in the opposite directions

$$i^x : H^* (\text{Stab}_H(i(x))) \to H^* (\text{Stab}_G(x))$$

and

$$i^y : H^* (\text{Stab}_H(i(y))) \to H^* (\text{Stab}_G(y))$$

we defined $i^* F(x) = i^x F(i(x))$ and $i^* F(y) = i^y F(i(y))$.

Condition 2 of definition 6 for $i^* F$ amounts to showing that the map induced by the isomorphism $\text{Stab}_G(x) \xrightarrow{\phi^*} \text{Stab}_G(y)$ in cohomology, $\phi^* : H^* (\text{Stab}_G(y), S) \to H^* (\text{Stab}_G(x), S)$, gives $i^* F(x) = \phi^* (i^* F(y))$.

Using the identities above, the last equality boils down to showing that

$$i^x (i(\phi)^* F(i(y))) = \phi^* (i^y F(i(y)))$$

i.e. that the following diagram commutes

$$\begin{array}{ccc}
H^* (\text{Stab}_H(i(y)), S) & \xrightarrow{i^y} & H^* (\text{Stab}_G(y), S) \\
i(\phi)^* & & \phi^* \\
\downarrow & & \downarrow \\
H^* (\text{Stab}_H(i(x)), S) & \xrightarrow{i^x} & H^* (\text{Stab}_G(x), S)
\end{array}$$

Since all the maps are induced from group homomorphism, it suffices (by functoriality of cohomology w.r.t. group homomorphisms) to show that the following diagram commutes:

$$\begin{array}{ccc}
\text{Stab}_H(i(y)) & \xleftarrow{i^y} & \text{Stab}_G(y) \\
i(\phi)^* & & \phi^* \\
\downarrow & & \downarrow \\
\text{Stab}_H(i(x)) & \xrightarrow{i^x} & \text{Stab}_G(x)
\end{array}$$

Let’s check this latter condition. Given $g \in \text{Stab}_G(x) = \text{Hom}_G(x, x)$, applying $\phi^*$ yields $\phi \circ g \circ \phi^{-1} \in \text{Hom}_G(y, y) = \text{Stab}_G(y)$, then applying $i^y$ gives $i(\phi \circ g \circ \phi^{-1}) = i(\phi) \circ i(g) \circ i(\phi)^{-1}$ by functoriality of $i$.

On the other hand, we can start by applying $i^x$ to $g$, getting $i(g) \in \text{Hom}_H(i(x), i(x)) = \text{Stab}_H(i(x))$. Then applying $i(\phi)^*$ yields $i(\phi) \circ i(g) \circ i(\phi)^{-1}$, which proves the claim.

Finally, it remains to check that if $F$ is supported on finitely many connected components of $H$ and $\pi_0 \bar{\phi}$ has finite fibers, then also $i^* F$ is supported on finitely many connected components of $G$. We can obviously assume that $F$ is supported on a unique connected component $C$ of $H$; then $i^* F(x) \neq 0$ implies that $F(i(x)) \neq 0$ and thus $i(x) \in C$. Hence, the connected components $C_x$ of $x$ varies among the finite set $(\pi_0 \bar{\phi})^{-1}(C)$, which concludes the proof.

$\Box$

**Proposition 31** (Well-posedness of pushforward). Let $i : G \to H$ be a finite covering morphism of groupoids. Then pushforward is well-defined as a map $\mathbb{H}^*_x(G) \to \mathbb{H}^*_x(H)$. 

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Proof. Recall that for $F \in H^*_c(G)$ and $y \in \text{Ob}(H)$ we defined

$$(i_\ast F)(y) = \sum_{H_y \backslash x \in i^{-1}(y)} \text{cores}^\text{Stab}_G(y)_{\text{Stab}_G(x)} F(x).$$

We notice that corestriction is well-defined because $i$ is a finite covering, and also that the sum

is well-defined, i.e. that the summand corresponding to a particular $H_y$-orbit on the fiber $i^{-1}(y)$

does not depend on the particular representative $\phi$ where the last equality is due to invariance of $F$.

To prove that the components of $G$ have only finitely many nonzero summands, since $F$ is supported

on finitely many connected components of $G$ (above $y$, if we only assume that $F$ is $i$-fiberwise compactly supported).

To prove that $i_\ast F$ satisfies the invariance condition of the definition of groupoid cohomology, fix

$\phi \in \text{Hom}_H(y, y')$, and notice that this induces a bijection between the $H_y$-orbits on $i^{-1}(y)$ and

the $H_y'$-orbits on $i^{-1}(y')$ via $i^{-1}(y) \ni x \mapsto \tilde{\phi}(x)$ where $\tilde{\phi}$ is the unique morphism of $G$ having source $x$ and

mapping to $\phi$ under $i$. Therefore, we have

$$(i_\ast F)(y' = \phi(y)) = \sum_{H_y \backslash x \in i^{-1}(y)} \text{cores}^\text{Stab}_G(y)_{\text{Stab}_G(x;\phi)} F(\tilde{\phi} x) = \sum_{H_y \backslash x \in i^{-1}(y)} \text{cores}^\text{Stab}_G(y)_{\text{Stab}_G(x;\phi)} \tilde{\phi}_* F(x)$$

where the last equality is due to invariance of $F$. To prove that $(i_\ast F)(\phi y) = \tilde{\phi}^* (i_\ast F)(y)$ it suffices

then to show that the diagram

$\begin{array}{ccc}
H^* (G_x) & \xrightarrow{\phi^*} & H^* (G_{\tilde{\phi} x}) \\
\text{cores} & & \text{cores} \\
H^* (H_y) & \xrightarrow{\phi^*} & H^* (H_{\phi y})
\end{array}$

commutes. This follows from the fact that the group isomorphisms induced by $\phi$ and $\tilde{\phi}$ via

‘conjugation’ respect the inclusions $G_x \hookrightarrow H_y$ and $G_{\tilde{\phi} x} \hookrightarrow H_{\phi y}$, in the sense that the diagram

$\begin{array}{ccc}
G_x & \xrightarrow{\phi^*} & G_{\tilde{\phi} x} \\
\downarrow & & \downarrow \\
H_y & \xrightarrow{\phi^*} & H_{\phi y}
\end{array}$

is commutative.

