1 April 1

Definition 1 (Topological group). A topological space $G$ with two continuous maps

$$m : G \times G \to G \quad i : G \to G$$

satisfying the usual axioms for a group, is said to be a topological group.

Example 1. $G = \text{GL}_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. The inverse is given by Cramer formula, so it’s clearly a continuous mapping.

Example 2. If $G$ and $G'$ are topological groups, so is $G \times G'$ with the product topology. An important example is

$$\mathbb{C}^\times \cong \mathbb{R}^\times_0 \times S^1$$

via the correspondence $z = re^{i\theta} = (r, \theta)$.

Example 3. If $H \subset G$ is a subgroup, then it is a topological group with the subspace topology. We will mainly be interested in closed subgroups.

Example 4. Let $G = \text{GL}_n(\mathbb{R})$. Then $H = \text{SL}_n(\mathbb{R}) = \{\det = 1\}$ is closed but not compact.

We are in particular interested in subgroups preserving bilinear forms.

Example 5. Let $(,)$ be the standard inner product in $\mathbb{R}^n$. Then we define the orthogonal group

$$\text{O}_n(\mathbb{R}) = \{g \in \text{GL}_n(\mathbb{R}) \mid (gv, gw) = (v, w) \forall v, w \in \mathbb{R}^n\} = \{g \in \text{GL}_n(\mathbb{R}) \mid {}^tgg = \text{id}\}.$$ 

This is given by a huge system of polynomial equations in the coefficients of $g$, and the condition $^tgg = \text{id}$ makes it clear that if $g \in \text{O}_n(\mathbb{R})$ then $\det g = \pm 1$.

Hence

$$\text{SO}_n(\mathbb{R}) = \text{O}_n(\mathbb{R}) \cap \text{SL}_n(\mathbb{R})$$

is an open subgroup (of index 2) and

$$\text{O}_n(\mathbb{R}) = \text{SO}_n(\mathbb{R}) \times \mathbb{Z}/2.$$ 

Note that $\text{O}_n(\mathbb{R})$ is compact! In fact it is clearly closed in $M_n(\mathbb{R})$ by the condition $^tgg = \text{id}$ and it is bounded, because if $g \in \text{O}_n(\mathbb{R})$, $e_i$ a basis element, $g(e_i)$ still is a unit vector because $g$ preserves the inner product, hence preserves the length.
Example 6. Fix a finite-dimensional real vector space $V$. We have a correspondence

$\{ \text{symmetric bilinear forms } V \times V \rightarrow \mathbb{R} \} \leftrightarrow \{ \text{quadratic forms } q : V \rightarrow \mathbb{R}, q(x) = \sum_{i,j} a_{i,j} x_i x_j \}$

via the mappings

$B \mapsto q_B$ where $q_B(v) = B(v, v)$

and

$q \mapsto B_q$ where $B_q(v, w) = \frac{1}{2} (q(v + w) - q(v) - q(w))$.

So equivalently we can define $O_n(\mathbb{R})$ as the group of $g \in \text{GL}_n(\mathbb{R})$ preserving the usual length, that is $|gv|^2 = |v|^2$ for all $v \in \mathbb{R}^n$.

But we can also substitute $q$ with another nondegenerate quadratic form on $\mathbb{R}^n$: this is uniquely determined (up to isomorphism) by its signature, so if

$q(x) = x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_n^2$

we define the orthogonal groups

$O_n(q) = O(r, n-r) = \{ g \in \text{GL}_n(\mathbb{R}) | q(gv) = q(v) \forall v \in \mathbb{R}^n \}$

Remark. When $q$ is not definite ($0 < r < n$) then $O(r, n-r)$ is not compact!

Now we may want to ask ourselves which of these groups are connected. This is sometimes not easy to determine, but we have the following results.

Proposition 1. $\text{GL}_n(\mathbb{R})$ and $O_n(\mathbb{R})$ are not connected. But $\text{SL}_n(\mathbb{R})$ is connected, hence also $\text{SO}_n(\mathbb{R})$ and $\text{GL}_n^+(\mathbb{R}) = \mathbb{R}_0^+ \times \text{SL}_n(\mathbb{R})$ are connected. Notice, however, that $\text{GL}_n(\mathbb{C})$ is connected. For any $0 < r < n$, $\text{SO}(r, n-r)$ is not connected.

Example 7. Let $G^0$ be the connected component of the identity in any topological group $G$. This is always a closed connected subgroup!

We now want to treat stuff in a slightly more general way, without using coordinates when possible.

Proposition 2. Let $A$ be any finite-dimensional, associative, $\mathbb{R}$-algebra. Then the invertible elements $A^\times$ are an open subset which form a topological group.

Proof. Given $a \in A$, consider the left multiplication map

$l_a : A \rightarrow A \quad x \mapsto ax$.

Then if $aa' = 1$ we have $l_a \cdot l_a' = \text{id}$. This shows that

$A^\times = \{ a \in A \text{ having a 2-sided inverse} \} = \{ a \in A | \det(l_a) \neq 0 \}$.

Now $\det(l_a)$ is an open condition given by polynomials (in some $\mathbb{R}$-basis of $A$). So we embedded

$A \rightarrow \text{End}_\mathbb{R}(A) \quad a \mapsto l_a$

and showed that under this embedding

$A^\times = A \cap (\text{End}_\mathbb{R}(A))^\times$ is an open set in $\text{End}_\mathbb{R}(A)$.

□
1.1 Quotients

Consider the following setup. Let $G$ be a topological group, $H \triangleleft G$ a subgroups with the subspace topology. Give $G/H = \{gH\}$ the quotient topology inherited from $G$ via the quotient map

$$G \xrightarrow{p} G/H.$$ 

Thus $U \subset G/H$ is open if and only if $p^{-1}(U) \subset G$ is open. As for any open set $V \subset G$ we have

$$p^{-1}(p(V)) = \bigcup_{h \in H} Vh$$

we get a bijection

$$\{\text{open sets in } G/H\} \leftrightarrow \{\text{open sets in } G \text{ that are invariant by } H\text{-multiplication on the right}\}.$$ 

Exercise 1. 1. Check that if $H' \subset G'$ is another such setup, then

$$(G \times G')/(H \times H') \longrightarrow G/H \times G'/H'$$

is a homeomorphism.

2. If $H \triangleleft G$ is a normal subgroup, check that $G/H$ with the quotient topology is a topological group.

Example 8. We have $\mathbb{R}/\mathbb{Z} \cong S^1$ the circle and more generally $\mathbb{R}^n/\mathbb{Z}^n \cong T^n$ the $n$-torus.

Consider now the following situation: let $G$ be a top. group acting continuously on a top. space $X$ (for example $SO_n(\mathbb{R})$ acting on $S^{n-1} \subset \mathbb{R}^n$). Fix $x_0 \in X$, we have the *stabilizer*

$$\text{Stab}_X(x_0) = G_{x_0} = \{g \in G \mid g(x_0) = x_0\}.$$

Then the orbit map

$$G \longrightarrow X \quad g \mapsto g(x_0)$$

is continuous and invariant with respect to right multiplication by $G_{x_0}$, so we get a continuous injection (of top. spaces)

$$G/G_{x_0} \hookrightarrow X$$

onto the orbit of $x_0$.

Remark. In general, this injection is not an homeomorphism onto its image! But in nice cases it is.

Example 9. Let $G = SO_n(\mathbb{R})$ acting on $S^{n-1} \subset \mathbb{R}^n$. Let $x_0 = (1,0,\ldots,0) \in S^{n-1}$, so we have

$$G_{x_0} \cong SO_{n-1}(\mathbb{R}) \subset G$$

and as the action of $SO_n$ is transitive, we obtain a continuous bijection

$$SO_n(\mathbb{R})/SO_{n-1}(\mathbb{R}) \longrightarrow S^{n-1}.$$ 

Now it is a known fact from topology that a continuous bijection between two compact topological spaces is an homeomorphism if the target space is Hausdorff. This is the case for $S^{n-1}$, so the map above is in fact an homeomorphism.
Remark. The above homeomorphism gives a quick proof (by induction on \(n\)) on the connectedness of \(\text{SO}_n(\mathbb{R})\).

Now we consider a more interesting example. Define the quaternions
\[
\mathbb{H} = \left\{ \left( \frac{a}{b} \right) \in M_2(\mathbb{C}) \mid a, b \in \mathbb{C} \right\} = \mathbb{R}\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \oplus \mathbb{R}\left( \begin{array}{cc} i & 0 \\ 0 & i \end{array} \right) \oplus \mathbb{R}\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \oplus \mathbb{R}\left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right)
\]
a 4-dimensional algebra over \(\mathbb{R}\). Denote moreover
\[
i = \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right), j = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), k = \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right)
\]
so we have the usual identities
\[
i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j.
\]

Notice that moreover
\[
\det g = \det \left( \frac{a}{b} \right) = |a|^2 + |b|^2 \neq 0 \text{ if } g \neq 0.
\]

Exercise 2. Check that every \(g \in \mathbb{H} - \{0\}\) is invertible inside \(\mathbb{H}\).

\(\mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k\) are called the pure quaternions, giving some analogy with the pure imaginary complex numbers.

We have in fact a conjugation
\[
h = a + bi + cj + dk \mapsto \overline{h} = a - bi - cj - dk
\]
such that \(h\overline{h} = \overline{h}h = \det h\).

Example 10. Let \((,)\) be the standard hermitian form on \(\mathbb{C}^n\), conjugate linear on the right slot. Then we can define the unitary group
\[
U_n(\mathbb{R}) = \{ g \in \text{GL}_n(\mathbb{C}) \mid (gv, gw) = (v, w) \forall v, w \in \mathbb{C}^n \} = \{ \overline{g}g = \text{id} \}.
\]
This is compact.

Definition 2 (Lie Group). A Lie group is a topological group with a \(C^\infty\)-manifold structure such that multiplication and inversion are smooth maps.

2 April 3

Let’s tie up a loose end from last class. Consider a sesquilinear form on a finite-dimensional complex vector space \(V\), i.e. a bilinear form
\[
h : V \times V \to \mathbb{C} \text{ with } h(v, w) = \overline{h(w, v)}, h(v, v) \geq 0 \forall v, w \in V
\]
and such that \(h\) is linear in the first slot and conjugate linear in the second.

We moreover want \(h\) to be nondegenerate, which means that the induced conjugate linear map
\[
V \to V^* \quad w \mapsto h(-, w)
\]
should be an isomorphism.

We can then define the unitary group preserving \(h\):
\[
U_n(h) = \{ g \in \text{GL}(V) \mid h(gv, gw) = h(v, w) \forall v, w \in V \}.
\]
Claim 3. Preserving \( h \) is equivalent to preserving the \( h \)-length. That is, \( g \in U_n(h) \) if and only if \( h(gv, gv) = h(v, v) \) for every \( v \in V \).

Proof. One direction is obvious. For the other, prove that for all \( c \in \mathbb{C} \) one has
\[
\Re (ch(gv, gw)) = \Re (ch(v, w))
\]
which then implies \( h(gv, gw) = h(v, w) \).

We can now define the special unitary group
\[
SU_n(\mathbb{C}) = \ker \left( U_n(\mathbb{C}) \xrightarrow{\text{det}} S^1 \right) = \{ g \in U_n | \text{det } g = 1 \}.
\]
This is, in some sense, a compact cousin of \( \text{SL}_n(\mathbb{R}) \), just like \( \text{SO}_n(\mathbb{R}) \) is a compact cousin of \( \text{SO}(r, n - r) \) for \( 0 < r < n \).

Let us now try to find a correspondent group in the quaternionic setting.

Definition 3. Let \( V \) be a finite-dimensional vector space over some field \( k \). A symplectic form is a nondegenerate alternating bilinear form \( \psi : V \times V \rightarrow k \), where alternating means \( \psi(v, v) = 0 \). If \( \text{char } k \neq 2 \), we have \( \psi(v, w) = -\psi(w, v) \) for all \( v, w \in V \).

A useful fact from linear algebra is that once \( \psi \) is a symplectic form on \( V \), we must have \( \dim V = 2n \) is even, and there exists a symplectic basis
\[
\{ e_1, \ldots, e_{2n} \}
\]
for which the matrix of \( \psi \) is
\[
\begin{pmatrix}
0 & 1_n \\
-1_n & 0
\end{pmatrix}
\]
where \( 1_n \) stands for the identity \( n \)-by-\( n \) matrix.

Furthermore this means that \( \{ e_1, \ldots, e_n \} \) and \( \{ e_{n+1}, \ldots, e_{2n} \} \) span maximal isotropic subspaces and in fact
\[
\psi(e_i, e_{n+j}) = \delta_{i,j}, \forall 1 \leq i, j \leq n.
\]
We call the span of \( \{ e_i, e_{n+i} \} \) a hyperbolic plane.

Remark. Ultimately, linear algebra proves that up to isomorphism there is only one symplectic form on every even-dimensional vector space. So sometimes we just forget \( \psi \) and talk about the dimension of \( V \).

Definition 4. The symplectic group of a symplectic form \( \psi \) is
\[
\text{Sp}(\psi) = \text{Sp}_{2n}(k) = \{ g \in \text{GL}(V) | \psi(gv, gw) = \psi(v, w) \forall v, w \in V \}.
\]

Once we pick a symplectic basis, we find that
\[
\text{Sp}_{2n}(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | t \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}
\]
which gives us necessary and sufficient conditions for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) belonging to \( \text{Sp}_{2n} \) in terms of the matrices \( a, b, c, d \in M_n(k) \).
Exercise 3. Write explicitly these conditions.

Example 11. For \( n = 1 \), we get \( \text{Sp}(1) = \text{SL}_2 \)

Remark. Homework 2 shows that \( \text{SL}_n(\mathbb{R}) \) and \( \text{Sp}_{2n}(\mathbb{R}) \) are connected.

To define a compact variant of \( \text{Sp}_{2n}(\mathbb{R}) \) we use the quaternions.

Recall that \( \mathbb{H}^* = \mathbb{H} - \{0\} \) and that \( \mathbb{H} \) is a division algebra over the reals.

Fact 4. The only finite-dimensional division algebras over the reals (that is, \( \mathbb{R} \) commutes with everything) are \( \mathbb{R}, \mathbb{C} \) and \( \mathbb{H} \).

Proposition 5. The quaternionic conjugation swaps under commutation, that is

\[
\overline{h_1 \cdot h_2} = \overline{h_2} \cdot \overline{h_1} \quad \forall h_1, h_2 \in \mathbb{H}.
\]

Proof. It is an \( \mathbb{R} \)-bilinear identity, so just check it for the basis elements \( \{1, i, j, k\} \).

We had the norm

\[
N(h) = h\overline{h} = \overline{h}h \in \mathbb{R}, \quad h = a + bi + cj + dk \mapsto a^2 + b^2 + c^2 + d^2 \in \mathbb{R}.
\]

For computational purposes, it is useful to consider

\[
\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j \hookrightarrow M_2(\mathbb{C})
\]

where \( \mathbb{C} \) embeds in \( \mathbb{H} \) as

\[
\mathbb{C} \ni z = a + bi \mapsto a + bi = \begin{pmatrix} a + bi & 0 \\ 0 & a - bi \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}.
\]

Now consider \( \mathbb{H}^n \) as a left \( \mathbb{H} \)-module, and define \( M_n(\mathbb{H}) = \text{End}_{\mathbb{H}}(\mathbb{H}^n) \). What are the endomorphisms of \( \mathbb{H}^n \) as a left \( \mathbb{H} \)-module? Let \( T \in \text{End}_{\mathbb{H}}(\mathbb{H}^n) \), it is \( H \)-linear and we have

\[
T : e_j \mapsto \sum_i h_{i,j} e_i \quad \text{hence} \quad T \left( \sum_j c_j e_j \right) = \sum_i \left( \sum_j c_j h_{i,j} \right) e_i.
\]

Now if \( [T'] = (h'_{i,j}) \) is another \( H \)-linear map, we have

\[
[T' \circ T] = \left( \sum_k h_{k,j} h'_{i,k} \right)
\]

so the operator composition (seen in terms of product of matrices) has subscripts swapped with respect to the usual matrix multiplication.

We can then define

\[
\text{GL}_n(\mathbb{H}) = \{ \text{invertible elements in } M_n(\mathbb{H}) \}
\]

and this is an open subset, as we can check invertibility on the underlying \( R \)-linear level.

The key point is that as \( \mathbb{H} = \mathbb{C} \oplus \mathbb{C}j \), we also have \( \mathbb{H}^n = \mathbb{C}^n \oplus \mathbb{C}^n j \cong \mathbb{C}^{2n} \) so an \( \mathbb{H} \)-linear map is a special kind of \( \mathbb{C} \)-linear endomorphism of \( \mathbb{C}^{2n} \), that is, an element \( T \) of \( M_{2n}(\mathbb{C}) \) which also commutes with left multiplication by \( j \) on \( \mathbb{H}^n \cong \mathbb{C}^{2n} \). Notice though that left multiplication by \( j \) is NOT a \( \mathbb{C} \)-linear map on \( \mathbb{C}^{2n} \). We obtain

\[
M_n(\mathbb{H}) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in M_{2n}(\mathbb{C}) \right\}
\]

and for \( n = 1 \) this is just the usual embedding of \( \mathbb{H} \) in \( M_2(\mathbb{C}) \).
**Definition 5** (Quaternionic-hermitian form). Define

\[ <\cdot,\cdot>: \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H} \quad \langle v, w \rangle = \sum_r v_r w_r \]

which has the properties

- \( \langle v, v \rangle \geq 0 \) and is 0 if and only if \( v = 0 \).
- \( \langle cv, w \rangle = c \langle v, w \rangle \) and \( \langle v, cw \rangle = \langle v, w \rangle c \) for all \( v, w \in \mathbb{H}^n \) and all \( c \in \mathbb{C} \).
- \( \langle v, w \rangle = \overline{\langle w, v \rangle} \) (this is quaternionic conjugation!).

It is then natural to define the symplectic group over the quaternions as

\[ \text{Sp}(n) = \{ g \in \text{GL}_n(\mathbb{H}) | \langle gv, gw \rangle = \langle v, w \rangle \quad \forall v, w \in \mathbb{H}^n \}. \]

Again, the condition of preserving the form \( <\cdot,\cdot> \) is equivalent to preserving the quaternionic length \( |v|_H = \langle v, v \rangle^{1/2} \).

**Remark.** \( \text{Sp}(n) \) is compact, because it is bounded and closed with a completely analogue argument to compactness of \( \text{O}_n(\mathbb{R}) \). Note that the condition \( \langle gv, gw \rangle = \langle v, w \rangle \) forces \( g \) to be invertible, so in fact \( \text{Sp}(n) \) is closed in \( M_n(\mathbb{H}) \).

**Corollary 6** (1.12 in the book). Once we embed \( \text{GL}_n(\mathbb{H}) \hookrightarrow \text{GL}_{2n}(\mathbb{C}) \) we have

\[ \text{Sp}(n) = \mathbb{U}_{2n}(\mathbb{R}) \cap \text{Sp}_{2n}(\mathbb{C}). \]

### 3 April 5

**Definition 6** (\( C^\infty \)-manifold). A \( C^\infty \)-manifold \( M \) is a paracompact and Hausdorff smooth manifold over a finite-dimensional \( \mathbb{R} \)-algebra.

**Remark.** In particular this means that there exist \( C^\infty \)-partitions of unity, so we can integrate differential forms on the manifold \( M \).

We will denote from now on

\[ \text{O}(n) = \text{O}_n(\mathbb{R}), \quad \text{U}(n) = \text{U}_n(\mathbb{R}) \]

and similarly for the special groups \( \text{SO}(n) = \text{SO}_n(\mathbb{R}) \) and \( \text{SU}(n) = \text{SU}_n(\mathbb{R}) \).

**Remark.** Consider a nondegenerate hermitian form \( h: V \times V \to \mathbb{C} \) on a complex vector space \( V \) of dimension \( n > 0 \). We have two different isomorphisms

- the conjugate linear map \( V \to V^* \quad v \mapsto h(-,v) \)

and

- the linear map \( V \to \overline{V}^* \quad v \mapsto h(v,-) \).

**Example 12.** A significative example of nondegenerate hermitian form is the following. Let \( V = \mathbb{C}^n, n = r + s \) for \( r, s > 0 \). Define

\[ h_{r,s}(z, w) = \sum_{j=1}^r z_j \overline{w}_j - \sum_{j=r+1}^n z_j \overline{w}_j. \]
Proposition 7. Every pair \((V, h)\) where \(V\) is a finite-dimensional complex vector space and \(h\) a nondegenerate hermitian form on \(V\) is isomorphic to exactly one \((\mathbb{C}^n, h_{r,s})\).

Proof. Take \(n = \dim V\). Using Gram-Schmidt orthonormalization one can get an isomorphism which keeps tracks of the signs of \(h(e_k, e_k)\) and this gives an orthonormal basis \(\{e_j\}\), then \(r\) is just the number of positive \(h(e_k, e_k)\). \(\square\)

Remark. Once we identify \(\mathbb{R}^{2n} \cong \mathbb{C}^n\), the form \(h_{r,s}\) becomes a symmetric form with signature \((2r, 2s)\).

Proposition 8. Let \(G\) be a topological group, \(H < G\) a subgroup. If \(H\) is closed, then \(G/H\) is Hausdorff.

Proof. Recall that an equivalent condition for a topological space \(X\) being Hausdorff is that the diagonal \(\Delta(X) \subset X \times X\) is closed. We want then

\[
\Delta(G/H) \text{ to be closed in } G/H \times G/H \cong (G \times G)/(H \times H).
\]

This is equivalent to \(\Delta(G/H)\) having a closed preimage in \(G \times G\). Now

\[
\Delta(G/H) = \{(gH, g'H) | g^{-1}g' \in H\} \subset (G \times G)/(H \times H)
\]

and the preimage in \(G \times G\) is

\[
\{(g, g') \in G \times G | g^{-1}g' \in H\} = \alpha^{-1}(H)
\]

under the continuous map

\[
\alpha : G \times G \to G, (g, g') \mapsto g^{-1}g'.
\]

As \(H\) is closed, this concludes the proof. \(\square\)

Exercise 4. The converse is also true: if \(G/H\) is Hausdorff, then \(H < G\) is closed. Prove it.

Example 13. We saw that we had an homeomorphism

\[
\text{SO}(n)/\text{SO}(n-1) \to S^{n-1}\text{ sending } g \mapsto g(e_1)
\]

where \(e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n\). We will see that the domain has a \(C^\infty\)-manifold structure that turns this map into a diffeomorphism.

Note that the map above reduce the question about the connectedness of \(\text{SO}(n)\) for any \(n \geq 3\) to the connectedness of \(\text{SO}(2)\). Now

\[
\text{SO}(2) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ such that } a^2 + b^2 = 1 \right\} \cong S^1 \text{ is commutative}
\]

while \(\text{SO}(n)\) for \(n \geq 3\) is highly non-commutative. Thus \(\text{SO}(2)\) is much easier to study, and many questions about the general \(\text{SO}(n)\) can be reduced to easier question about \(\text{SO}(2)\).

On Homework 2 it is shown via a similar method with the unit sphere \(S^{2n-1} \subset \mathbb{C}^n\) that \(\text{SU}(n)\) is connected for \(n \geq 2\). Note that \(U(1) \cong S^1\) so \(\text{SU}(1) = \{1\}\) is trivial. We also have

\[
\text{SU}_n(\mathbb{H}) = \{ h \in \mathbb{H}^n \text{ with } N(h) = 1 \} = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in \text{GL}_{2n}(\mathbb{C}) \text{ such that } |a|^2 + |b|^2 = 1 \right\}
\]

is non-commutative. So the same reasoning about connectedness could be applied to \(\text{Sp}(n)\), using the unit sphere \(S^{4n-1} \subset \mathbb{H}^n\).

One needs objects more complicated then spheres, in order to apply the same reasoning to \(\text{SL}_n(\mathbb{R})\) and \(\text{Sp}_{2n}(\mathbb{R})\).
Remark. For any Zariski closed $\mathbb{R}$-subgroup $G \subset \text{GL}(n, \mathbb{R})$ one finds that $G(\mathbb{R})$ has always finitely many connected components, but can be disconnected (i.e. more than one connected components) even when $G$ is Zariski-connected. Groups that have the above feature are, for instance, $\text{GL}_n$ and the nondefinite $\text{SU}(p, q)$.

**Proposition 9.** Let $G$ be a $C^\infty$-manifold with a topological group structure such that $m : G \times G \rightarrow G$ is a smooth map. Then the inversion $i : G \rightarrow G$ is also a $C^\infty$-map. 

**Proof.** Check the handout on smoothness of inversion. 

### 3.1 Left-invariant vector field

Let $G$ be a Lie group and denote the tangent space at the identity as $T_e(G) = \text{Lie}(G) = \mathfrak{g}$.

**Definition 7** (Left-invariant vector field). A (set-theoretic) vector field $X$ on $G$ is left-invariant if for any $g, g' \in G$ we have that the differential of the left multiplication

$$d l_g : T_{g'}(G) \rightarrow T_{gg'}(G)$$

sends $X(g')$ to $X(gg')$.

**Exercise 5.** Prove that the above condition is equivalent to

$$X(g) = c \quad \forall g \in G.$$ 

Moreover, if $X$ is a $C^\infty$-vector field, this is the same to say that $l_g^* X$ verifies (as an operator on $C^\infty(G)$)

$$X \circ l_g^* = X \quad \forall g \in G.$$ 

In homework 2 you will prove that $X(g) = d l_g (X(e))$ is in fact a $C^\infty$-vector field

**Example 14.** $G = \text{GL}_n(\mathbb{R}) \subset M_n(\mathbb{R})$ is open, so $\text{Lie}(G) = T_1(M_n(\mathbb{R})) \cong M_n(\mathbb{R})$ where the isomorphism is canonical and given by

$$\sum_{i,j} a_{i,j} \theta_{x_{i,j}}|_1 \mapsto (a_{i,j})$$ 

**Proposition 10.**

1. $\theta_{x_{i,j}}|_1$ extends to the left-invariant vector field $\sum_k x_{k,j} \theta_{x_{k,j}}$.

2. If $A, B \in M_n(\mathbb{R})$ are left-invariant vector fields, then the commutator $AB - BA$ is a left invariant vector field in $\text{Lie}(G)$.

**Proof.** Homework 2. 

### 4 April 8

If we wish to study the classical groups over finite fields, e.g. $\text{SL}_n(\mathbb{F}_q)$, $\text{Sp}_n(\mathbb{F}_q)$ and so on, then some techniques from algebraic geometry are required. In particular for every classical group $G$ we define

$$G(\overline{\mathbb{F}_q}) = \bigcup_{n>0} G(\mathbb{F}_{q^n})$$

which is 'smooth' (in some sense) and has a Lie algebra. Then Lie algebra techniques can be applied to $G(\overline{\mathbb{F}_q})$, leading to some similarities between finite groups and classical groups.
Proposition 11. Let $G$ be a $C^\infty$-manifold with a group structure such that $m : G \times G \to G$ is a smooth map. Then inversion is a smooth map.

Proof. The main idea is using homogeneity of $G$ to reduce the argument to a study of the inverse function theorem around the origin $e \in G$. Then one finds out that multiplication has differential

$$d m(e, e) : g \times g \to g \quad (v, w) \mapsto v + w$$

while inversion $i : G \to G$ has differential

$$d i(e) : g \to g \quad v \mapsto -v.$$

For the details, see the handout. □

4.1 Lie algebras

Let $G$ be any Lie group. Recall from last time that we have a correspondence

$$g = T_e(G) \leftrightarrow \{\text{left-invariant vector fields on } G\}$$

given by

$$v \mapsto (g \mapsto d l_g(e)(v)).$$

Recall that any left-invariant vector field on $G$ is $C^\infty$ and that

$$X \text{ is left-invariant } \iff d l_g(X(g')) = X(g g') \forall g, g' \in G \iff X \circ l^*_g = X \forall g \in G$$

where $l^*_g$ is the induced action on $C^\infty(G)$.

The rightmost characterization of a left-invariant vector field makes it obvious that the commutator of two vector fields is also a vector field:

$$[X, Y] \circ l^*_g = (X \circ Y - Y \circ X) \circ l^*_g = X \circ Y \circ l^*_g - Y \circ X \circ l^*_g = X \circ Y - Y \circ X = [X, Y].$$

So we can transport back $[X, Y]$ to the left side of the correspondence, and hence we defined an operation

$$g \times g \to g \quad (v, w) \mapsto [\tilde{v}, \tilde{w}](e)$$

where $\tilde{v}$ and $\tilde{w}$ are the left-invariant vector fields associated to $v$ and $w$.

Example 15. Let $G = \text{GL}_n(\mathbb{R}) \subset \text{M}_n(\mathbb{R})$ be the open subset of invertible matrices. When we compute the tangent space at every point, this is canonically identified with the underlying vector space, thus $g = \text{M}_n(\mathbb{R})$ with the brackets $[A, B]_{\text{GL}_n(\mathbb{R})} = AB - BA$.

Definition 8 (Lie algebra). A Lie algebra over a field $k$ is a vector space $g$ equipped with a $k$-bilinear mapping

$$[,] : g \times g \to g$$

satisfying

1. the alternating property: $[X, X] = 0$ for all $X \in g$. If $\text{char} k \neq 2$, this is equivalent to $[X, Y] = -[Y, X]$ for all $X, Y \in g$. 

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2. the Jacobi identity:

\[
[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \forall X,Y,Z \in \mathfrak{g}.
\]

Remark. In general, a Lie algebra does not have a unit!

Example 16. $\mathbb{R}^3$ is a Lie algebra where the brackets are given by the usual cross product.

Example 17. Let $A$ be an associative $k$-algebra and define $[a,b] = ab - ba$, then $(A,[,])$ is always a Lie algebra.

For instance, when $k = \mathbb{R}$ one can take

- $A = \{C^\infty$-differential operators on $C^\infty(M)\}$ for $M$ some smooth manifold;
- $A = \text{End}(V)$ for any vector space $V$, even infinite-dimensional.

If in the last instance we set $V = C^\infty(G)$ where $G$ is a Lie group, we get a Lie algebras embedding

\[
\mathfrak{g} \hookrightarrow \text{End}(C^\infty(G)),
\]

this is telling us that left-invariant vector fields are differential operators and the commutators of two vector fields is again a left-invariant vector field (we already checked that).