Finally, the fact that $i_\ast F$ is supported on finitely many connected components of $H$ is a con-

sequence of the same fact for $F$ and the fact that morphisms of groupoids preserve connectedness:

more in details, if $i_\ast F$ is nonzero at $y$ and $y'$ it means that there exists objects $x \in i^{-1}(y)$ and

$x' \in i^{-1}(y')$ where $F$ is nonzero. Assuming by linearity that $F$ is supported on only one connected

component yields that there exists an element $g : x \longrightarrow x'$, but then $i(g) : y \longrightarrow y'$.
Lemma 10. Let

\[
\begin{array}{ccc}
X & \xrightarrow{k} & Z \\
\downarrow i & & \downarrow j \\
Y & & \\
\end{array}
\]

be a commutative triangle of topological groupoid morphisms. Suppose that \( i \) is a finite covering morphism and that \( j \) induces inclusions between isotropy groups.

Let then \( F \in \mathbb{H}^\bullet(X) \) be \( i \)-fiberwise compactly supported and \( G \in \mathbb{H}^\bullet(Z) \), we have

\[
i_*F \cup j^*G = i_* (F \cup k^*G) \text{ in } \mathbb{H}^\bullet(Y).
\]

Moreover, if \( j \) and \( k \) induce finite-fibers maps on connected components, then each operation on cohomology preserves the compact support and thus if \( F \in \mathbb{H}^\bullet(X) \) and \( G \in \mathbb{H}^\bullet(Z) \), the formula above holds in \( \mathbb{H}^\bullet(Y) \).

Proof. This is obviously an equivariant version of the well-known formula for group cohomology

\[
(\text{cores}_H^G \alpha) \cup \beta = \text{cores}_H^G (\alpha \cup \text{res}_H^G \beta).
\]

We give all details since we use this result repeatedly in the paper.

First of all, notice that since \( i \) is a covering morphism, it induces continuous inclusions on isotropy groups. Letting \( x \in \text{Ob}(X) \), \( y = i(x) \in \text{Ob}(Y) \) and \( z = j(y) = k(x) \in \text{Ob}(Z) \) we have thus the commutative triangle of group homomorphisms

\[
\begin{array}{ccc}
\text{Stab}_X(x) & \xrightarrow{k_*} & \text{Stab}_Z(z) \\
\downarrow i_* & & \downarrow j_* \\
\text{Stab}_Y(y) & & \\
\end{array}
\]

and since \( i_* \) and \( j_* \) are inclusions by assumption, so is \( k_* \): this shows that \( k \) also induces inclusions between isotropy groups. We also denote \( j_y^* : H^\bullet(Z_{j(y)}) \to H^\bullet(Y_y) \) and \( k_x^* : H^\bullet(Z_{k(x)}) \to H^\bullet(X_x) \) the induced maps on group cohomology.

Notice also that since \( F \in \mathbb{H}^\bullet(X) \) is \( i \)-fiberwise compactly supported - and the space of such cohomology classes is an ideal of \( \mathbb{H}^\bullet(X) \) under pointwise cup product - we obtain that \( F \cup k^*G \) is also \( i \)-fiberwise compactly supported, thus making sense of the right hand side of the formula we aim to prove.

Let \( y \in \text{Ob}(Y) \), then we have

\[
(i_*F \cup j^*G)(y) = i_*F(y) \cup j^*G(y) = \left( \sum_{Y_{y} \setminus i^{-1}(y) \ni x} \text{cores}_{X_x}^{Y_y} F(x) \right) \cup \left( j_y^*G(j(y)) \right) = \sum_{Y_{y} \setminus i^{-1}(y) \ni x} \left( \text{cores}_{X_x}^{Y_y} F(x) \cup j_y^*G(j(y)) \right)
\]

The assumptions on the maps guarantees that for each \( y \in \text{Ob}(Y) \) and each \( x \in i^{-1}(x) \) we have \( X_x \subset Y_y \subset Z_{j(y)} \). We apply the abovementioned formula 11 for group cohomology for each single summand, where \( G = Y_y \supset X_x = H \), \( \alpha = F(x) \) and \( \beta = j_y^*G(j(y)) \). We obtain

\[
\sum_{Y_{y} \setminus i^{-1}(y) \ni x} \text{cores}_{X_x}^{Y_y} \left( F(x) \cup \text{res}_{X_x}^{Y_y} j_y^*G(j(y)) \right) = \sum_{Y_{y} \setminus i^{-1}(y) \ni x} \text{cores}_{X_x}^{Y_y} \left( F(x) \cup k_x^*G(k(x)) \right)
\]

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The final assertion of the lemma is immediate, since pushforward along \( i \) always preserves the compactly supported cohomology, and under the additional assumption that \( j \) and \( k \) induce finite-fibers map in cohomology, the pullbacks along \( j \) and \( k \) preserve that too.

\[ = \sum_{y \in \text{Ob}(Y)} \text{cores}_{X}^{Y} \left( ((F \cup k^*G)(x)) \right) = (i^*_a(F \cup k^*G))(y). \]

\[ \text{Lemma 11. Let} \]

\[ \begin{array}{ccc}
Z & \longrightarrow & X \\
\tilde{i} & \downarrow & \downarrow i \\
Y & \longrightarrow & A
\end{array} \]

be a pullback square of topological groupoids, where \( i \) is a finite covering morphism. Suppose also that \( \pi \) is continuous and induces open injections of isotropy groups at all objects (e.g. \( \pi \) is also a finite covering morphism, but in fact it suffices that every morphisms in \( A \) has at most a unique lift under \( \pi \)).

Let \( F \in \mathbb{H}^*(X) \) be \( i \)-fiberwise compactly supported, then \( \tilde{\pi}^*F \) is \( i \)-fiberwise compactly supported and we have

\[ (\tilde{i})^*_a(\tilde{\pi}^*F) = \pi^*(i^*_aF). \]

If moreover \( \pi \) induces a finite-fibers map on connected components, then given \( F \in \mathbb{H}^*_c(X) \) the same formula above holds in \( \mathbb{H}^*_c(Y) \).

\[ \text{Proof. This is clearly an equivariant version of the restriction-corestriction formula for group cohomology. We give all details, since we use this lemma several times.} \]

Firstly notice that by proposition 2.8 in \([2]\), \( \tilde{i} \) is also a covering morphism, and is continuous as seen in the definition 2 of pullback and homotopy pullback.

We start by proving that \( \tilde{\pi} \) also induces an injection on isotropy groups. Let \( z \in \text{Ob}(Z) \), and let \( x = \tilde{\pi}(z) \), \( y = \tilde{i}(z) \) and \( \pi(\tilde{i}(z)) = a = i(\tilde{\pi}(z)) \). By theorem 2.2(v) in \([4]\), the isotropy group \( \text{Stab}_Z(z) \) is the pullback of the diagram

\[ \begin{array}{ccc}
\text{Stab}_X(x) & \longrightarrow & \text{Stab}_A(a) \\
\downarrow i^*_a & & \\
\text{Stab}_Y(y) & \longrightarrow & \text{Stab}_A(a)
\end{array} \]

and on the other hand \( i_x : \text{Stab}_X(x) \longrightarrow \text{Stab}_A(a) \) is an injection since \( i \) is a covering morphism, while \( \pi_y : \text{Stab}_Y(y) \longrightarrow \text{Stab}_A(a) \) is an injection by assumption on \( \pi \).

This proves that the pullback of the diagram above is simply the intersection \( \text{Stab}_Y(y) \cap \text{Stab}_X(x) \subset \text{Stab}_A(a) \), where we identify \( \text{Stab}_Y(Y) \) with its image under \( \pi_y \) inside \( \text{Stab}_A(a) \), and same for \( \text{Stab}_X(x) \). In particular, \( \text{Stab}_Z(z) \cong \text{Stab}_Y(y) \cap \text{Stab}_X(x) \subset \text{Stab}_A(a) \) inside \( \text{Stab}_A(a) \), so that \( \tilde{\pi} \) also induces injection of isotropy groups. Moreover, \( \text{Stab}_Z(z) = \text{Stab}_Y(y) \cap \text{Stab}_X(x) \) is open in \( \text{Stab}_Y(y) \), since \( \text{Stab}_X(x) \subset \text{Stab}_A(a) \) is open by assumption on \( i \) being a finite covering morphism, and the property of being an open map is preserved under pullback by the continuous map \( \pi_y : \text{Stab}_Y(y) \longrightarrow \text{Stab}_A(a) \).

The above reasoning also shows that \( \tilde{i} \) is a finite covering morphism: indeed we have shown that \( \text{Stab}_Z(z) \cong \text{Stab}_Y(y) \cap \text{Stab}_X(x) \), and on the other hand the map of coset spaces

\[ \text{Stab}_Y(y)/(\text{Stab}_Y(y) \cap \text{Stab}_X(x)) \longrightarrow \text{Stab}_A(a)/\text{Stab}_X(x) \]

\[ g.(\text{Stab}_Y(y) \cap \text{Stab}_X(x)) \longrightarrow g.\text{Stab}_X(x) \]
is easily seen to be injective, so the target Stab$_A(a)/$Stab$_X(x)$ being finite by assumption on $i$ proves the same finiteness result for $\tilde{i}$.

The same argument when we switch the roles of $\pi$ and $i$ shows that, as $\pi$ is assumed to inducing open injections on stabilizers, so does $\tilde{\pi}$.

We also show that under the additional assumption that $\pi$ induces a finite fibers map on connected components so does $\tilde{\pi}$, proving that the pullback $\tilde{\pi}^*$ preserves compactly supported cohomology. The pullback square induces a commutative diagram on connected components:

\[
\begin{array}{ccc}
\pi_0Z & \xrightarrow{\pi_0} & \pi_0X \\
\downarrow{\tilde{i}_0} & & \downarrow{i_0} \\
\pi_0Y & \xrightarrow{\pi_0} & \pi_0A
\end{array}
\]

Fix a component $\mathcal{X} \in \pi_0X$. Then $i(\mathcal{X}) \in \pi_0A$ has finitely many connected components of $Y$ above it, by the assumption on $\pi$: denote them by $\{y_1, \ldots, y_m\}$. Then clearly

\[
\tilde{\pi}_0^{-1}(\mathcal{X}) = \bigsqcup_{i=1}^m \tilde{\pi}_0^{-1}(\mathcal{X}) \cap \tilde{i}_0^{-1}(y_i)
\]

and hence it suffices to prove that each of $\tilde{\pi}_0^{-1}(\mathcal{X}) \cap \tilde{i}_0^{-1}(y_i)$ is finite.

We fix then one such $y_i$ and denote it by $\check{y}$ from now on. Let $z \in \text{Ob}(Z)$ be in one of the components of $\tilde{\pi}_0^{-1}(\mathcal{X}) \cap \tilde{i}_0^{-1}(\check{y})$, and denote $x = \tilde{\pi}(z) \in \mathcal{X}$, $y = \tilde{i}(z) \in \check{y}$ and $i(x) = a = \pi(y)$. The ‘Mayer-Vietoris’ sequence of theorem 2.2 in [2] is

\[
\begin{array}{ccc}
\text{Stab}_Z(z) & \xrightarrow{\tilde{\pi}} & \text{Stab}_X(x) \\
\downarrow{i} & & \downarrow{i} \\
\text{Stab}_Y(y) & \xrightarrow{\pi} & \text{Stab}_A(a)
\end{array}
\]

\[
\begin{array}{ccc}
\pi_0Z & \xrightarrow{\pi_0} & \pi_0A \\
\downarrow{\tilde{i}_0} & & \downarrow{i_0} \\
\pi_0Y & \xrightarrow{\pi_0} & \pi_0A
\end{array}
\]

and part (ii) of the statement of the theorem is that the image of $\Delta$ inside $\pi_0Z$ is the intersection $\tilde{\pi}_0^{-1}(\mathcal{X}) \cap \tilde{i}_0^{-1}(y)$.

On the other hand, part (iv) of the same theorem says that the image of $\Delta$ is in bijection with the double coset space $\pi(\text{Stab}_Y(y)) \setminus \text{Stab}_A(a)/i(\text{Stab}_X(x))$. The assumption on $i$ inducing finite-index inclusions on isotropy groups implies that this double coset space is finite, and this concludes the proof that $\tilde{\pi}_0$ also has finite fibers.

Before proving the main formula, it remains to show that if $F$ is $i$-fiberwise compactly supported, then $\tilde{\pi}^*F$ is $\tilde{i}$-fiberwise compactly supported, so that the left hand side of the formula is well-defined. Fix then $y \in \text{Ob}(Y)$: we want to show that $\tilde{\pi}^*F$ is supported on finitely many connected components of $Z$ above $y$.

Proposition 2.1 in [3] says that we have a bijection between $\text{Stab}_Y(y)$-orbits on $\tilde{i}^{-1}(y)$ and connected components of $Z$ intersecting the fiber $\tilde{i}^{-1}(y)$, so it suffices to show that $\tilde{\pi}^*F$ is only nonzero on finitely many $\text{Stab}_Y(y)$-orbits on $\tilde{i}^{-1}(y)$.