From now on, we will sometimes denote Lie algebras with gothic letters, for example

\[
\mathfrak{gl}_n(\mathbb{R}) = \text{Lie}(\text{GL}_n(\mathbb{R})), \quad \mathfrak{sl}_n(\mathbb{R}) = \text{Lie}(\text{SL}_n(\mathbb{R})), \quad \mathfrak{so}(n) = \text{Lie}(\text{SO}(n)), \quad \mathfrak{u}(n) = \text{Lie}(\text{U}(n)).
\]

Fact 12. We will prove in a few that given a map of Lie groups $f : G \rightarrow H$, the induced map on the tangent spaces at the identity

\[
d f(e) : \mathfrak{g} \rightarrow \mathfrak{h}
\]

is a map of Lie algebras, i.e. it preserves the brackets.

Remark. This is NOT completely obvious from the definition, as we cannot pushforward vector fields.

To prove the previous fact, as well as that $\text{Lie}(\text{GL}_n(\mathbb{C})) = M_n(\mathbb{C})$ and $\text{Lie}(\text{GL}_n(\mathbb{H})) = M_n(\mathbb{H})$, we need a way to think about $[,]$ that is not as "global" as the definition, but depends only on local properties around the identity element.

Remark. 1. Every Lie algebra $\mathfrak{g}$ is in a natural way a Lie subalgebra of an associative unital algebra denoted $U(\mathfrak{g})$, called the universal enveloping algebra.

2. When $\mathfrak{g}$ is finite-dimensional, it is possible to prove that we have an embedding of Lie algebras

\[
\mathfrak{g} \hookrightarrow \text{End}(k^n) = M_n(k).
\]

Ado proved this when $\text{char} k > 0$ while Harish-Chandra handled the positive-characteristic case.
We now reformulate the Jacobi identity. For a Lie algebra \( g \) and \( X \in g \), we define 
\[
\text{ad}_g(X) : g \rightarrow g \quad Y \mapsto [X, Y]
\]
thus we have a mapping
\[
\text{ad} : g \rightarrow \text{End}(g) \quad X \mapsto \text{ad}_g(X) = [X, -].
\]
The Jacobi identity is exactly the statement that \( \text{ad} \) is a Lie algebras map!

In the Lie group setting, we can use a variant of the construction above. For every \( g \in G \), the conjugation by \( g c_g \) is an automorphism of \( G \), so we can differentiate it at the identity and get
\[
\text{Ad}_G : G \rightarrow \text{GL}(g) \quad g \mapsto d(c_g)(e)
\]
called the \textit{adjoint representation} of \( G \). In fact \( c_g' \circ c_g = c_g'g \) so \( \text{Ad}_G \) is a group homomorphism and in fact is also a smooth map.

\textit{Remark}. As all the ”interesting” Lie groups are non-commutative, this is in fact a very useful tool as conjugation is non-trivial.

Now \( \text{Ad}_G \) is smooth and takes \( e \in G \) to \( \text{id} \in \text{GL}(g) \), hence differentiating gives us a map of Lie algebra
\[
d(\text{Ad}_G)(e) : g \rightarrow \text{End}(g)
\]
which turns out to be exactly \( \text{ad}_g \).

We now go on to describe a ”local” formula for the brackets \([.,.\] \).

**Theorem 13** (Key theorem). Let \( g \) be the Lie algebra of a Lie group \( G \). For all \( X \in g \) there exists a unique Lie group map
\[
\alpha_X : \mathbb{R} \rightarrow G \text{ such that } \alpha_X(0) = e \in G, \alpha_X'(0) = X \in g.
\]
\( \alpha_X \) is an instance of a so called 1-parameter subgroup of diffeomorphism.

**Proof.** Use ODEs theory and differential geometry (Frobenius theorem). See the handout. \( \square \)

**Example 18.** Let \( G = \text{GL}_n(\mathbb{R}) \), then \( \alpha_X(t) = \exp(tX) \) is a group map, and in fact those are the only possible group maps \( \mathbb{R} \rightarrow G \).

**Example 19.** Let \( G = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2 \). Then \( X \in g \) is uniquely identified with an angle \( \theta \), when we use the square-description of the torus with the opposite edges glued together, and \( X \) is a vector from the bottom-left vertex which forms an angle \( \theta \) with the bottom edge.

Then if \( \theta \in \mathbb{Q}\pi \) the image of the flow given by \( X \) on the torus is a closed image, thus a circle, while if \( \theta \notin \mathbb{Q}\pi \), the image of the flow given by \( X \) is dense on the torus, and the corresponding group map \( \mathbb{R} \rightarrow G \) is injective.

**Fact 14** (Key Formula). This is proved in the handout on the adjoint representation. Let \( X,Y \in g \), then
\[
[X,Y]_G = \frac{d}{dt}\big|_{t=0}(\text{Ad}_G(\alpha_X(t))(Y))
\]
Notice that as \( t \rightarrow \text{Ad}_G(\alpha_X(t))(Y) \) is a parametric curve in \( T_e(G) = g \), it makes sense to take its derivative at \( t = 0 \) as another element of \( \mathfrak{g} \).

The above fact has two consequences:

1. If \( G \) is commutative, then \( \text{Ad}_G \) is trivial so \([.-.-] = 0 \). The converse is true too if \( G \) is connected!

2. The brackets \([.,.\] are functorial in \( G \), because we’ve been able to describe the bracket operation in local terms.
The goals for this and next lecture is to define the exponential map $\exp_G : g \to G$ and tie up loose ends on 1-parameter subgroups.

Soon we will also define integration on $G$ and $G/H$, which will turn out to be a very useful tool in the representation-theoretic setting.

Compact Lie groups will be our main topic of study. These are a crucial tool even in the non-compact case, because of the following

**Theorem 15.** Let $G$ be a Lie group with $|\pi_0(G)| < \infty$, i.e. $G$ has finitely many connected components. Then

1. Every compact Lie subgroups lies inside a maximal one and all the maximal ones are conjugate.

2. Let $K$ be such a maximal compact subgroups, then $K$ meets every connected components, and in fact

$$G^0 \cap K = K^0.$$

3. We have an isomorphism of manifolds

$$K^0 \times V = G^0$$

for some vector space $V$.

**Example 20.** We have $\text{SL}_n(\mathbb{R}) = \text{SO}_n(\mathbb{R}) \times \{\text{symmetric matrices}\}$.

**Remark.** Thanks to the theorem, any topological question about $G$ can be reduced to a question about $K$.

Last time we had the following theorem about 1-parameter subgroups:

**Theorem 16.** Let $G$ be a Lie group, $v \in g$. Then there exists a unique Lie group map $\alpha_v : \mathbb{R} \to G$ such that $\alpha'_v(0) = v$.

**Remark.** The uniqueness part gives functoriality (in $G$) of the 1-parameter subgroup. That is, if $f : G \to G'$ is a Lie group map, and $v' = df(e)(v) \in g'$, then

$$\alpha_{v'} = f \circ \alpha_v : \mathbb{R} \to G'.$$

**Example 21.** Let $G = \text{GL}_n(\mathbb{R})$, it is easy to check that $\alpha_v(t) = \exp(tv)$ works, so by uniqueness this must be the group map that the theorem talks about. Note that the same works for the general linear group on $\mathbb{C}$ or $\mathbb{H}$.

By the previous remark, we can then compute $\alpha_v$ for any closed subgroup of the general linear group.

We had the key formula

$$[v, w] = \left. \frac{d}{dt} \right|_{t=0} (\text{Ad}_G(\alpha_v(t))(w)) \quad \forall v, w \in g.$$

Recall also the adjoint representation

$$\text{Ad}_G : G \to \text{GL}(g) \quad \text{Ad}_G(g) = d(c_g)(e) : g \to g.$$
Proposition 17. The key formula gives functoriality of the Lie brackets.

Proof. Let \( G \rightarrow G' \) be a Lie group map. For \( v, w \in \mathfrak{g} \) and \( v' = df_e(v) \in \mathfrak{g}' \), \( w' = df_e(w) \in \mathfrak{g}' \), we want to show that

\[
(v') = df_e([v, w]_G) = [v', w']_{G'},
\]

that is, \( df_e \) is a Lie algebras map.

Now \([v, w]_G\) is the velocity at time \( t = 0 \) of the parametric curve

\[
t \mapsto \Ad_G(\alpha_v(t))(w) \in \mathfrak{g},
\]

so \( df_e([v, w]_G) \) is the velocity at \( t = 0 \) of the parametric curve pushed forward:

\[
t \mapsto df_e(\Ad_G(\alpha_v(t)))(w).
\]

Recall that \( \Ad_G(g) = d(c_g)(e) \), so by the chain rule

\[
df_e \circ \Ad_G(g) = df_e \circ d(c_g)(e) = df_e \circ df_e \circ \Ad_G(f(g)) \circ df_e
\]

because \( f \circ c_g = c_{f(g)} \circ f \) as functions \( G \rightarrow G' \). So \([v, w]_G\) is the velocity at time \( t = 0 \) of the parametric curve

\[
t \mapsto df_e \Ad_G(\alpha_v(t))(w) = \Ad_{G'}(f(\alpha_v(t)))(df_e(w)) = \Ad_{G'}(\alpha_{v'}(t))(w').
\]

By the key formula, the velocity at \( t = 0 \) of the rightmost side is exactly \([v', w']_{G'}\), thus proving the claim.

Now we try to answer a deeper question: why does \( \alpha_v \) exist? The key inputs for the proof are the theory of integral curves to \( C^\infty \)-vector fields on manifolds, and the existence and uniqueness theorems for first order nonlinear vector-valued ODEs. In fact, we need even more than that, because we need a smooth dependence (of the flow) on initial conditions.

Consider the following setup. \( M \) is a \( C^\infty \)-manifold and \( X \) a smooth vector field on \( M \). Pick \( p \in M \), an integral curve to \( X \) at \( p \) is

\[
c : I \rightarrow M \text{ such that } c(0) = p, c'(t) = X(c(t)) \forall t \in I
\]

where \( I \) is some real interval around 0.

Remark. In local coordinates around \( p = c(0) \in M \), we are looking for a solution \( \Phi \) of the vector-valued ODE

\[
\Phi'(t) = f(t, \Phi(t)) \quad \Phi(0) = \upsilon_0 \in \mathbb{R}^n
\]

where \( f \) is some function where the definition of \( X \) is encoded.

The existence and uniqueness theorems for ODEs give us that there exists a unique solution \( \Phi \) on a maximal interval of definition, so

\[
c^\text{max}_p : I(p) \rightarrow M \text{ is the integral curve to } X \text{ at } p.
\]

We want to be able to vary \( p \) and make sure that this integral curve depends smoothly on \( p \).

Theorem 18 (Miracle). Let \( \Omega = \{(t, p) \in \mathbb{R} \times M \mid t \in I(p)\} \), then \( \Omega \subset \mathbb{R} \times M \) is open and the flow

\[
\Phi : \Omega \rightarrow M \quad (t, p) \mapsto c^\text{max}_p(t) = \Phi_t(p) \text{ is a smooth map.}
\]
Notice that the uniqueness of the integral curve given $\Phi(0) = p$ tells us that if $t \in I(p)$, $t' \in I(\Phi_t(p))$ then $t + t' \in I(p)$ and in fact

$$\Phi_t'(\Phi_t(p)) = \Phi_{t+t'}(p).$$

Now we would like to extend the integral curve to the whole real numbers, but if $M$ is a non-compact manifold it is unclear how to define an interval that works for every $p$, even less how to stretch this interval. Here is where the left-translation feature of the Lie group situation comes in handy.

Let $M = G$ be a Lie group, $v \in \mathfrak{g}$, and consider the associated left-invariant vector field

$$X(g) = d l_g(e)(v).$$

Consider the associated flow $\Phi : \Omega \rightarrow G$.

**Claim 19.** The maximal interval of definition of an integral curve $I(p)$ is independent of $p$, and in fact we have $I(p) = \mathbb{R}$.

**Proof.** The fact that the interval is independent of $p$ is an easy consequence of the left invariance of $X$ and the fact that $l_g$ carries the identity element $e$ to $g \in G$. Now we have

$$\Phi_{t+t'} = \Phi_t \circ \Phi_{t'} \quad \forall t, t' \in I = I(p)$$

hence also $t + t' \in I$, so as $I$ is a non-empty open interval, it must be the whole real line $\mathbb{R}$. \qed

We get then a *global flow*

$$\Phi : \mathbb{R} \times G \rightarrow G$$

depending only on $v \in \mathfrak{g}$.

And note that at the identity, this flow is an homomorphism because we have

$$g\Phi_t(p) = \Phi_t(gp)$$

since $X \circ l_g^* = X$.

For $v \in \mathfrak{g}$ and $\Phi : \mathbb{R} \times G \rightarrow G$ the associated flow, define then

$$\alpha_v(t) := \Phi_t(e).$$

**Claim 20.** $\alpha_v : \mathbb{R} \rightarrow G$ is a Lie group map with velocity vector $v$.

**Proof.** As

$$\frac{d\Phi}{dt}(t, g) = X(g)$$

we have $\alpha'_v(0) = X(0) = v$, moreover $\alpha_v$ is clearly smooth so we only have to check that it is a group map. But we have

$$\alpha_v(t + t') = \Phi_{t+t'}(e) = \Phi_t'(\Phi_t(e)) = \Phi_t'(\alpha_v(t)) =$$

$$= \Phi_t'(\alpha_v(t) \cdot e) = \alpha_v(t) \cdot \Phi_t'(e) = \alpha_v(t) \cdot \alpha_v(t')$$

where in the first equality of the last line we used the left-translation property by $\alpha_v(t)$.

Conversely, if $\alpha' : \mathbb{R} \rightarrow G$ is a Lie group map and $v = \alpha'(0) \in \mathfrak{g}$, we can create

$$\psi(t, g) = g \cdot \alpha(t) : \mathbb{R} \times G \rightarrow G,$$

which turns out to be a flow for the left-invariant vector field $X$ defined by $X(0) = v$. Thus by uniqueness of integral curves, this flow must be the only flow, thus $\alpha$ is exactly the integral curve $\alpha_v$ to $X$ at $e$. 

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Today we will define the exponential map, study some of its properties and its relation with Lie algebras.

Let’s first tie up a loose end from last time, regarding the uniqueness of 1-parameter subgroups. Let \( \alpha : \mathbb{R} \to G \) be a Lie group map, with \( \alpha'(0) = v \in \mathfrak{g} \). We want to show that \( \alpha = \alpha_v \), so that \( \alpha \) must be the 1-parameter subgroup: to do so, we’ll show both \( \alpha \) and \( \alpha_v \) satisfy the same first order ODE with the same initial data.

Fix \( t \) and consider \( \alpha(t + \tau) = \alpha(t)\alpha(\tau) = l_{\alpha(t)}(\alpha(\tau)) \) as an equality of parametric curves in \( \tau \). We can then compute the velocity vector for \( \tau = 0 \) in two different ways, and we get

\[
\frac{d}{d\tau} (\alpha(t + \tau)) = \alpha'(t + \tau)
\]

and

\[
\frac{d}{d\tau} (l_{\alpha(t)}(\alpha(\tau))) = dl_{\alpha(t)}(\alpha(\tau))(\alpha'(\tau))
\]

and the two terms are equal as elements of \( T_{\alpha(\tau)}(G) \). Setting \( \tau = 0 \), we get

\[
\alpha'(t) = dl_{\alpha(t)}(\alpha(0))(\alpha'(0)) = dl_{\alpha(t)}(e)(v) = \tilde{v}(\alpha(t))
\]

where \( \tilde{v} \) is the flow which propagates by left-translation \( v \in T_e(G) = \mathfrak{g} \). So we have the ODE

\[
\alpha'(t) = \tilde{v}(\alpha(t)), \quad \alpha(0) = e
\]

which implies that \( \alpha \) is the integral curve for \( \tilde{v} \) at \( e \), and thus it must the only such integral curve defined on the whole \( \mathbb{R} \). But by definition, the only integral curve for \( \tilde{v} \) at \( e \) defined on \( \mathbb{R} \) is \( \alpha_v \), thus \( \alpha = \alpha_v \).

**Definition 9 (Exponential map).** We are then ready to define the exponential map as

\[
\exp_G : T_e(G) = \mathfrak{g} \to G \quad \exp_G(v) = \alpha_v(1).
\]

**Remark.** By the uniqueness considerations for 1-parameters subgroups we immediately get \( \alpha_{tv}(\tau) = \alpha_v(t\tau) \) for every \( t, \tau \in \mathbb{R} \), as both are 1-parameter subgroups (in \( \tau \)) with the same velocity at \( e \).

Setting \( \tau = 1 \) we obtain

\[
\alpha_v(t) = \alpha_{tv}(1)
\]

which means

\[
\exp_G(tv) = \alpha_v(t) \quad \forall t \in \mathbb{R}, \forall v \in \mathfrak{g}.
\]

Geometrically, the map \( \exp_G \) carries the line \( tv \) to the curve \( \alpha_v(t) \).

**Exercise 6.** Prove that \( \exp_G \) is a smooth map. This is a manifestation of smoothness of solutions of ODEs depending on initial conditions.

**Claim 21.** \( \exp_G \) is a local isomorphism near the identity. So when \( G \) is connected, this gives information on the whole \( G \).
**Fact 22.** Let \( f : G \to G' \) be a Lie group map, then the following diagram commutes.

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\text{Lie}(f)} & \mathfrak{g}' \\
\exp_G & \downarrow & \exp_{G'} \\
G & \xrightarrow{f} & G'
\end{array}
\]

This ultimately boils down to the fact that \( \alpha_v : \mathbb{R} \to G \) is functorial in \( G \), as we already verified.

**Example 22.** Let \( G = \text{GL}_n, \mathfrak{g} = M_n \) so we have

\[
\exp_G(v) = e^v \quad \forall v \in M_n,
\]

and in fact \( \alpha_t(v) = e^{tv} \), as one verifies using the uniqueness of the 1-parameter subgroup.

**Corollary 23.** When \( H \subset G \) is a closed Lie subgroup, we have that

\[
\mathfrak{h} \xrightarrow{\exp_H} H
\]

is just the restriction to \( \mathfrak{h} \) of \( \exp_G : \mathfrak{g} \to G \).

Hence everything we have to know is a description of \( \mathfrak{h} \) embedded in \( \mathfrak{g} \).

**Lemma 24.** The map

\[
d\exp_G(0) : T_0(\mathfrak{g}) \to T_0(G) = \mathfrak{g}
\]

is the identity, once we identify canonically \( T_0(\mathfrak{g}) \cong \mathfrak{g} \) via directional derivatives.

**Proof.** We differentiate \( \exp_G(tv) = \alpha_v(t) \): we have

\[
d\exp_G(tv)(v) = \alpha'_v(t)
\]

so by setting \( t = 0 \) we get

\[
d\exp_G(0)(v) = \alpha'_v(0) = v
\]

thus \( d\exp_G(0) = \text{id} \).

**Corollary 25.** By the inverse function theorem, \( \exp_G \) is a local diffeomorphism near 0 and \( e \). But in general \( \exp_G \) is NOT an homomorphism!

**Corollary 26.** Let \( G \) be a connected Lie group. Then every Lie group map \( f : G \to G' \) is uniquely determined by the induced map \( \text{Lie}(f) : \mathfrak{g} \to \mathfrak{g}' \).

**Proof.** We have the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\text{Lie}(f)} & \mathfrak{g}' \\
\exp_G & \downarrow & \exp_{G'} \\
G & \xrightarrow{f} & G'
\end{array}
\]

and we’ve seen that \( \exp_G \) is a local isomorphism between \( 0 \in \mathfrak{g} \) and \( e \in G \). Thus \( \text{Lie}(f) \) completely determines what \( f|_U \) for some open neighborhood \( U \) of \( e \) in \( G \). But \( f \) is a group map, so the above also determines \( f \) on \( U^{-1} \), and hence \( f \) is determined on the subgroup generated by \( U \). We proved in the homework that this subgroup must be \( G \), by connectedness. 

\[\square\]
**Example 23.** Let \( G = V / \Gamma \) be a Lie group, where \( V \) is a vector space and \( \Gamma \) a lattice in it. Then \( \exp_G \) is a local diffeomorphism on [WHICH OPEN SET?]

*Remark.* The converse is usually false, that is: not every Lie algebra map \( \mathfrak{g} \rightarrow \mathfrak{g}' \) comes from a map of Lie groups \( G \rightarrow G' \). There are topological obstructions.

*Remark.* We had that \( \text{Ad}_G : G \rightarrow \text{GL}(\mathfrak{g}) \) is a \( C^\infty \) map, sending \( g \mapsto d_c g(e) \). In the handout on the adjoint representation, it is shown that \( d(\text{Ad}_G)(e) : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}) \) works as \( X \mapsto [X,-] \).

If \( G \) is commutative, the map \( \text{Ad}_G \) is constant and equal to \( \text{id} \in \text{GL}(\mathfrak{g}) \), so its derivative is trivial, and \( \text{ad}_g = 0 \), i.e. \( [\cdot,\cdot] = 0 \), i.e. the Lie algebra is commutative.

Conversely, if \( G \) is connected and \( \mathfrak{g} \) is commutative (i.e. \( \text{ad}_g = 0 \)), by the uniqueness of the induced map \( \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}) \) we must have \( \text{Ad}_G \equiv 1_{\text{GL}(\mathfrak{g})} \), because this map induces the right \( \text{ad}_g \). Thus for any fixed \( g \in G \), we have

\[
\text{Lie}(c_g) = d(c_g)(e) = \text{id} \in \text{GL}(\mathfrak{g}),
\]

and as \( \text{id}_G \) induces exactly this map, again by uniqueness of the induced map we must have \( c_g = \text{id}_G \), so \( G \) is commutative.

**Lemma 27.** Let \( G \) be commutative, then \( \exp_G : \mathfrak{g} \rightarrow G \) is a homomorphism.

*Proof.* Consider the multiplication map \( m : G \times G \rightarrow G \). As \( G \) is commutative, this is a group homomorphism! Hence it makes sense to look at the induced effect on the level of Lie algebras:

\[
\begin{array}{ccc}
\mathfrak{g} \times \mathfrak{g} & \xrightarrow{+} & \mathfrak{g} \\
\exp_G \times \exp_G & | & | \exp_G \\
G \times G & \xrightarrow{m} & G
\end{array}
\]

because we already proved that multiplication induces the sum on the tangent space. This implies

\[
\exp_G(v + w) = \exp_G(v) \cdot \exp_G(w)
\]

thus \( \exp_G \) is a homomorphism. \( \square \)

**Corollary 28.** Suppose \( G \) is commutative and connected. Then \( \exp_G \) is a surjective homomorphism, as the image \( \exp_G(\mathfrak{g}) \) is an open subgroup of \( G \).

**Corollary 29.** \( \ker(\exp_G) \) is discrete, because around \( 0 \) we showed that \( \exp_G \) is a local diffeomorphism, hence around \( 0 \) the kernel is just \( 0 \). By translation, every point in the kernel is an isolated point. In particular \( \ker(\exp_G) \) is a discrete subgroup in \( \mathfrak{g} \).

**Fact 30** (Lemma 38, ch. I). \( \Gamma \subset \mathbb{R}^n \) is a discrete subgroup if and only if \( \Gamma \) is the \( \mathbb{Z} \)-span of some \( \mathbb{R} \)-linearly independent set of vectors (i.e. a lattice in a subspace \( W \subset \mathbb{R}^n \)).

Let then \( \Gamma = \ker(\exp_G) \), a lattice in some subspace \( W \subset \mathfrak{g} \), thus \( \mathfrak{g} = W \oplus W' \), so that

\[
G = (W/\Gamma) \times W'
\]

and \( W/\Gamma \) is isomorphic (at least as an additive group) to \( \mathbb{R}^n / \mathbb{Z}^n \cong \mathbb{T}^n = (S^1)^n \), thus

\[
G \cong (S^1)^n \times V
\]

for some vector space \( V \).
Corollary 31. The only compact, connected, commutative Lie groups are the tori $T^n = (S^1)^n$.

This is particularly important, because the representation theory of any Lie group $G$ lies on the representation theory of its tori.

Let us now consider other important applications of the exponential map.

Fact 32 (Theorem 3.11, ch. I). Let $G$ be a Lie group, $H \subset G$ an abstract subgroup. Then $H$ is closed if and only if $H$ is a locally closed submanifold of $G$, when with "locally closed" we mean it is an embedded submanifold.

Remark. The upshot of the fact above is that any closed abstract subgroup of a Lie group is in fact smooth!

Corollary 33. Let $f : G \rightarrow G'$ be a continuous map between Lie groups. Then in fact $f$ is smooth.

Idea. Consider the graph $\Gamma_f \subset G \times G'$. As $f$ is a group map, $\Gamma_f$ is a closed subgroup, so it is an embedded submanifold. Now consider the projections on the two factors, both of which are smooth maps as $\Gamma_f$ is a Lie subgroups of $G \times G'$:

$$ pr_1 : \Gamma_f \rightarrow G, \quad pr_2 : \Gamma_f \rightarrow G'. $$

By using the inverse function theorem we can invert the first projection and then we get $f = pr_2 \cdot (pr_1)^{-1}$.

Another application is the following: we have a correspondence

$$ \{ \text{Lie connected subgroups } H \subset G \} \leftrightarrow \{ \text{Lie subalgebras } \mathfrak{h} \subset \mathfrak{g} \}. $$

See the handout on Frobenius theorem for details, but notice that the most difficult part is proving that a subalgebra induces a closed Lie subgroup.

Definition 10 (Lie subgroup). Let $G$ be a Lie group. A Lie subgroup of $G$ is an abstract subgroup $H$ equipped with an injective immersion $H \hookrightarrow G$ which is a group map. $H$ may not have the subspace topology!

Claim 34. Let $G$ be a Lie group, $H$ a subgroup. There exists a unique $C^\infty$-structure on $G/H$ making

$$ \pi : G \rightarrow G/H $$

a submersion of manifolds.

Proof. The existence is the hard part. First we use the previous application and write $\mathfrak{g} = \mathfrak{h} \oplus W$, then we use $\exp_G$ to put a $C^\infty$ structure on a neighborhood of $eH$ on $G/H$, and then we move it by left-translation. This gives a quotient structure on $G/H$ with good universal mapping properties (see also HW 3).

7 April 15 (Daniel)

We start by reviewing some character theory. Let $G$ be a finite group.

Theorem 35 (Maschke theorem). Let $\rho : G \rightarrow \text{GL}_n(\mathbb{C})$ be a representation. Then $\rho$ is semisimple.
Sketch. Fix $V \subset W$ a subrepresentation. Pick $(,)$ an inner product on $W$, replace it with a $G$-invariant inner product $(,)_G$ by averaging, and take $V' = V^\perp$ a direct sum complement.

Our goal is to replace the averaging given by $\sum_{g \in G}$ on a finite group, with the averaging given by an integral on the compact Lie group.

**Definition 11.** The *character* of a representation $\rho$ is $\xi_\rho(g) = \text{Tr}(\rho(g))$. For $f, h : G \to \mathbb{C}$ class functions, we define an inner product as

$$\langle f, h \rangle = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{h(g)}.$$ 

**Proposition 36 (Schur Lemma).** If $V$ is an irreducible representation, then $\text{Hom}_G(V, V) \cong \mathbb{C}$.

If $V, W$ are irreducible representations, then $\text{Hom}_G(V, W) = \begin{cases} \mathbb{C} & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases}$.

**Proposition 37.** Let $\rho$ be a representation and $\xi$ its character. Then

$$\langle 1, \xi \rangle = \dim \rho^G$$

where $\rho^G$ is the subrepresentation of $\rho$ given by the fixed elements.

Applying this to $\rho = \text{Hom}(V, W) = V^* \otimes W$ gives us

$$\frac{1}{|G|} \sum \xi_V \xi_W = \dim \text{Hom}_G(V, W).$$

**Corollary 38.** The characters for the irreducible representations of $G$ are an orthonormal basis for the class functions.

**Fact 39.** Finite groups have an obvious left-invariant measure given by

$$f \mapsto \frac{1}{|G|} \sum_{g \in G} f(g)$$

for every function $f : G \to \mathbb{C}$.

Suppose from now on that $G$ is a compact Lie group, and let $\rho : G \to \text{GL}(V)$ be a finite dimensional representation. We can associate to this a Lie algebra representation

$$\mathfrak{g} \to \mathfrak{gl}(V) = \text{End}(V),$$

and conversely such a Lie algebra representation determines $\rho$ if $G$ is connected. If we extend scalars to $\mathbb{C}$, we can get

$$\mathfrak{g}_\mathbb{C} \to \mathfrak{gl}(V) \quad x \mapsto d\rho(x)$$

where we want the identity

$$d\rho[x, y] = [d\rho(x), d\rho(y)].$$

Notice that associating a Lie algebra representation to a Lie group representation is a functorial process, in particular it preserves subrepresentations.
Theorem 40. Let \( G \) be simply connected, then
\[
\{\text{subrepresentations of } \rho\} \leftrightarrow \{\text{subrepresentations of } d\rho\}
\]
is a bijective correspondence. We will prove this later.

Remark. Not every representation of Lie algebras is induced by a representation of Lie groups. For example let \( SU(2) = (\mathbb{H}^\times)^1 \) be the norm-1 quaternions, with the standard representation \( SU(2) \hookrightarrow GL_2(\mathbb{C}) \). Notice that we also have a double cover \( SU(2) \twoheadrightarrow SO(3) \) which is a Lie group map, but this induces an isomorphism on the Lie algebras, as it is a covering map. In particular for \( SO(3) \) the Lie groups representations are not the Lie algebra representations. Notice that \( SO(3) \) is connected but not simply connected, and \( SU(2) \) is its universal cover.

Remark. The topological obstructions are in the first homology group , because when we get a representation of \( G \) from a representation of \( g \), we are integrating on 1-parameter subgroups, so it is crucial to check if loops are (or are not) trivial.

Example 24. Let \( SO(3) \sim \mathbb{R}^3 \) in the obvious way, so we have an induced action \( SO(3) \sim C^\infty(\mathbb{R}^3, \mathbb{C}) \) which preserves the polynomials \( C[x_1, x_2, x_3] \), a subspace. Take the laplacian operator \( \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \) which is invariant under the action of \( SO(3) \). Now the metric on \( \mathbb{R}^3 \) is a section \( \langle, \rangle \in \Gamma(\mathbb{R}^3, T\mathbb{R}^3 \otimes T\mathbb{R}^3) \), because \( T^*\mathbb{R}^3 \cong T\mathbb{R}^3 \) as bundles, and as the metric is symmetric, in fact we have \( \langle, \rangle \in \Gamma(\mathbb{R}^3, Sym^2(T\mathbb{R}^3)) \). Any section of the last bundle we wrote is a degree-2 differential operator.

Define then \( P_n = \{ \text{homogeneous degree } n \text{ polynomials } f \in C[x_1, x_2, x_3] \text{ such that } \Delta f = 0 \} \); these are called harmonic functions on the sphere.