Let now $a = \pi(y) \in \text{Ob}(A)$. An explicit model for the pullback groupoid $Z$ gives that the objects $\text{Ob}(Z)$ are pairs in $\text{Ob}(Y) \times \text{Ob}(X)$ mapping to the same object of $A$.  

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In particular, the objects of the fiber \( \tilde{i}^{-1}(y) \) consists of pairs \( \{(y, x), |a = \pi(y) = i(x)\} \) - this is obviously in bijection with the fiber \( i^{-1}(a) \) via the map \( (y, x) \mapsto x \).

**Claim 32.** This bijection respects the \( \text{Stab}_Y(y) \)-actions on each side: the action on \( \tilde{i}^{-1}(y) \) is the natural one, while the action on \( i^{-1}(a) \) is the restriction to \( \text{Stab}_Y(y) \) of the natural action of \( \text{Stab}_A(a) \) (recall that by assumption \( \pi \) induces an injection \( \text{Stab}_Y(y) \hookrightarrow \text{Stab}_A(a) \)).

**Proof of claim.** Fix \( (y, x) \in \text{Ob}(\tilde{i}^{-1}(y)) \) and let \( g \in \text{Stab}_Y(y) \). As \( \tilde{i} \) is a covering, \( g \) lifts to a unique morphism in \( Z \): denote it \( \tilde{g} : (y, x) \mapsto (y, x') \). We thus want to show that the \( \text{Stab}_A(a) \)-action of \( \pi(g) \) on \( x \in i^{-1}(a) \) maps it to \( x' \).

Under the map \( \pi \), we have that \( \pi(g) \in \text{Stab}_A(a) \) and as such it acts on the fiber \( i^{-1}(a) \): its unique lift under the covering \( i \) having source \( x \) is the element \( \pi(\tilde{g}) : x \mapsto \tilde{x}' \), and thus the \( \text{Stab}_Y(y) \)-action of \( g \) sends \( x \) to \( \tilde{x}' \).

Now notice that

\[
\pi(g) = \pi(\tilde{g}(i)) = i(\pi(\tilde{g}))
\]

so that \( \tilde{\pi}(\tilde{g}) \) is another lift under \( i \) of \( \pi(g) \): the uniqueness statement of the covering means that we must have \( \tilde{\pi}(\tilde{g}) = \pi(\tilde{g}) \). But \( \tilde{\pi}(\tilde{g}) : x \mapsto x' \), so that \( x' = \tilde{x}' \). This proves that the two \( \text{Stab}_Y(y) \)-actions coincide under the given bijection, and completes the proof of the claim. \( \square \)

Finally, notice that as \( F \) is \( i \)-fiberwise compactly supported, \( F(x) \neq 0 \) only on finitely many connected components of \( X \) in the fiber \( i^{-1}(a) \). By proposition 2.1 of [3], these are in bijection with \( \text{Stab}_A(a) \)-orbits on the fiber \( i^{-1}(a) \), and since \( \text{Stab}_Y(y) \hookrightarrow \text{Stab}_A(a) \) is a finite index inclusion by assumption on \( \pi \), we obtain that \( F(x) \neq 0 \) only on finitely many \( \text{Stab}_Y(y) \)-orbits.

Since \( F(x) \neq 0 \) is a necessary condition for \( \tilde{\pi}^*F((y, x)) \neq 0 \), the statement of the claim completes the proof that \( \tilde{\pi}^*F \) is \( i \)-fiberwise compactly supported.

We are finally ready to prove the main formula of the lemma. Notice that the assumption on \( \pi \) and \( \tilde{\pi} \) inducing open injections of isotropy groups at all objects means that the formula for the pullback becomes

\[
(\tilde{\pi})^*F(z) = (\tilde{\pi}(z))^*F(\tilde{\pi}(z)) = \text{res}_{\text{Stab}_Z(\pi(z))}^{\text{Stab}_X(\pi(z))} F(\tilde{\pi}(z))
\]

and similarly for \( \pi \).

Fix \( y \in \text{Ob}(Y) \), then by definitions we have

\[
(\tilde{\pi})_*((\tilde{\pi})^*F)(y) = \sum_{\text{Stab}_Y(y) \backslash z \in \tilde{i}^{-1}(y)} \text{cores}_{\text{Stab}_Z(\pi(z))}^{\text{Stab}_X(\tilde{\pi}(z))} F(\tilde{\pi}(z)) = \sum_{\text{Stab}_Y(y) \backslash z \in \tilde{i}^{-1}(y)} \text{cores}_{\text{Stab}_Y(y) \backslash \text{Stab}_Z(\pi(z))}^{\text{Stab}_X(\tilde{\pi}(z))} F(\tilde{\pi}(z))
\]

On the other hand, we have

\[
\pi^*(i_*F)(y) = \text{res}_{\text{Stab}_Y(y)}^{\text{Stab}_A(\pi(y))} (i_*F)(\pi(y)) = \text{res}_{\text{Stab}_Y(y) \backslash \text{Stab}_A(\pi(y)) \backslash \tilde{i}^{-1}(\pi(y))} \sum_{\text{Stab}_A(\pi(y) \backslash \tilde{i}^{-1}(\pi(y)))} \text{cores}_{\text{Stab}_X(\pi(z))}^{\text{Stab}_A(\pi(y))} F(x)
\]

Moving the restriction map inside the summation and applying the restriction-corestriction formula for group cohomology yields

\[
\sum_{\text{Stab}_A(\pi(y) \backslash \tilde{i}^{-1}(\pi(y)))} \sum_{\text{Stab}_Y(y) \backslash \text{Stab}_A(\pi(y)) \backslash \text{Stab}_X(x) \ni g} \text{cores}_{\text{Stab}_Y(y) \backslash \text{Stab}_X(x) \backslash \text{Stab}_A(\pi(y))}^{\text{Stab}_X(x) \backslash \text{Stab}_A(\pi(y))} \text{res}_{\text{Stab}_Y(y) \backslash \text{Stab}_X(x) \backslash \text{Stab}_A(\pi(y))} F(x)
\]

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Since $\pi$ is a covering morphism, each double coset representative $g$, which is an element of $\text{Stab}_A(\pi(y))$, has a well-defined lift to a morphism in $X$ with source $x$; in particular $g x \in \text{Ob}(X)$ is well-defined, and its stabilizer is obviously $\text{Stab}_X(gx) = g\text{Stab}_X(x)g^{-1}$. Invariance of $F$ also gives that the single summand is $\text{cores}^\text{Stab}_X(Y)\text{res}^\text{Stab}_X(Y)F(gx)$. Notice that the intersection $\text{Stab}_Y(y) \cap \text{Stab}_X(gx)$ is taken inside $\text{Stab}_A(\pi(y))$, and since both maps between isotropy groups are inclusions (by assumption on $\pi$ and $i$), theorem 2.2(v) in [4] yields that this intersection is precisely $\text{Stab}_Z(z)$.