Claim 41. \( P_n \) are all the finite-dimensional representations of \( SO(3) \).

In particular, they are representations of \( SU(2) \) and denoting \( \rho_n : SU(2) \twoheadrightarrow GL(P_n) \) we have in fact
\[
\rho_n = Sym^{2n} (SU(2) \hookrightarrow GL_2(\mathbb{C})),
\]
that is, the \( 2n \)-symmetric power of the standard representation.

Next time, we will adapt Maschke theorem to the compact Lie groups: we will have that any finite dimensional representation \( \rho : G \twoheadrightarrow GL(V) \) split into a direct sum of irreducibles.

Example 25. Take a torus \( T = (S^1)^r \twoheadrightarrow \mathbb{C}^* \) and a one-dimensional representation of the form \( (z_1, \ldots, z_r) \mapsto z_1^{c_1} \cdots z_r^{c_r} \). Then every representation of the torus will be a direct sum of one-dimensional representation of this form.

Suppose we have a torus \( T \subseteq G \), so a representation of \( G \) gives a representation of the torus
\[
T \hookrightarrow G \twoheadrightarrow GL(V),
\]
thus we have a splitting as \( T \)-representations:
\[
V = \bigoplus V(c_1, \ldots, c_r)
\]
where each \( V(c_1, \ldots, c_r) \) is called a weight space.

Fact 42. Large tori exist in any compact Lie group, so we can usually find a decomposition like the one above.
Example 26. Examples of large tori are

\[
\left\{ \begin{array}{ccc}
    r_{\sigma_1} & \cdots & \\
    \cdots & \ddots & \\
    r_{\sigma_n}
\end{array} \right\} \subset \text{SO}(2n)
\]

where each \( r_{\sigma_i} \in \text{SO}(2) \) is a rotation of angle \( \sigma_i \) in a plane. Similarly

\[
\left\{ \begin{array}{ccc}
    e^{\theta_1} & \cdots & \\
    \cdots & \ddots & \\
    e^{\theta_n}
\end{array} \right\} \subset \text{U}(n)
\]

is a large torus in the unitary group.

Claim 43. Let \( G \) be a topological group, locally simply connected. Then the universal covering

\[
(\tilde{G}, \tilde{e}) \xrightarrow{\pi} (G, e)
\]

admits a unique Lie group structure such that \( \pi \) is a Lie group map.

Let now \( \Gamma \subset G \) be a normal discrete subgroup. Then \( \Gamma \) is abelian.

Proof. One can prove the second statement by showing that \( \pi_1 \) of any topological group is abelian, and \( \Gamma \) is a quotient of one such. \( \square \)

8 April 17 (Daniel)

We want to generalize to the compact Lie group case the following facts from the character theory of finite groups.

1. Maschke theorem;
2. Schur lemma;
3. \( \xi_\rho(g) = \text{Tr}(\rho(g)) \) is the character and then \( \frac{1}{|G|} \sum_{g \in G} \xi_\rho(g) = \dim \rho^G \);
4. \( \xi_\rho = \overline{\xi_\rho} \), \( \xi_{\rho_1 \otimes \rho_2} = \xi_{\rho_1} \cdot \xi_{\rho_2} \) and \( \xi_{\text{Hom}(\rho_1, \rho_2)} = \overline{\xi_{\rho_1}} \cdot \xi_{\rho_2} \);
5. \( \text{Hom}(\rho_1, \rho_2)^G = \text{Hom}_G(\rho_1, \rho_2) \);
6. if we define \( \langle f, h \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \overline{h(g)} \), as an inner product on class functions, then \( \langle \xi_{\rho_1}, \xi_{\rho_2} \rangle = \dim \text{Hom}_G(\rho_1, \rho_2) \).

To generalize 1. we need a way to "average" on the compact Lie group.

2. goes through without any work, because the proof just make use of the definition of irreducible representation, which is the same in the finite case and in the compact case.

3. and 4. are proved using just linear algebra, so they go through in the compact case, and so does 5. as it is a just a formal definition.

We need to generalize 6.
8.1 Character theory for compact Lie groups

First of all, recall we have the following left-invariant measure on a finite group $G$:

$$f : G \rightarrow \mathbb{C} \rightarrow \frac{1}{|G|} \sum_{g \in G} f(g).$$

Suppose now $G$ is a Lie group, and choose $\bar{\omega} \in \bigwedge^{\dim G} g^*$, where $g^*$ is the cotangent space at $e \in G$. Let $\omega$ be the left-invariant differential form associated to $\bar{\omega}$, we set

$$\omega(g) := d l_g^*(\bar{\omega}) \in \bigwedge^{\dim G} T_g^*$$

where $d l_g^*$ is the map induced on $\bigwedge^{\dim G} g^*$ by the left translation by $g$. Then define

$$\mu_\omega(U) := \int_U |\omega|,$$

from now on we will denote $\omega = d g$ abusing notation.

**Claim 44.** $d g$ is left-invariant, that is

$$\int_G f(g) \, d g = \int_G f(g'g) \, d g \quad \forall g' \in G.$$

To prove, just plug in the measure $\mu$ and check the equality.

In fact, we showed that any locally compact topological group admits a left-invariant measure (Haar measure).

**Remark.** The construction above works for every Lie group, even the non-compact ones.

**Example 27.** Let $G = S^1$, $\omega = d \theta$ for $\theta$ the coordinate on $S^1$. This is manifestly left-invariant.

**Example 28.** Let $G = \text{GL}_n(\mathbb{R})$, then

$$\omega = \frac{d x_{11} \wedge \ldots \wedge d x_{nn}}{\det(x_{ij})}$$

is left-invariant, as proved in homework 2.

**Proposition 45.** For a connected compact Lie group, left-invariant and right-invariant measures coincide.

**Proof.** Consider a left-invariant form $\omega$ and the pullback via right-translation $r_g^*(\omega)$, we claim they are equal. Recall that $r_g^*(\omega)(e) \in \bigwedge^{\dim G} T_e$ which is a one-dimensional vector space, thus $r_g^*(\omega)(e) = c_g \cdot \omega(e)$, but $r_g^*(\omega)$ is also left-invariant, as $(gx)g' = g(xg')$. 

**Fact 46.** The map

$$G \rightarrow \mathbb{R}^* \quad g \mapsto c_g$$

is a group map called the modulus character

This is clear as $r_{g_1 g_2}^* = r_{g_1}^* \cdot r_{g_2}^*$, so we have

$$c_{g_1 g_2} = c_{g_1} \cdot c_{g_2}.$$ 

Roughly speaking, the modulus character tells us how much the measure changes as we shift it via right translation.
Remark. As the modulus character is a continuous mapping, if $G$ is compact and connected then $c : G \to \mathbb{R}^*$ is constant and equal to 1, because the image must be a compact connected subgroup of $\mathbb{R}^*$, and $\{1\}$ is the only one such.

Remark. In the disconnected case, it is not true that left-invariant measures are also right-invariant, even if $G$ is compact! For example for $O_n(\mathbb{R})$, the modulus character $c$ takes value 1 on $SO_n(\mathbb{R})$ and $-1$ on the other connected component.

Now we are ready to generalize Maschke’s theorem. Let $G$ be a compact Lie group, we want to construct an invariant inner product.

Let $\rho : G \to \text{GL}(V)$ be a finite-dimensional representation for the Lie group $G$. Then (by definition of representation of a Lie group), $\rho$ is continuous and hence smooth by a previous result.

Claim 47. $V$ admits a $G$-invariant inner product (i.e. an hermitian pairing).

Proof. Pick some hermitian inner product $h : V \times V \to \mathbb{C}$ and define

$$h'(v,w) := \int_G h(gv,gw) \, dg.$$ 

This is $G$ invariant as the integral is left-invariant. Compactness makes sure that the integral is well-defined, as the map 

$$g \mapsto h(gv,gw)$$

is continuous for any fixed $v, w \in V$. Finally, this is positive definite because

$$h'(v,v) = \int_G h(gv,gv) \, dg \geq 0$$

and $\int_G h(gv,gv) \, dg = 0$ if and only if $gv = 0$ almost everywhere, which implies $v = 0$ by the continuity of the representation and the fact that $ev = v$. But having proved left-invariance of the integral, we can also say

$$h'(v,v) = \int_G h(gv,gv) \, dg = \int_G h(g^{-1}gv,g^{-1}gv) \, dg = \int_G h(v,v) \, dg = h(v,v)\mu(G)$$

and as $\mu(G) > 0$, we have

$$h'(v,v) = 0 \iff h(v,v) = 0 \iff v = 0.$$

\[ \square \]

Theorem 48 (Maschke theorem for compact Lie groups). Every finite-dimensional continuous representation $\rho : G \to \text{GL}(V)$ is semisimple.

Proof. Exactly as in the finite group case, using now the invariant inner product above. \[ \square \]

Corollary 49. Let $G$ be compact and commutative. Then every finite-dimensional irreducible $G$-representation is 1-dimensional.

Proof. Works as in the finite group case: let $\rho : G \to \text{GL}(V)$ be a finite-dimensional representation: by commutativity of $G$, every $\rho(g) : V \to V$ is a $G$-invariant map, hence a scalar multiplication by Schur lemma. Irreducibility then implies $\dim V = 1$. \[ \square \]
**Corollary 50.** Let $G$ be connected, compact and commutative, i.e. $G = (S^1)^r$ is a torus as proved before. Then every finite-dimensional irreducible representation of $G$ is of the form

$$(z_1, \ldots, z_r) \mapsto z_1^{n_1} \cdots z_r^{n_r} \text{ for some } (n_1, \ldots, n_r) \in \mathbb{Z}^r.$$ 

**Proof.** Suppose $\rho : (S^1)^r \longrightarrow \mathbb{C}^\times$ is a finite-dimensional irreducible representation, which we know is one-dimensional by the corollary above. By compactness of $G$, it must map into $S^1 \subset \mathbb{C}^\times$, so we have the following commutative diagram using the universal covers of $G$ and $S^1$:

$$
\begin{array}{ccc}
\mathbb{R}^r & \xrightarrow{\tilde{\rho}} & \mathbb{R} \\
\downarrow{\pi^r} & & \downarrow{\pi} \\
(S^1)^r & \xrightarrow{\rho} & S^1
\end{array}
$$

Clearly $\tilde{\rho}$ must respect the map between the kernels of the two projections, i.e. there’s a map

$$\ker(\pi^r) = \mathbb{Z}^r \longrightarrow \mathbb{Z} = \ker(\pi)$$

and thus $\tilde{\rho}$ must be of the form

$$(f_1, \ldots, f_r) \mapsto f_1 n_1 + \ldots + f_r n_r \text{ for some integers } (n_1, \ldots, n_r) \in \mathbb{Z}^r.$$ 

$\square$

**Remark.** Most of the theory we will develop will come out by finding big tori in a compact Lie group $G$, so the result above is actually really useful.

### 9 April 19 (Daniel)

Recall the following facts from character theory and last time.

1. Maschke theorem;
2. Schur lemma: given $\rho_1, \rho_2$ irreducibles, we have
   $$\dim \text{Hom}_G(\rho_1, \rho_2) = \begin{cases} 
1 & \text{if } \rho_1 \cong \rho_2 \\
0 & \text{otherwise}
\end{cases}$$
3. $\xi_\rho(g) = \text{Tr}(\rho(g))$ is the character and then $\frac{1}{|G|} \sum_{g \in G} \xi_\rho(g) = \dim \rho^G$;
4. $\xi_{\rho^r} = \overline{\xi_\rho}$, $\xi_{\rho_1 \otimes \rho_2} = \xi_{\rho_1} \cdot \xi_{\rho_2}$ and $\xi_{\text{Hom}(\rho_1, \rho_2)} = \overline{\xi_{\rho_1}} \xi_{\rho_2}$;
5. $\text{Hom}(\rho_1, \rho_2)^G = \text{Hom}_G(\rho_1, \rho_2)$;
6. if we define $\langle f, h \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \overline{h(g)}$, as an inner product on class functions, then $\langle \xi_{\rho_1}, \xi_{\rho_2} \rangle = \dim \text{Hom}_G(\rho_1, \rho_2)$.

Today we’ll finish generalizing these facts to the compact Lie group case. On Monday, we constructed all the irreducible representations of $\text{SO}(3)$, today we’ll show that they are irreducible and that in fact every finite-dimensional irreducible representation of $\text{SO}(3)$ is isomorphic to one of those.
Proposition 51. Let $\xi : S^1 \to \mathbb{C}^\times$ be a representation, which we know factors through $\xi^1 \to \mathbb{C}^\times$. Then
$$\xi : z \mapsto z^n \text{ for some } n \in \mathbb{Z}.$$ 

Proof. We have the following commutative diagram
\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\text{Lie}(\xi)} & \mathbb{R} \\
\exp \downarrow & & \exp \downarrow \\
S^1 & \xrightarrow{\xi} & S^1
\end{array}
\]
where the map $\text{Lie}(\xi) : \mathbb{R} \to \mathbb{R}$ is a group map which extends to a group map between the kernels $\mathbb{Z} \to \mathbb{Z}$. Any such group map is 'multiplication by $n$' for some $n \in \mathbb{Z}$, hence $\xi(z) = z^n$. 

Proposition 52. Let $G_1, G_2$ be two groups satisfying Maschke theorem. Let $\pi_i : G_1 \times G_2 \to G_i$ be the two projections and set $\pi_i^* \rho = \rho \circ \pi_i$ where $\rho : G_i \to \text{GL}(V)$ is a $G_i$-representation. Then every irreducible representation of the product is of the form
$$\rho = \pi_1^* \rho_1 \otimes \pi_2^* \rho_2$$
for some irreducible representation $\rho_1, \rho_2$ of $G_1, G_2$. Moreover, such a $\rho$ satisfies Maschke theorem.

Proof. Let $\rho$ be an irreducible representation of $G_1 \times G_2$. Now $G_1$ is a subgroup of the product, so we consider $G_2 \rhd \text{Res}_{G_1 \times G_2}^G \rho$, which is a $G_1$-representation as $G_2$ and $G_1$ commutes with each other, as subgroups of $G_1 \times G_2$.

Now $G_1$ satisfies Maschke theorem, hence
$$\text{Res}_{G_1}^{G_1 \times G_2} \rho = \bigoplus G_1\text{-irreps}.$$ 

By Schur lemma, if two different irreducibles showed up, $G_2$ must also preserve them and then $\rho$ would but not irreducible. This proves that in the decomposition of $\text{Res}_{G_1}^{G_1 \times G_2} \rho$ there is only one isotypic component. We then have a map
$$\rho_2 : G_2 \to \text{Aut}_{G_1}(\text{Res}_{G_1}^{G_1 \times G_2} \rho) = \text{GL}_n(\mathbb{C})$$
and we also denote by $\rho_1$ one of the irreducible representation appearing in the decomposition of $\text{Res}_{G_1}^{G_1 \times G_2} \rho$, we showed above they are all isomorphic so we lose no generality in picking one.

We now claim that $\rho = \pi_1^* \rho_1 \otimes \pi_2^* \rho_2$. To prove it, choose a basis for the decomposition in irreducibles and define a map between the two representations, which turns out to be an isomorphism (exercise). 

Corollary 53. Any irreducible representation of $(S^1)^r$ is of the form
$$(z_1, \ldots, z_r) \mapsto z_1^{n_1} \cdots z_r^{n_r}.$$ 

Proof. Irreducible representations of $S^1$ are given by $\xi_n : z \mapsto z^n$, so the tensor product of the pullbacks $\pi_i^* \xi_{n_i}$ is exactly
$$(z_1, \ldots, z_r) \mapsto z_1^{n_1} \cdots z_r^{n_r}.$$
Recall that we have a covering 2 : 1 map

\[ \text{SU}(2) \rightarrow \text{SO}(3) \]

with kernel \( \{ \pm 1 \} \). Thus a representation of SO(3) is the same thing of a representation of SU(2) which kills \( \{ \pm 1 \} \). Recall also the standard representation \( \text{SU}(2) \rightarrow \text{GL}_2(\mathbb{C}) \).

**Claim 54.** Every finite-dimensional irreducible representation of \( \text{SU}(2) \) is isomorphic to \( \text{Sym}^n \rho \) for some \( n \in \mathbb{N} \). But which ones kill \( \{ \pm 1 \} \), thus becoming irreducible representations of \( \text{SO}(3) \)?

We can describe \( \text{Sym}^n \rho \) as the space of homogeneous degree \( n \) polynomials in two variables such that \( \text{SU}(2) \) acts by

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} = (az_1 + bz_2, cz_1 + dz_2).
\]

Then clearly \( \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \) acts trivially on \( \text{Sym}^n \rho \), so we can view these as representations of \( \text{SO}(3) \).

**Proof.** We now prove the claim, and we start by showing that each \( \text{Sym}^n \rho \) is an irreducible \( \text{SU}(2) \)-representation for \( n \geq 1 \). Consider the two tori of \( \text{SU}(2) \):

\[
T_1 = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in S^1 \right\} \quad \text{and} \quad T_2 = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}.
\]

Now we decompose \( \text{Sym}^n \rho \) in irreducible \( T_1 \)-representations: the action is given by

\[
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (t z_1, t^{-1} z_2)
\]

therefore

\[
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}
\begin{pmatrix} z_1^{n_1} z_2^{n_2} \\ z_1^{-n_1} z_2^{-n_2} \end{pmatrix} = \left( t^{n_1-n_2} z_1^{n_1} z_2^{n_2} \right).
\]

We then get

\[
\text{Sym}^n \rho \cong \bigoplus_{n_1,n_2 | n_1+n_2=n} \mathbb{C}(z_1^{n_1} z_2^{n_2}) \quad \text{as } T_1\text{-representation},
\]

because the underlying vector space of \( \text{Sm}^n \rho \) is given by the homogeneous polynomials in two variables of degree \( n \).

Now, to prove that \( \text{Sym}^n \rho \) is irreducible it is enough to show that no proper direct sum of those \( T_1 \)-weight spaces is stable under the action of \( T_2 \). Suppose some weight space \( V_{n_1,n_2} \) belongs to a subrepresentation \( W \subset V \), we have

\[
\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\begin{pmatrix} z_1^{n_1} z_2^{n_2} \\ z_1^{-n_1} z_2^{-n_2} \end{pmatrix} = \left( \cos \theta z_1 - \sin \theta z_2 \right)^{n_1} \left( \sin \theta z_1 + \cos \theta z_2 \right)^{n_2},
\]

by setting \( \theta = 0 \) we get the original monomial but varying \( \theta \) we get a nonzero intersection with any other weight space, thus \( W_{n_1,n_2} \neq 0 \) for every weight \( (n_1, n_2) \) which proves \( W = V \) and ultimately show irreducibility.

**Remark.** The proof above clearly illustrates the power of maximal tori!

**Theorem 55.** 1. \( \{ 1, \text{Sym}^n \rho \} \) are all the finite-dimensional irreducible representations of \( \text{SU}(2) \).
2. \(\{\text{Sym}^{2n}\rho\}\) are all the finite-dimensional irreducible representations of \(\text{SO}(3)\).

Clearly, (2) follows immediately from (1) and the \(2:1\) covering \(\text{SU}(2) \rightarrow \text{SO}(3)\).

Remark. \(\text{Sym}^{2n}\rho\) are isomorphic to the "spherical harmonic representations" we discussed on Monday.

**Definition 12.** Let \(\rho : G \rightarrow \text{GL}(V)\) be a finite-dimensional representation of a Lie group \(G\). Then
\[
\xi_\rho(g) := \text{Tr}(\rho(g))
\]
is the character of \(\rho\).

Remark. Clearly, \(\xi_\rho\) is a class function, and is smooth because taking the trace is a \(C^\infty\)-operation.

**Example 29.** Let \(\rho_1 : G \rightarrow \text{GL}(V_1)\) and \(\rho_2 : G \rightarrow \text{GL}(V_2)\) be representations, then \(\xi_{\rho_1 \otimes \rho_2} = \xi_{\rho_1} \xi_{\rho_2}\). This is clear, as for two matrices \(T\) and \(T'\) one has \(\text{Tr}(T \otimes T') = \text{Tr}(T) \cdot \text{Tr}(T')\).

**Proposition 56.** Let \(G\) be compact and \(\rho : G \rightarrow \text{GL}(V)\) be a finite-dimensional representation. Then \(\xi_{\rho^*} = \overline{\xi_\rho}\).

**Proof.** First of all, notice that the eigenvalues of \(\rho(g)\) have absolute value 1: if not, then without loss of generality (that is, up to replacing \(g\) with \(g^{-1}\), we can assume that \(\rho(g)\) has an eigenvalue \(\lambda\) with \(|\lambda| > 1\). Now the image of \(\rho\) is compact but \(\rho(g^n)\) has eigenvalue \(\lambda^n\) whose norm \(|\lambda^n|\) goes to \(+\infty\) as \(n \rightarrow +\infty\), and this is a contradiction.

Now \(\lambda \overline{\lambda} = 1\), hence \(\overline{\lambda} = \lambda^{-1}\) and thus \(\text{Tr}(\rho(g^{-1})) = \overline{\text{Tr}(\rho(g))}\), which is the claim. \(\square\)

Suppose from now on that \(G\) is a compact Lie group.

**Definition 13.** Rescale the bi-invariant measure \(dg\) such that \(\int_G dg = 1\), for example if \(G = S^1\) take \(dg = \frac{d\theta}{2\pi}\). Then for every representation \(\rho : G \rightarrow \text{GL}(V)\) we can define an *averaging operator*
\[
R(\rho) : V \rightarrow V \quad v \mapsto \int_G \rho(g).v \, dg,
\]
also called the Reynolds operator.

Observe that \(R(\rho)(V) \subset V^G\), the fixed part of the representation, by left invariant of the integral:
\[
g'.R(\rho)(v) = g'.\int_G \rho(g).v \, dg = \int_G \rho(g'g).v \, dg = \int_G \rho(g).v \, dg = R(\rho)(v).
\]
Furthermore, \(R(\rho)|_{V^G}\) is the identity on \(V^G\), thanks to the normalization of the measure.

**Proposition 57.**
\[
\int_G \xi_\rho(g) \, dg = \dim V^G.
\]

**Proof.** We have
\[
\int_G \text{Tr}(\rho(g)) \, dg = \text{Tr} \left( \int_G \rho(g) \, dg \right) = \text{Tr}(R(\rho))
\]
but by what we said above, \(R : V \rightarrow V^G\) is the projection operator on \(V^G\), hence its trace is exactly \(\dim V^G\). \(\square\)

We’re finally ready to generalize 6. of the original list.

**Corollary 58.** Apply the above to \(\rho = \text{Hom}(V,W)\): once we define
\[
\langle f, h \rangle = \int_G f(g)\overline{h(g)} \, dg
\]
then we get \(\langle \xi_V, \xi_W \rangle = \dim \text{Hom}_G(V,W)\).

**Proof.** Apply the previous proposition to \(\xi_V \xi_W = \xi_{\text{Hom}_G(V,W)}\). \(\square\)
Today we’re going to introduce Weyl’s unitary trick, which underlines the key role of compact connected Lie groups in the representation theory of general Lie groups.

Imagine a setting as follows: we may have two (real) Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ which become isomorphic once we extend scalars to $\mathbb{C}$:

$$\mathfrak{g} \cong \mathfrak{g}' \mathbb{C}.$$ 

Example 30. For $\mathfrak{su}(n)$ and $\mathfrak{sl}_n(\mathbb{R})$ we have the situation above, because $\text{SU}(n) \subset \text{SL}_n(\mathbb{C})$ implies $\mathfrak{su}(n) \subset \mathfrak{sl}_n(\mathbb{C})$ and in fact

$$\mathfrak{su}(n) \mathbb{C} \cong \mathfrak{sl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{R}) \mathbb{C}.$$ 

Details for $n = 2$ are worked out in Homework 4. Thus morally $\text{SU}(n)$ and $\text{SL}_n(\mathbb{R})$ have the same ”complexification” $\text{SL}_n(\mathbb{C})$.

The original Lie group may not be isomorphic: $G \neq G'$ and a natural question to ask ourselves is: what is the group version of ”complexification”? One should use algebraic geometry on $\mathbb{R}$ and $\mathbb{C}$, via linear algebraic groups.

We will instead discuss how to transform a representation of $G$ into a representation of $\mathfrak{g}$.

Theorem 59. Let $G$, $G'$ be two Lie groups with $G$ connected and simply connected (examples of such Lie groups can be found in Homework 5, for example $\text{SU}(n)$ and $\text{Sp}(n)$). Then

$$\text{Hom}(G, G') \cong \text{Hom}_{\text{Lie}}(\mathfrak{g}, \mathfrak{g}').$$

Proof. See the handout on the Frobenius theorem. In general the map above is always injective, but we need the topological conditions on $G$ to make sure it is also surjective. \qed

Example 31. Let $G' = \text{GL}_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then if $G$ is connected and simply connected we have an isomorphism

$$\text{Hom}(G, \text{GL}_n(\mathbb{K})) \cong \text{Hom}_{\text{Lie}}(\mathfrak{g}, \mathfrak{gl}_n(\mathbb{K})).$$

But $\mathfrak{gl}_n(\mathbb{K}) = \text{End}_\mathbb{K}(\mathbb{K}^n)$ thus a map $f : \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{K})$ is equivalent to a bilinear mapping $\mathfrak{g} \times \mathbb{K}^n \to \mathbb{K}^n$ where $[X,Y]$ acts as $f(X)f(Y) - f(Y)f(X)$, and this is exactly the definition of a Lie algebra representation! We then have

$$\rho \in \text{Hom}(G, \text{GL}_n(\mathbb{K})) \mapsto \left( g \ni X \mapsto \left( v \mapsto \frac{d}{dt} \rho(\exp tX)(v) \right) \right),$$

where $\frac{d}{dt} \rho(\exp tX) \in \text{End}_\mathbb{K}(\mathbb{K}^n)$. Clearly this is a statement about representations, thus taking isomorphism classes of $\mathbb{K}$-linear representations we get

$$\{\text{finite-dim., } \mathbb{K}\text{-linear group reps of } G\} \leftrightarrow \{\text{finite-dim., } \mathbb{K}\text{-linear Lie algebra reps of } \mathfrak{g}\}.$$ 

Take from now on $\mathbb{K} = \mathbb{C}$.

Fact 60. Suppose $G$ and $G'$ are complex Lie groups, with $G$ connected and simply connected. Then

$$\text{Hom}_{\text{hol}}(G, G') \cong \text{Hom}_{\text{C-Lie}}(\mathfrak{g}, \mathfrak{g}'),$$

that is we have an isomorphism between the space of holomorphic, complex Lie group homomorphisms from $G$ to $G'$ and the space of complex Lie algebras homomorphisms from $\mathfrak{g}$ to $\mathfrak{g}'$. 

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Take then $G' = \text{GL}_n(\mathbb{C})$, with a similar reasoning as the one above we get

\[ \{\text{finite-dim., holomorphic group reps of } G\} \leftrightarrow \{\text{finite-dim., } \mathbb{C}\text{-Lie algebra reps of } g\}. \]

From now on, denote the left hand side as $\text{Rep}^\mathbb{C}_\text{hol}(G)$ and the right hand side as $\text{Rep}^\mathbb{C}_\text{C-Lie}(g)$.

**Example 32.** Let $G = \text{SL}_n(\mathbb{C})$ and let us try to understand its representation theory. We have

\[ \text{Rep}^\mathbb{C}_\text{hol}(\text{SL}_n(\mathbb{C})) = \text{Rep}^\mathbb{C}_\text{C-Lie}(\text{sl}_n(\mathbb{C})) = \text{Rep}^\mathbb{C}_\text{C-Lie}(\text{sl}_n(\mathbb{R})_\mathbb{C}) = \text{Rep}^\mathbb{C}_\text{C-Lie}(\text{su}(n)_\mathbb{C}) = \text{Rep}^\mathbb{C}_\mathbb{R}\text{-Lie}(\text{su}(n)) \]

where the last equality comes from the fact that

\[ \text{su}(n) \rightarrow \text{gl}_N(\mathbb{C}) \]

is a real Lie algebras map if and only if the map we obtain via extension of scalars

\[ \text{su}(n)^{\mathbb{C}\text{-linear}} \rightarrow \text{gl}_N(\mathbb{C}) \]

is a complex Lie algebras map.

Notice that we also have $\pi_1(\text{SU}(n)) = 1$, thus

\[ \text{Rep}^\mathbb{C}_\mathbb{R}\text{-Lie}(\text{su}(n)) = \text{Rep}^\mathbb{C}(\text{SU}(n)). \]

Hence, we proved that the holomorphic, $\mathbb{C}$-linear representations of $\text{SL}_n(\mathbb{C})$ are equivalent to the $\mathbb{C}$-linear representation of its maximal compact subgroup $\text{SU}(n)$ (see Homework 5).

**Remark.** For compact Lie groups, we can use integration to decompose every representation into a direct sum of irreducible representations. Thus as the previous bijection is in fact a bijection of Hom-sets, we also proved that the finite-dimensional, $\mathbb{C}$-linear representations of $\text{SL}_n(\mathbb{C})$ and $\text{sl}_n(\mathbb{C})$ are completely reducible, even if we could not have directly used integration to prove it!

What we just did above is an example of the *unitary trick*. In fact, the trick works with $\text{SL}_n(\mathbb{C})$ replaced by any Zariski-closed subgroup $G \subset \text{GL}_n(\mathbb{C})$, as long as $G$ is Zariski-connected and has no normal nontrivial unipotent subgroups.

It remains to understand why the Lie algebra representation theory of $\text{sl}_n(\mathbb{C})$ is not too hard. We’ll treat the case $n = 2$ and see that $\text{sl}_2(\mathbb{C})$ and $\text{su}(2)$ in fact play a central role in the general case, so their understanding is very useful even for a general $n$.

**Remark.** Everything we said above is false over the reals, as in that case we have more serious constraints, for example

\[ \text{SL}_{2n+1} \rightarrow \text{deg}_{2n+1} \rightarrow \text{PGL}_{2n+1} \text{ is an isomorphism on } \mathbb{R}\text{-points.} \]

In Homework 4 one works out the irreducible representations of $\text{SU}(2)$, hence also of $\text{su}(2)$ and of $\text{sl}_2(\mathbb{C})$ by what we said above. In particular there exists a unique (up to isomorphism) irreducible representation $V_n$ of dimension $n + 1$ for each $n \geq 0$. We saw that we can realize $V_n = \text{Sym}^n((\mathbb{C}^2)^*)$ as the space of homogeneous, degree $n$ polynomials in two variables, with the action of $\text{SU}(2)$ given by the standard map

\[ \text{SU}(2) \hookrightarrow \text{SL}_2(\mathbb{C}) \subset \text{GL}(\mathbb{C}^2). \]
We want to rediscover this by working directly with $\mathfrak{sl}_2(\mathbb{C})$.
Using the representation theory of Lie algebras have some advantages, e.g. it works for every field $k$ of characteristic 0. Let then $k$ a field of characteristic 0,
\[ \mathfrak{sl}_2 = \mathfrak{sl}_2(k) = kX^- \oplus kH \oplus kX^+ \]
where
\[ X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]
and the brackets turn out to be
\[ [H, X^+] = 2X^+, \quad [H, X^-] = -2X^-, \quad [X^+, X^-] = -H. \]
Our goal is to describe the finite-dimensional, $k$-linear Lie algebra representations of $\mathfrak{sl}_2$.