We can wrap up the two summation in a single one by defining $\tilde{x} = gx$ to be representatives for the $\text{Stab}_Y(y)$-orbits on $i^{-1}(\pi(y))$, where the action is the restriction to $\text{Stab}_Y(y)$ of the natural action of $\text{Stab}_A(\pi(y))$:

$$
\pi^*(i_* F)(y) = \sum_{i^{-1}(\pi(y))\ast \tilde{x}} \text{cores}^\text{Stab}_Y(y)\text{res}^\text{Stab}_X(Y)F(\tilde{x})
$$

where we notice that an explicit model of the objects of $Z$ is given by pairs in $\text{Ob}(Y) \times \text{Ob}(X)$ which map to the same object in $A$ - and each choice of $(y, \tilde{x})$ in the sum satisfies this requirement.

To finish the proof of the lemma, it remains to show that the last summation coincides with

$$
\sum_{\text{Stab}_Y(y) \setminus \text{Ob}(Y)} \text{cores}^\text{Stab}_Y(y)\text{res}^\text{Stab}_X(Z)F(\tilde{x})
$$

It is immediate - again by the description of the objects of the pullback $Z$ as pairs $\text{Ob}(X) \times \text{Ob}(Y)$ mapping to the same object of $A$ - that we have $\text{Ob}(\tilde{i}^{-1}(y)) = \{(y, \tilde{x}), \lambda_1(y) = \lambda_2(\tilde{x})\}$ in bijection with $\text{Ob}(i^{-1}(\pi(y)))$ via $(y, \tilde{x}) \mapsto \tilde{x}$.

It remains to check that the two actions of $\text{Stab}_Y(y)$ correspond under this identification, as then it will follow that the orbits correspond as well: the two summations will be indexed by the same set and will coincide summand by summand. This is precisely the result of claim 32. 

We now describe a very general setup for derived Hecke algebras. Let $G$ be a topological group acting transitively on a set $X$, for instance $X = G/K$ with the action by left multiplication.

Consider the topological groupoid $(G \rightharpoonup X^2)$ where $g.(x_1, x_2) = (gx_1, gx_2)$ whose topology is defined as in example 2. We denote this groupoid by $G_X$. Similarly, let $(G \rightharpoonup X^3)$ be another topological groupoid, where $G$ acts diagonally as above. Let $i_{1,2} : (G \rightharpoonup X^3) \to G_X$ be the projection onto the first and second factors, and similarly define $i_{2,3}$ and $i_{1,3}$. Fix a base point $x_0 \in X$ and denote $H = \text{Stab}_G(x_0)$.

**Fact 33.** Suppose that the following conditions hold:

1. For each $F_1, F_2 \in \mathbb{H}^*_c(G_X)$, the cohomology class $i_{1,2}^* F_1 \cup i_{2,3}^* F_2$ is $i_{1,3}$-fiberwise compactly supported;

2. For all $x, z \in H \setminus G/H$, the subset $H x H z H$ is a finite union of $(H, H)$-cosets.

3. $i_{1,3}$ is a finite covering morphism.

Then the compactly supported cohomology $\mathbb{H}^*_c(G_X)$ admits the structure of a derived Hecke algebra under convolution, as follows: given $F_1, F_2 \in \mathbb{H}^*_c(G_X)$ we define their convolution to be the
cohomology class resulting from the following diagram

\[
\begin{array}{ccc}
G_X^{F_1} & \xrightarrow{i_{1,2}} & G_X^{F_2} \\
\downarrow i_{1,3} & & \downarrow i_{2,3} \\
(G \acts X^3) & \xrightarrow{i_{1,3}} & G_X
\end{array}
\]

where we pullback \( F_1 \) and \( F_2 \) to \((G \acts X^3)\), cup them, then pushforward along \( i_{1,3} \) to \( G_X \). We denote this algebra by \( \mathcal{H}(G, X) \).

**Remark.** Before proving the fact, we notice that if \( H \) is compact and open in \( G \) then all conditions are automatically satisfied. Indeed, \( i_{1,3} \) is always a covering, so it is a finite covering as long as the index \([G_{x,z} : G_{x,y,z}]\) is finite, which holds once \( H \) is compact open. Similarly, condition 2 follows immediately since \( H \times H \cdot z \) is compact if \( H \) is. Finally, condition 1 is implied by \( G_{x,z} \) having finitely many orbits on the set of \( y \in X \) such that \( F_1(x, y) \neq 0 \neq F_2(y, z) \): since \( F_1 \) is compactly supported, \( G_x \) has finitely many orbits on that set, so finiteness of the index \([G_x : G_{x,z}]\) implies condition 1.

**Proof.** Conditions 1 guarantees that we can pushforward along \( i_{1,3} \). To show that this pushforward is compactly supported, we notice that by linearity we can assume that \( F_1 \) is supported on the \( G \)-orbit of \((x_0, x_0)\) and \( F_2 \) on the \( G \)-orbit of \((x_0, z x_0)\) - or in other words, \( F_1 \) is supported on the connected component \( H x H \) and \( F_2 \) on \( H z H \). Then unraveling the formulas for pullback, cup product and pushforward shows that the final cohomology class is supported on \( H x H \cdot z H \): condition 2 assures that this is a finite union of \((H, H)\)-cosets - or in other words, it consists of finitely many connected components.

It remains to prove that this operation is associative and defines thus an algebra structure: notice that the unit under this operation is the cohomology class supported on the \( G \)-orbit of \((x_0, x_0)\) and taking value \( 1(x_0, x_0) = 1 \in S \cong H^0(H, S) \), the unit element of the ring \( S \).

It remains to show that this operation is associative: given \( A, B, C \in H_c^*(G_X) \) we need to check that the following two diagrams output the same cohomology class:

\[
\begin{array}{ccc}
G_X^A & \xrightarrow{i_{1,2}} & G_X^B \\
\downarrow i_{1,3} & & \downarrow i_{2,3} \\
(G \acts X^3) & \xrightarrow{i_{1,3}} & G_X^C
\end{array}
\]

\[
\begin{array}{ccc}
G_X^{A \circ B} & \xrightarrow{i_{1,2}} & G_X^C \\
\downarrow i_{1,3} & & \downarrow i_{2,3} \\
(G \acts X^3) & \xrightarrow{i_{1,3}} & G_X^{(A \circ B) \circ C}
\end{array}
\]
We start with the following fact.