Remark. Infinite-dimensional representations are totally different and we will not treat them.

let then $V$ be a nonzero, finite-dimensional $k$-linear representation of $\mathfrak{sl}_2$, i.e. a map of Lie algebras
\[ \rho : \mathfrak{sl}_2 \rightarrow \text{End}_k(V). \]

Remark. If $\mathfrak{g}$ is a Lie algebra over $k$, we will call $\mathfrak{g}$-module a $k$-linear representation of $\mathfrak{g}$ on a $k$-vector space, i.e.
\[ \mathfrak{g} \rightarrow \text{End}_k(V) \text{ with } \rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X) \forall X, Y \in \mathfrak{g}. \]

We can creat the universal enveloping algebra $U(\mathfrak{g}) = T(\mathfrak{g})/I$ where $T(\mathfrak{g})$ is the tensorial algebra over $\mathfrak{g}$ and $I = (X \otimes Y - Y \otimes X - [X, Y])$. Then $U(\mathfrak{g})$ is an associative algebra, and a Lie algebra with the bracket given by commutators. For semisimple Lie algebras $\mathfrak{g}$, $U(\mathfrak{g})$ has nice properties.

Fact 61. Giving a Lie algebra map
\[ \mathfrak{g} \rightarrow \text{gl}(V) \]
is equivalent to endowing $V$ with a left $U(\mathfrak{g})$-module structure (over $k$).

Proposition 62. Let $V$ be an $\mathfrak{sl}_2$-module of dimension $d > 0$. Then $\ker(X^+) \neq 0$ and contains a nonzero $v_0$ such that $H.v_0 = mv_0$ for some $m \in \mathbb{Z}_{\geq 0}$.

Pick such a $v_0$ and consider the submodule it generates $V_0 = \mathfrak{sl}_2 \cdot v_0$. Then
1. $\ker(X^+|_{V_0}) = kv_0$.
2. Defining
\[ v_j = \frac{(-1)^j}{j!}(X^-)^j.v_0 \text{ for every } 0 \leq j \leq m, \]
then $kv_j$ is an eigenline for $H$ with eigenvalue $m - 2j$: we call these eigenvalues the weights.
3. \[ V_0 = \bigoplus_{j=0}^{m} kv_j \]
and thus $\dim V_0 = m + 1$.  

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4. The actions of the basis elements work as follows:

\[ X^-(v_j) = -(j + 1)v_{j+1}, \quad X^+(v_j) = (m - j + 1)v_{j-1}, \quad H(v_j) = (m - 2j)v_j, \]

in particular \( H \) preserves every eigenline, \( X^- \) sends \( kv_j \) to \( kv_{j+1} \) and \( X^+ \) sends \( kv_j \) to \( kv_{j-1} \).

**Proof.** See the handout.

**Corollary 63.** Let \( V \) be a finite-dimensional, irreducible representation of \( \mathfrak{sl}_2 \). Then \( V \cong V_0 \) constructed as above, and thus \( V \) is uniquely determined (up to isomorphism) by its dimension \( \dim V \).

**Corollary 64.** Every finite-dimensional representation is self-dual.

**Remark.** The highest weight of a representation \( V \) is \( m = \dim V - 1 \). Moreover, for every \( m \geq 0 \) we can construct an irreducible representation of \( \mathfrak{sl}_2 \) of dimension \( m + 1 \) using degree \( m \) homogeneous polynomials in two variables.

What we described above is in fact a general phenomenon: on every semisimple Lie algebra there exists a commutative subalgebra (called Cartan subalgebra) acting on it, and this give weights.

**Remark.** Using the fact \( \text{SO}(3) \cong \text{SU}(2)/\pm 1 \), one can recover the \( \mathbb{C} \)-linear irreducible representations of \( \text{SO}(3) \), and relate them to "spherical-harmonic functions" and Legendre polynomials.

### 11 April 24

Today we will see some examples and applications regarding the following conjugacy theorem, due to Weyl:

**Theorem 65** (Conjugacy theorem). Let \( G \) be a connected compact Lie group. Then

1. All maximal tori \( T \subset G \) are conjugate to each other.
2. Every \( g \in G \) lies in a maximal torus. In particular, there are many such maximal tori.

**Remark.** By a dimension argument and connectedness of \( G \), the first claim immediately implies that any torus \( T_0 \subset G \) lies in a maximal one, via a Lie algebra argument: once \( \dim \text{Lie}(T) \) is maximal for \( T \) a torus, then \( T \) is maximal.

In fact, the proof of the theorem shows that \( \bigcup_{g \in G} gTg^{-1} = G \), where \( T \) is a fixed maximal torus.

**Remark.** On Homework 4, the above statement is proved directly for \( \text{SU}(n) \).

The proof of the theorem uses \( G \)-invariant integration on \( G/T \), so we need to discuss \( G \)-invariant integration on the coset space \( G/H \) when \( H \) is a closed subgroup. This is done in details in Homework 5.

**Remark.** It could happen that \( G/H \) is not orientable, and in such cases there cannot exist such an invariant integral. But when \( H \) is compact and connected, everything works.

We will discuss the theorem next time, today we show some applications and introduce the Weyl group.

For the rest of this lecture, let \( G \) be a compact connected Lie group.
Corollary 66. 1. If $T \subset G$ is a torus, then $T$ is maximal if and only if $Z_G(T) = T$ (i.e. $T$ is self-centralizing).

2. The center of $G$ is

$$Z_G = \bigcap_{\text{maximal tori}} T.$$ 

Proof. We first prove the second claim. Choose $z \in Z_G$, by the theorem we have $z \in T_0$ for some maximal torus $T_0$. As all the maximal tori are conjugate, we have

$$z = g_0 z g_0^{-1} \in g_0 T_0 g_0^{-1} = T$$

for any conjugate of $T_0$, i.e. any maximal torus. Hence

$$Z_G \subseteq \bigcap_{\text{maximal tori}} T.$$ 

Vice versa, let $g \in T$ for any maximal torus $T$ and take any $g' \in G$. Now $g'$ lies in some maximal torus $T_0$, which is commutative and also contains $g$, thus $gg' = g'g$, that is $g$ commutes with $g'$. Hence

$$Z_G \supseteq \bigcap_{\text{maximal tori}} T$$

thus proving the claim.

Now we consider the first claim. As any torus $T$ is commutative, it is obvious that if $Z_G(T) = T$, then $T$ must be maximal as any torus containing $T$ would also centralize it. Vice versa, suppose $T$ is maximal and choose $g \in Z_G(T)$, we want to show $g \in T$.

Claim 67. Any $g \in G$ satisfies $g \in Z_G(g)^0$.

For example, consider $g = (-1, e^{2\pi i \theta}) \in S^1 \times S^1$ for $\theta \notin \mathbb{Q}$, then $(g)^0 = \{1\} \times S^1 \not\subset g$: this shows it is not obvious, in general, that any element is in the center of a connected closed subgroup.

Back to our corollary. As $T$ is commutative, $T \subset Z_G(g)$ and as $T$ is connected we have in fact $T \subset Z_G(g)^0$. Now we apply the second claim of the corollary, which we already proved, to the compact connected group $Z_G(g)^0$: this says

$$Z_{Z_G(g)^0} = \bigcap T$$

as $T$ runs among the maximal tori in $Z_G(g)^0$.

Obviously $T$ is also a maximal tori in $Z_G(g)^0$, and thus

$$g \in Z_{Z_G(g)^0} \subset T,$$

proving the opposite direction. 

Remark. The key point of the proof is of course given by the claim, which is quite deep. The idea is that if $g \notin Z_G$, then obviously $\dim Z_G(g)^0 < \dim G$ and we often use inductive arguments on the dimension. Moreover, it is a general technique to take centralizers of subgroup and impose connectedness.

We are also interested in the case of non-maximal tori.

Corollary 68. Let $S$ be any torus in $G$. Then $Z_G(S)$ is connected, and if $S \neq \{1\}$, then

$$\dim Z_G(S)/S < \dim G.$$
Proof. Choose \( g \in Z_G(S) \), we want to show \( g \in Z_G(S)^0 \). As \( S \) is commutative and connected, it obviously centralizes \( g \) hence \( S \subset Z_G(g)^0 \) with obviously \( g \) central in \( Z_G(g)^0 \). Clearly we have \( S \subset S' \) for some maximal torus \( S' \) of \( Z_G(g)^0 \), hence \( g \in S' \) by the previous corollary (part 2 applied to \( Z_G(g)^0 \)). As \( S' \) is commutative, and contains \( S \), we have \( S' \subset Z_G(S) \), and as \( S' \) is connected, we have \( S' \subset Z_G(S)^0 \). Hence \( g \in S' \subset Z_G(S)^0 \) and the claim is proved. The second part of the claim follows trivially, as no nontrivial torus has zero dimension. \( \square \)

Remark. This corollary underlines the key point that the centralizer \( Z_G(S) \) of a torus \( S \) is already connected, and thus in any inductive argument when we take centralizers we do not really need to impose connectedness.

Corollary 69. Let \( f : G \longrightarrow \tilde{G} \) be a surjective Lie groups map. Suppose \( f \) maps the torus \( S \subset G \) onto the torus \( \tilde{S} \). Then \( Z_G(S) \) surjects onto \( Z_{\tilde{G}}(\tilde{S}) \).

In particular, if \( S \subset G \) is maximal, then \( S = Z_G(S) \) by a previous corollary, so that \( Z_{\tilde{G}}(\tilde{S}) = \tilde{S} \) and \( \tilde{S} \) is maximal too.

Remark. The striking feature of this corollary is that usually centralizers does not behave well under lifting, but they do in case of tori.

Proof. First of all, notice that the image of \( S \) under a surjective Lie group map is necessarily a torus, because the image is connected and compact (as \( f \) is continuous) and is a commutative subgroup (as \( f \) is a group map). Certainly \( Z_G(S) \) maps into \( Z_{\tilde{G}}(\tilde{S}) \), as \( f \) is surjective. Now \( Z_{\tilde{G}}(\tilde{S}) \) is connected, so by Homework 3, exercise 5.(iii), the map

\[
 f : Z_G(S) \longrightarrow Z_{\tilde{G}}(\tilde{S})
\]

is surjective if and only if it is surjective on Lie algebras, where to prove the ”if” part we use the submersion theorem, the exponential map and connectedness of the centralizers. We are then in the following situation

\[
 \begin{array}{ccc}
 \mathfrak{g} & \overset{\text{Lie}(f), \text{su}}{\longrightarrow} & \mathfrak{g}' \\
 \downarrow & & \downarrow \\
 \text{Lie}(Z_G(S)) & \overset{\text{Lie}(f|_{Z_G(S)})}{\longrightarrow} & \text{Lie}(Z_{\tilde{G}}(\tilde{S}))
\end{array}
\]

and we want to prove the surjectivity of the horizontal lower map.

In section 4 of the handout on the Frobenius theorem, we are given the following

Theorem 70. Let \( H \subset H' \) be Lie groups, with \( H \) a connected closed subgroup. Then

\[
 \text{Lie}(Z_{H'}(H)) = (\mathfrak{h}')^H
\]

where the action \( H \sim \mathfrak{h}' \) is the adjoint action.

In our case, we have

\[
 S \sim \mathfrak{g} \rightarrow \mathfrak{g} \sim \tilde{S} \text{ where both actions are the adjoint actions,}
\]

and we want to show we have the surjectivity \( \mathfrak{g}^S \rightarrow \mathfrak{g}^{\tilde{S}} \). Viewing \( \mathfrak{g} \) as a \( S \)-representation via \( S \rightarrow \tilde{S} \), we find that the surjection \( \mathfrak{g} \rightarrow \mathfrak{g} \) is in fact a map of \( S \)-representations, so passing to the Lie algebras we want the map between the invariant submodules \( \mathfrak{g}^S \rightarrow \mathfrak{g}^{\tilde{S}} \) to be surjective. This follows from the following lemma. \( \square \)
Lemma 71. Let $S$ be a torus, $V$ and $V'$ be finite-dimensional representations of $S$ over the reals. Then any $S$-equivariant surjection $V' \twoheadrightarrow V$ induces a surjection between the $S$-invariant subspaces: $(V')^S \twoheadrightarrow V^S$.

Proof. Extending scalars, we have $(V^S)_\mathbb{C} = (V_S)^S \subset V_S$, so it suffices to treat the case for $\mathbb{C}$-linear representations. We know that any $\mathbb{C}$-linear representation of $S$ is completely reducible, so we have

$$V' = \bigoplus_i \mathbb{C}(\xi'_i)^{\oplus e_i}$$

as the sum varies among 1-dimensional representations

$$\xi'_i: S \to S' \subset \mathbb{C}^\times$$
denoted $\mathbb{C}(\xi'_i)$

and the isotypic component is denoted $\mathbb{C}(\xi'_i)^{\oplus e_i}$. Similarly, we decompose

$$V = \bigoplus_j V_{\xi_j}$$
in its isotypic components.

As the formation of any $\xi$-isotypic piece is functorial, any given isotypic piece of $V'$ can only map into the same isotypic piece in $V$. In fact, for any irreducible representation $\xi$ of $S$, we have that $V'_\xi$ maps into $V_\xi$:

$$V' \twoheadrightarrow \bigoplus V'_\xi$$
$$\quad \downarrow f \quad \downarrow \oplus f_\xi$$
$$V \twoheadrightarrow \bigoplus V_\xi$$

As $V' \twoheadrightarrow V$ is surjective, so must be $\oplus f_\xi$ which means every $f_\xi$ is surjective. Taking then $\xi = 1$ the trivial representation, we have

$$V'_1 = (V')^S \twoheadrightarrow V^S = V_1$$

and the claim is proved. \qed

Remark. We showed that for every connected Lie group $H$ whose $\mathbb{C}$-linear finite-dimensional representation theory is completely reducible, if the situation is as in the following diagram:

$$G \xrightarrow{f, \text{su}} \rightarrow G$$
$$\downarrow \quad \downarrow \text{f|}_H, \text{su} \quad \downarrow$$
$$H \xrightarrow{} \rightarrow H$$

then

$$Z_G(H)^0 \twoheadrightarrow Z_G(H)^0$$
is also surjective.

The key point of the lemma lies in representation theory and Schur lemma, not in Lie theory.

In general, we’ll see that for any torus $S$ in a compact, connected Lie group $G$, we have the Weyl group of $S$

$$W(G, S) = N_G(S)/Z_G(S),$$
a finite group.

As $Z_G(S)$ is connected by a corollary above, the finiteness of the quotient is equivalent to $Z_G(S) = N_G(S)^0$.