**Claim 34.** The square

\[
\begin{array}{c}
(G \hookrightarrow X^3) \\
\downarrow^{i_{1,3}} \\
G_X^{A \circ (B \circ C)}
\end{array} \quad \text{is a pullback square of groupoids. Moreover, it satisfies the additional assumptions of lemma 11, that is to say: } i_{1,3} \text{ is a finite covering morphism and } i_{1,2} \text{ induces an injection of isotropy groups at all objects.}
\]

**Proof of claim.** By assumption \(i_{1,3}\) is a finite covering morphisms, and it is immediate to see that \(i_{1,2}\) induces an injection of isotropy groups at all objects.

By proposition 4.4 in [3], the pullback square of a diagram of groups acting on sets is constructed by taking the pullback of the groups acting on the pullback of the sets. It is immediate that the pullback group is again \(G\). Moreover, since \(i_{1,3}\) is a covering, the pullback and the homotopy pullback coincide, hence a model for the object set of the pullback is

\[
\{(x, y, z), (a, b, c)\} \in X^3 \times X^3 | \ i_{1,3}(x, y, z) = i_{1,2}(a, b, c) = (a, b, c) = X^4
\]

with the maps \(i_{1,3,4}\) and \(i_{1,2,3}\) as above.

By assumption on \(G \hookrightarrow X\) the cohomology class \(F = i_{1,2}^* A \cup i_{2,3}^* B\) is \(i_{1,3}\)-fiberwise compactly supported, thus lemma 11 applies and we can use the above pullback squares to modify the two original diagrams without impacting the final result in cohomology. The first diagram becomes
A biproduct of lemma 11 was that \( F_1 = i_{1,2,3}^* F \) is \( i_{1,3,4} \)-fiberwise compactly supported, thus we can apply lemma 10 to the subdiagram

\[
\begin{array}{c}
(G \leadsto X^4) \xrightarrow{i_{1,4}} G_X \\
\downarrow \quad \downarrow i_{2,3} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Then the derived Hecke algebra $\mathcal{H}(G, X)$ acts on the compactly supported cohomology $\mathbb{H}_c^*(G \rhd X \times Y)$, as follows: given $F \in \mathcal{H}(G, X)$ and $C \in \mathbb{H}_c^*(G \rhd X \times Y)$, we define their convolution to be the cohomology class resulting from the following diagram

$$
\begin{array}{c}
\xymatrix{
G_X^F \ar[rr]_{i_{1,2}} \ar[dr]^{i_{1,3}} & & (G \rhd X^2 \times Y) \ar[dl]^{i_{2,3}} \ar[r] & (G \rhd X \times Y)^C \\
(G \rhd X \times Y) & & & (G \rhd X \times Y) \\
(G \rhd X \times Y) & & & (G \rhd X \times Y)
}\end{array}
$$

where we pullback $F$ and $C$ to $(G \rhd X^2 \times Y)$, cup them, then pushforward along $i_{1,3}$ to $(G \rhd X \times Y)$.

**Remark.** Before proving the fact, we notice that if $H$ is compact and open in $G$ and $L$ is closed in $G$ then all conditions are automatically satisfied. Indeed, conditions 2 follows from $H$ being compact open, as then $H \times H$ is a finite union of left $H$-cosets. Since $i_{1,3}$ is always a covering, we notice that it is a finite covering as long as the index $[G_{x,z} : G_{x,y,z}]$ is finite for all $x, z \in X$ and $y \in Y$. This holds once $H$ is compact open and $L$ is closed because $G_{x,y}$ is compact and intersecting with the open $G_z$ yields that $G_{x,y,z}$ is open in $G_{x,y}$, thus finite-index. Finally, condition 1 is implied by $G_{x,y}$ having finitely many orbits on the set of $z \in X$ such that $F(x, z) \neq 0 \neq C(z, y)$: assuming $F$ is supported on a single connected component $HzH$, and that (by $G$-invariance) $x = x_0$, it suffices to show that the set of such $z$’s is finite: this set is in bijection with $HzH/H$ which is finite as $H$ is open and compact.

**Proof.** Conditions 1 guarantees that we can pushforward along $i_{1,3}$. To show that this pushforward is compactly supported, we notice that by linearity we can assume that $F$ is supported on the $G$-orbit of $(x_0, xx_0)$ and $C$ on the $G$-orbit of $(x_0, zy_0)$ - or in other words, $F$ is supported on the connected component $H \times H$ and $C$ on $HzL$. Then unraveling the formulas for pullback, cup product and pushforward shows that the final cohomology class is supported on $H \times HzL$: condition 2 assures that this is a finite union of $(H, L)$-cosets - or in other words, it consists of finitely many connected components.

It remains to prove that this defines indeed a left-action, in the sense that given $F_1, F_2 \in \mathcal{H}(G, X)$ and $C \in \mathbb{H}_c^*(G \rhd X \times Y)$, we have $F_1 \cdot (F_2, C) = (F_1 \circ F_2) \cdot C$. In other words, we need to check that the following two diagrams output the same cohomology class:

$$
\begin{array}{c}
\xymatrix{
G_X^{F_1} \ar[r]_{i_{1,2}} & (G \rhd X^3) \ar[r]_{i_{2,3}} & (G \rhd X^2 \times Y) \ar[r]_{i_{2,3}} & (G \rhd X \times Y)^C \\
G_X^{F_1 \circ F_2} \ar[rr]_{i_{1,2}} \ar[dr]^{i_{1,3}} & & (G \rhd X \times Y)^{(F_1 \circ F_2), C} \\
(G \rhd X \times Y) & & & (G \rhd X \times Y)
}\end{array}
$$

50
We start with the following fact.

**Claim 36.** The square

\[
\begin{array}{c}
(G \rightsquigarrow X^2 \times Y) \\
i_{1,2} \\
(G \rightsquigarrow X \times Y)^C
\end{array}
\]

is a pullback square of groupoids. Moreover, it satisfies the additional assumptions of lemma 11, that is to say: \(i_{1,3}\) is a finite covering morphism and \(i_{1,2}\) induces an injection of isotropy groups at all objects.

**Proof of claim.** By assumption on \(G \rightsquigarrow X\), \(i_{1,3}\) is a finite covering morphisms, and moreover it is immediate that \(i_{1,2}\) induces an injection of isotropy groups at all objects.

By proposition 4.4 in [3], the pullback square of a diagram of groups acting on sets is constructed by taking the pullback of the groups acting on the pullback of the sets. It is immediate that the pullback group is again \(G\). Moreover, since \(i_{1,3}\) is a covering, the pullback and the homotopy pullback coincide, hence a model for the object set of the pullback is

\[
\left\{ ((x, y, z), (a, b, c)) \in X^3 \times (X^2 \times Y) \mid i_{1,3}(x, y, z) = i_{1,2}(a, b, c) \right\} =
\]

\[
\left\{ ((x, y, z), (a, b, c)) \in X^3 \times (X^2 \times Y) \mid (x, z) = (a, b) \right\} = \{ (x = a, y, z = b, c) \} = X^3 \times Y
\]

with the maps \(i_{1,3,4}\) and \(i_{1,2,3}\) as above.

By assumption on \(G \rightsquigarrow X\), the cohomology class \(F = i_{1,2}^*F_1 \cup i_{2,3}^*F_2\) is \(i_{1,3}\)-fiberwise compactly supported, thus lemma 11 applies and we can use the above pullback squares to modify the two original diagrams without impacting the final result in cohomology. The first diagram becomes
A biproduct of lemma 11 was that $\tilde{F} = i_{1,2,3}^*F$ is $i_{1,3,4}$-fiberwise compactly supported, thus we can apply lemma 10 to the subdiagram

$$(G \ract X^3 \times Y) \xrightarrow{\; i_{1,2,3} \;} (G \ract X \times Y) \xrightarrow{\; i_{1,3,4} \;} (G \ract X^2 \times Y)$$

to obtain that for each $\tilde{F} \in \mathbb{H}^*_c ((G \ract X^3 \times Y))$ that is $i_{1,3,4}$-fiberwise compactly supported and each $C \in \mathbb{H}^*_c ((G \ract X \times Y))$, we have

$$(i_{1,3,4})_* \tilde{F} \cup i_{2,3}^*C = (i_{1,3,4})_* \left( \tilde{F} \cup i_{3,4}^*C \right)$$

and hence we can replace the first diagram with

$$
\begin{array}{c}
\xymatrix{ 
G_X^{F_1} 
\ar[rr]^{i_{1,2}} \ar[d]_{\; i_{1,2,3} \;} & & G_X^{F_2} 
\ar[rr]^{i_{3,4}} \ar[d]_{\; i_{1,3,4} \;} & & (G \ract X \times Y)^C 
\ar[d]_{\; i_{1,3} \;} & \\
(G \ract X^3) & & (G \ract X^3 \times Y) & & (G \ract X^2 \times Y) & & (G \ract X \times Y)^{(F_1 \circ F_2)_*} 
\end{array}
\end{array}
$$

which is equivalent to

$$
\begin{array}{c}
\xymatrix{ 
G_X^{F_1} 
\ar[d]_{\; i_{1,2} \;} & & G_X^{F_2} 
\ar[d]_{\; i_{3,4} \;} & & (G \ract X \times Y)^C 
\ar[d]_{\; i_{1,4} \;} & \\
(G \ract X^3 \times Y) & & (G \ract X \times Y)^{(F_1 \circ F_2)_*} 
\end{array}
\end{array}
$$

This last diagram is obviously symmetric in $F_1, F_2, C$, which proves the claim. □

Obviously the compactly supported cohomology $\mathbb{H}^*_c ((G \ract X \times Y))$ also admits a right action of the derived Hecke algebra $\mathcal{H}(G, Y)$ in a completely analogous way, as soon as the action $G \ract Y$ satisfies the conditions of fact 33.

**Fact 37.** Suppose the actions $G \ract X$ and $G \ract Y$ both satisfy the conditions of fact 33. Then the compactly supported cohomology $\mathbb{H}^*_c ((G \ract X \times Y))$ is a $(\mathcal{H}(G, X), \mathcal{H}(G, Y))$-bimodule.

**Proof.** We already described how the two actions are defined, so it remains to show that they commute, i.e. that given $F_1 \in \mathcal{H}(G, X)$, $F_2 \in \mathcal{H}(G, Y)$ and $C \in \mathbb{H}^*_c ((G \ract X \times Y))$ we have $(F_1 \cup C).F_2 = F_1.(C.F_2)$. This boils down to the following two diagrams outputting the same cohomology class:
We start with the following fact.

**Claim 38.** The square

$$
\begin{align*}
& (G \acts X^2 \times Y) & \xrightarrow{i_{1,3}} & (G \acts X \times Y)^{F_1, C} & \xrightarrow{i_{1,2}} & (G \acts X \times Y)^{F_1, C, F_2} \\
& \downarrow & & \downarrow & & \downarrow \\
& (G \acts X \times Y) & \xrightarrow{i_{2,3}} & (G \acts X \times Y^2) & \xrightarrow{i_{1,3}} & (G \acts X \times Y)(^{(F_1, C), F_2})
\end{align*}
$$

is a pullback square of groupoids. Moreover, it satisfies the additional assumptions of lemma 11, that is to say: $i_{1,3}$ is a finite covering morphism and $i_{1,2}$ induces an injection of isotropy groups at all objects.

**Proof of claim.** By assumption on the actions $G \acts X$ and $G \acts Y$, $i_{1,3}$ is a finite covering morphisms, and it is immediate that $i_{1,2}$ induces an injection of isotropy groups at all objects.

By proposition 4.4 in [3], the pullback square of a diagram of groups acting on sets is constructed by taking the pullback of the groups acting on the pullback of the sets. It is immediate that the pullback group is again $G$. Moreover, since $i_{1,3}$ is a covering, the pullback and the homotopy pullback coincide, hence a model for the object set of the pullback is

$$
\begin{align*}
\{(x, y, z), (a, b, c)\} &\in (X^2 \times Y) \times (X \times Y^2) | i_{1,3}(x, y, z) = i_{1,2}(a, b, c) = \\
\{(x, y, z), (a, b, c)\} &\in (X^2 \times Y) \times (X \times Y^2) | (x, z) = (a, b) = \{(x, x, y, z = b, c)\} = X^2 \times Y^2
\end{align*}
$$

with the maps $i_{1,3,4}$ and $i_{1,2,3}$ as above. \qed
By assumption on \( G \sim X \) and \( G \sim Y \), the cohomology class \( F = i_{1,2}^* F_1 \cup i_{2,3}^* C \) is \( i_{1,3} \)-fiberwise compactly supported, thus lemma 11 applies and we can use the above pullback squares to modify the two original diagrams without impacting the final result in cohomology. The first diagram becomes

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
G^F_X \ar[r]^{i_{1,2}} & (G \sim X \times Y)^C \\
(G \sim X^2 \times Y) \ar[r]^{i_{1,2,3}} & (G \sim X^2 \times Y^2) \ar[u]_{i_{2,3}}
}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
(G \sim X \times Y^2) \ar[u]_{i_{1,3}} \\
(G \sim X \times Y)^{(F_1 \cup C) \cdot F_2}
}
\end{array}
\end{array}
\]

A biproduct of lemma 11 was that \( \tilde{F} = i_{1,2,3}^* F \) is \( i_{1,3,4} \)-fiberwise compactly supported, thus we can apply lemma 10 to the subdiagram

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
(G \sim X^2 \times Y^2) \ar[r]^{i_{3,4}} & G_Y \\
(G \sim X \times Y^2) \ar[u]_{i_{2,3}} \\
(G \sim X \times Y)^{(F_1 \cup C) \cdot F_2}
}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
\ar[r]^{i_{1,3,4}} & (G \sim X \times Y)^{(F_1 \cup C) \cdot F_2}
}
\end{array}
\end{array}
\]

to obtain that for each \( \tilde{F} \in \mathbb{H}^* ((G \sim X^3 \times Y)) \) that is \( i_{1,3,4} \)-fiberwise compactly supported and each \( F_2 \in \mathbb{H}^* (G_Y) \), we have

\[
(i_{1,3,4})_* \tilde{F} \cup i_{2,3}^* F_2 = (i_{1,3,4})_* \left( \tilde{F} \cup i_{3,4}^* F_2 \right)
\]

and hence we can replace the first diagram with

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
G^F_X \ar[r]^{i_{1,2}} & (G \sim X \times Y)^C \\
(G \sim X^2 \times Y) \ar[r]^{i_{1,2,3}} & (G \sim X^2 \times Y^2) \ar[u]_{i_{2,3}}
}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
(G \sim X \times Y^2) \ar[u]_{i_{1,3}} \\
(G \sim X \times Y)^{(F_1 \cup C) \cdot F_2}
}
\end{array}
\end{array}
\]

which is equivalent to

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
G^F_X \ar[r]^{i_{1,2}} & (G \sim X \times Y)^C \\
(G \sim X^2 \times Y^2) \ar[u]_{i_{1,4}} \\
(G \sim X \times Y)^{(F_1 \cup C) \cdot F_2}
}
\end{array}
\end{array}
\]
B Various definitions of the derived Hecke algebra

In this section we discuss the relationship between our definition of the derived Hecke algebra as $G$-equivariant cohomology classes and the following "categorical" definition.

Let $G$ be a split, connected reductive group over a local, non-archimedean field $F$. Denote $G = G(F)$ and let $K$ be an open compact subgroup of $G$. Let $S$ be a coefficient ring, and denote by $1$ the trivial representation of $K$ with $S$-coefficients, so that $S[G/K] = \iota_K^G 1$ is the compactly induced representation to $G$.

The classical Hecke algebra can be defined as

$H(G, K) = \text{Hom}_{S[G]}(S[G/K], S[G/K])$, the endomorphism of $S[G/K]$ as a $G$-module. In [26], Venkatesh defines the derived Hecke algebra as a graded algebra over the classical one by replacing $\text{Hom}$ with its derived functor $\text{Ext}$:

$$\mathcal{H}(G, K) = \bigoplus_{n \geq 0} \text{Ext}^n_{S[G]}(S[G/K], S[G/K]).$$

When the residue characteristic $p$ is not invertible in the coefficient ring $S$ (for instance $S = \mathbb{Z}/p^n\mathbb{Z}$), we can also follow Schneider’s construction from [22]: let $S[G/K] \rightarrow I$ be an injective resolution and consider the differential graded algebra $\text{Hom}_{S[G]}(I, I)^{\text{op}}$. Its cohomology algebra is

$$\mathbb{H}^* \left( \text{Hom}_{S[G]}(I, I)^{\text{op}} \right) \cong \text{Ext}^*_{S[G]}(S[G/K], S[G/K])$$

so it coincides with the definition in formula 12 above.

Since remark 7 in [22] holds for any compact open $K \subset G$, it shows that for any injective resolution $S[G/K] \rightarrow I$ one has

$$\mathbb{H}^* \left( \text{Hom}_{S[G]}(I, I)^{\text{op}} \right) \cong H^* \left( K, S[G/K] \right).$$

We can use Mackey theory to decompose the $K$-module $S[G/K]$ as $\bigoplus_{g \in K \backslash G/K} \iota_K^K S$. Then section 2.6 in [25] shows that

$$H^n \left( K, \bigoplus_{g \in K \backslash G/K} \iota_K^K g, 1 \right) = \text{Ext}^n_{S[K]} \left( 1, \bigoplus_{g \in K \backslash G/K} \iota_K^K g, 1 \right) \cong$$

$$\cong \bigoplus_{g \in K \backslash G/K} \text{Ext}^n_{S[K]} \left( 1, \iota_K^K g, 1 \right) = \bigoplus_{g \in K \backslash G/K} H^n \left( K, \iota_K^K g, 1 \right) \cong \bigoplus_{g \in K \backslash G/K} H^n \left( K \cap gKg^{-1}, 1 \right)$$

where the last isomorphism holds by Shapiro’s lemma. In particular, this is exactly the double coset description given in 2.4 of [26]. Thus, the two possible definitions of a derived Hecke algebra ($G$-equivariant cohomology classes versus cohomology of the complex $\text{Hom}_{D(G)}(S[G/K], S[G/K])$) coincide as $S$-modules, and it remains to check that the two multiplication operations agrees. For the time being we take as definition of the derived Hecke algebra the one as $G$-equivariant cohomology classes given in section 4, which is much more explicit and therefore easier to work with, and we leave it as conjecture that the two coincide.
Conjecture 39. The cohomology algebra $\mathbb{H}^* (\text{Hom}_S[G](I,I)^{op})$ and the algebra of $G$-equivariant cohomology classes $\mathcal{H}_S(G,K)$ as defined in section 4 are isomorphic as graded algebras via their common double cosets description. More precisely, the isomorphism of $S$-modules given by

$$\mathcal{H}_S(G,K) \longrightarrow \bigoplus_{g \in K\backslash G/K} H^n(G \cap gKg^{-1},1) \quad F \mapsto (F(K,gK))_{g \in K\backslash G/K}$$

is in fact an algebra isomorphism, where the right hand side is canonically identified with the cohomology algebra $\mathbb{H}^* (\text{Hom}_S[G](I,I)^{op})$ as explained above.

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