When $T$ is a maximal torus, the Weyl group of $T$, $W(G, T) = N_G(T)/T$, is called the Weyl group
of $G$, because as any two maximal tori are conjugate, their Weyl groups are isomorphic. Consider the character lattice of $T$:

$$X(T) = \text{Hom}(T, S^1) = \text{Hom}((S^1)^r, S^1) \cong \mathbb{Z}^r,$$

the Weyl group acts faithfully on it. For example

$$W(U(n), T) = S^n \sim \mathbb{Z}^n$$

in the natural way.

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Let $G$ be a compact, connected Lie group, $S \subset G$ any torus. Today we want to prove that all tori are conjugate.

Recall that we had the Weyl group $W(G, S) = N_G(S)/Z_G(S)$ which acts on $S$ by conjugation, and this action is faithful because we quotiented modulo the centralizer of $S$. Thus

$$W(G, S) \cong \text{Aut}(S).$$

**Example 33.** Let $G = U(n) \subset \text{GL}_n(\mathbb{C})$, and take $T = (S^1)^n$ the diagonal maximal torus. Permutation matrices normalize the torus as they swap diagonal entries, and in fact we find

$$N_G(T) = S_n \times T$$

so we have a copy of $N_G(T)/T$ inside $N_G(T)$! Notice moreover that $S_n$ is finite, and $\{1\} \times T = N_G(T)^0$.

**Remark.** 1. For $SU(2)$ we get $W(G, T) = S_2$ using the same reasoning as above. In general for $SU(n)$ we find $W(G, T) = S_n$ but already when $n = 2$ we cannot lift the Weyl group into $N_G(T)$ as a group (see handout on Weyl groups), because in fact we cannot find representatives for $W(G, T)$ in $SU(n)$ having the same orders of the group elements of $W(G, T)$ that they represent.

2. We always have that $W(G, S)$ is finite, because this is equivalent to $Z_G(S) = N_G(S)^0$. Thus the finiteness of the Weyl group rests on the conjugacy theorem and the connectedness of the centralizer of a torus.

Ultimately, the Weyl group $W(G, T)$ is finite because of the compactness of the torus $T$ and the discreteness of the character lattice

$$X(T) = \text{Hom}(T, S^1) \cong \mathbb{Z}^r.$$ 

In general,

$$W(G, S) \cong \text{Aut}(S) \cong \text{GL}(X(S))^{opp} = \text{GL}_r(\mathbb{Z})^{opp}$$

thus the Weyl group is always “made of permutations”.

**Remark.** In general, we cannot embed $W(G, S) \hookrightarrow G$ as a subgroup! $G = U(n)$ is a particular case when this is possible, but for $G = SU(2)$ it is impossible, as explained above.

We now proceed to build up for the proof of the conjugacy theorem.
12.1 Invariant integration on homogeneous spaces for Lie groups

**Definition 14 (Homogeneous space).** We call a *homogenous space* a manifold $X$ where a Lie group $G$ acts transitively, so that we have $X = G/\text{Stab}_G(x)$ for some $x \in X$.

Let $G$ be a Lie group, $H \subset G$ a closed subgroup, and choose left-invariant measures $\mu_G, \mu_H$ on $G$ and $H$. We build them by choosing nonzero, left-invariant smooth differential forms $\omega$ on $G$ and $\nu$ on $H$ and then setting

$$\mu_G(\Sigma) := \int_\Sigma \mid \omega \mid \text{ for } \Sigma \subset G$$

and

$$\mu_H(\Sigma') := \int_{\Sigma'} \mid \nu \mid \text{ for } \Sigma' \subset H.$$  

Notice that the orientation is not an issue as we are integrating the absolute value of the differential form, but in doing so we lose linearity of the map $\omega \mapsto \mu_G$.

The following question arises naturally: can we find a $G$-invariant measure $\bar{\mu}$ on $G/H$ such that some sort of Fubini theorem holds, e.g. for $f \in C_c(G)$, defining

$$\bar{f}(\bar{g}) := \int_H f(gh) \, d\mu_H(h) \text{ where } g \text{ is a representative of the coset } \bar{g} = gH$$

gives the identity

$$\int_G f(g) \, d\mu_G(g) = \int_{G/H} \bar{f}(\bar{g}) \, d\bar{\mu}(\bar{g})?$$

Notice that in homework 5 it is proved that such a $\bar{f}$ is in $C_c(G/H)$.

In fact, we’d like such an identity as the last one to hold for any $L^1$ function.

Even better, we want to find a nonzero, $G$-invariant, smooth, top-degree differential form $\bar{\omega}$ on $G/H$, then we want to define

$$\bar{\mu}(\bar{\Sigma}) := \int_{\bar{\Sigma}} \mid \bar{\omega} \mid$$

and make sure the identity above holds for such a definition of the measure $\bar{\mu}$.

**Remark.** There may be no $G$-invariant form, but we can nonetheless find $G$-invariant forms up to a sign. Then the measure $\bar{\mu}$ defined above turns out to be well-defined and $G$-invariant. Anyway, when $G$ is connected and compact everything works fine, and we can produce a nonzero, $G$-invariant, top-degree, smooth differential form. Homework 5 addresses this issue in details.

**Example 34.** If $\mu_G$ is right-invariant but $\mu_H$ is not, then there cannot be such a $\bar{\mu}$. For instance take $G = \text{GL}_2(\mathbb{R})$, $H = \{(t \, 0 \, 0 \, t^*)\}$ then there is no such $\bar{\mu}$. But now consider $G' = \text{SL}_2(\mathbb{R})$, $H' = \{(t \, 0 \, 0 \, t^*)\}$, then as cosets spaces we have $G/H \cong G'/H'$ but as $H'$ is commutative, there now exists a $\bar{\mu}'$ satisfying our requests.

The key point of this example is that $G'/H'$ has a $\text{SL}_2(\mathbb{R})$-invariant measure, but on $G/H$ we can act with $(t \, 0 \, 0 \, t^*) \notin \text{SL}_2(\mathbb{R})$, so that $(t \, 0 \, 0 \, t^*)$ acts on $G'/H'$ by scaling with $\frac{1}{t}$, thus losing left-invariance.

**Example 35.** Let $G$ be compact, connected. We may ask for a differential form $\bar{\omega}$ on $G/H$ to have all the properties as above. Then such a $\bar{\omega}$ exists if and only if $G/H$ is orientable. The last condition may fail to happen!

In doing some integration, we will need to keep track of orientation (locally): this is why we really want the invariant differential form and not just the invariant measure, as the latter makes us lose any notion of orientation.
Example 36. Let $G = \text{SO}(n) \supset \text{SO}(n-1)$. Then for $n \geq 3$ we have
\[
\left\{ \left( \text{SO}(n-1) \atop 1 \right) \right\} \subset \left\{ \left( g \in \text{O}(n-1) \atop \frac{1}{\det g} \right) \right\} \subset \text{SO}(n)
\]
thus
\[
\text{SO}(n)/\text{SO}(n-1) \xrightarrow{\cong} S^{n-1} \quad g \mapsto g(e_n)
\]
and the quotient $\mathbb{Z}_2 \cong \text{O}(n-1)/\text{SO}(n-1)$ acts via the antipodal map on $S^{n-1}$, so that
\[
\text{SO}(n)/\text{O}(n-1) \cong \mathbb{R}^{p,n-1} \text{ which is not orientable for } n \geq 3.
\]
The issue is due to the fact that taking $G = \text{SO}(n)$ and $H = \text{O}(n-1)$ leaves us with a non-connected $H$.

Now suppose we are given the differential forms $\omega$ on $G$ and $\nu$ on $H$: how can we cook $\bar{\omega}$ on $G/H$?

The idea is the following: we seek a preferred $\bar{\omega}(\bar{e}) \in \det(T^*_\bar{e}(G/H))$ (where $\det V$ is a notation for the top-degree exterior power of the vector space $V$) such that $H$ acts invariant on $\bar{\omega}$. We should define $\bar{\omega}(\bar{e})$ in terms of $\omega(e) \in \det(T^*_e(G))$ and $\nu(e) \in \det(T^*_e(H))$ and then (this is the subtlety) $G$-translate it, to get $\bar{\omega}$ on $G/H$.

Suppose then $X \xrightarrow{f} Y$ is a submersion of manifolds (for example $G \rightarrow G/H$). Given $x \mapsto f(x) = y$, the submersion theorem says that there is a small neighborhood $x \in U \subset X$ such that $U \cong Y \times V$ so we have
\[
0 \rightarrow T_x(X_y) \rightarrow T_x(X) \rightarrow T_y(Y) \rightarrow 0,
\]
where $X_y$ is the fiber above $y \in Y$. In the case of a submersion $G \rightarrow G/H$ we get the exact sequence
\[
0 \rightarrow T_e(H) \rightarrow T_e(G) \rightarrow T_e(G/H) \rightarrow 0.
\]
Now every exact sequence of vector spaces
\[
0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0
\]
gives canonically an isomorphism among the top exterior powers
\[
\det(V) \cong \det(V') \otimes \det(V'')
\]
and similarly for the dual spaces. So in our case we get
\[
\Omega^\text{top}_G(e) \cong \Omega^\text{top}_H(e) \otimes \Omega^\text{top}_{G/H}(\bar{e}),
\]
so picking $\omega(e) \in \Omega^\text{top}_G(e)$ and $\nu(e) \in \Omega^\text{top}_H(e)$ identifies uniquely a single element of $\Omega^\text{top}_{G/H}(\bar{e})$, and we define this element to be $\bar{\omega}(\bar{e})$.

How to propagate $\bar{\omega}(\bar{e})$ to $G/H$? The left $H$-action fixes $\bar{e} = H \in G/H$, thus we have an $H$-action on $T^*_\bar{e}(G/H)$, that is
\[
\text{Ad}_{G/H} : h \mapsto d l^*_{h^{-1}}(\bar{e}) \in \text{GL}(T^*_\bar{e}(G/H)).
\]
Now the determinant of this representation
\[
\det(\text{Ad}_{G/H}) : H \rightarrow \mathbb{R}^x
\]
must be the trivial map, because we want $H$ to fix $\bar{\omega}(\bar{e})$. In homework 5, we show that this necessary condition is also sufficient to make sure that $H$ fixes $\bar{\omega}(\bar{e})$. 38
Remark. For $H$ compact and connected, the condition above is verified because the map $\det(\text{Ad}_{G/H})$ is clearly continuous and a group map. For $H$ compact but disconnected, we can have a quadratic character, as in the example above, so that the image of $\det(\text{Ad}_{G/H})$ is $\{\pm 1\}$.

In our main examples, $G$ will be compact and $H$ connected and closed, thus compact. So everything will work out just fine.

We finally start the proof of the conjugacy theorem: the important tool which we can now use is the Fubini theorem.

Remark. For $G$ compact and connected, $H$ a connected, closed subgroup, the existence of $\bar{\omega}$ implies that $G/H$ is orientable.

Let’s first recall the theorem:

**Theorem 72 (Conjugacy theorem).** Let $G$ be a connected compact Lie group. Then all maximal tori are conjugate, and every $g \in G$ lies in one of them.

**Proof.** We will study $w : (G/T) \times T \to G$, $(\bar{g}, t) \mapsto gtg^{-1}$ for $g$ a representative of the coset $\bar{g} \in G/T$.

Choose orientations, and fix $dg$ on $G$, $dt$ on $T$ to get $d\bar{g}$ on $G/T$. We have now a smooth map $q$ between oriented, connected manifolds of the same dimension, and to prove the second part of the theorem it is clearly enough to show that $q$ is surjective.

**Claim 73.** The first part of the theorem follows also by the surjectivity of $q$.

This follows from the following

**Lemma 74.** For any torus $T'$, there exists $t' \in T'$ such that the subgroup generated $(t')$ is dense in $T'$.

**Proof.** The textbook covers the details in chapter I, theorem 4.13, we present the idea. When $T = S^1$, take $t' = e^{2\pi i \theta}$ for some irrational $\theta$. In general we have to pick $\dim T$ angles that are $\mathbb{Q}$-linearly independent. An alternative proof is in the handout.

Now we prove the claim, that is given another maximal torus $T'$ we want to show $T' \subset gTg^{-1}$ for some $g \in G$. Choose then $t' \in T'$ such that $(t')$ is dense in $T'$, by the surjectivity of $q$, we have $t' = gtg^{-1}$ for some $g \in G$, $t \in T$. In particular $t' \in gTg^{-1}$ which is a closed subgroup, hence

$$T = \langle t' \rangle \subset gTg^{-1}$$

proving the first part of the theorem.

It remains to prove that $q$ is surjective. In general, there is a criterion to check the surjectivity for a smooth map between same-dimensional, orientable manifolds $M \to M'$ when $M$ and $M'$ are connected and compact. Let’s unfold this criterion.

We have the pullback maps on differential forms:

$$
\begin{array}{ccc}
H^d(M', \mathbb{Z}) & \xrightarrow{f^*} & H^d(M, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^d(M', \mathbb{R}) & \xrightarrow{f^*} & H^d(M, \mathbb{R}) \\
\cong & & \cong \\
H^d_{dR}(M') & \xrightarrow{f^*} & H^d_{dR}(M)
\end{array}
$$
where the isomorphisms $H^d(M, \mathbb{R}) \cong H^d_{dR}(M)$ and $H^d(M', \mathbb{R}) \cong H^d_{dR}(M')$ are due to the features of de Rham cohomology.

As $d = \dim M = \dim M'$, the top degree cohomology is one-dimensional so
\[ \mathbb{Z} \cong H^d(M', \mathbb{Z}) \xrightarrow{f^*} H^d(M, \mathbb{Z}) \cong \mathbb{Z} \]
is a group map from $\mathbb{Z}$ to $\mathbb{Z}$, hence is multiplication by some number $m \in \mathbb{Z}$ and we define the degree of $f$ to be that number:
\[ \deg(f) := m. \]
Thus,
\[ \int_{M'} f^*(\omega) = \deg(f) \int_M \omega \] see theorem 5.19, chapter I of the textbook.
Moreover if $\deg(f) \neq 0$ we must have $f(M) = M'$, or otherwise choosing a differential form $\omega$ supported in $M' - f(M)$ would give a contradiction to the previous equality between integrals.

Next time we will compute $\deg(q)$ which turns out to be $|W(G, T)|$.

13 April 29

Today we will prove the conjugacy formula.

Remark. It is easy to see that if $G$ is compact, connected and nontrivial, then there exists a nontrivial torus $T \subset G$.

Proof. Choose $0 \neq X \in \mathfrak{g}$ and get a 1-parameter subgroup $\alpha_X : \mathbb{R} \rightarrow G$ with $\alpha'_X(0) = X \neq 0$. Take then $T = \alpha_X(\mathbb{R}) \subset G$: $T$ is connected, closed (hence compact) and commutative (because closure preserves commutativity), thus $T$ is a torus and is clearly non trivial as $\alpha_X \neq 1$.  

The problem with this proof is that we have no clue on the internal structure of $T$.

Let now $G$ be a compact, connected Lie group and $T \subset G$ a maximal torus (i.e. a torus not strictly contained in a larger one). We had the map
\[ q : (G/T) \times T \xrightarrow{\text{C}^\infty} G \quad (\bar{g}, t) \mapsto g t g^{-1} \]
and we introduced an integer invariant called degree of a map. The last time we also showed that in order to prove the conjugacy theorem it only remains to prove that $q$ is surjective. We don’t know (yet) that $Z_G(T) = T$ but one can show (by elementary means) that $N_G(T)^0 = T$, which is equivalent to $N_G(T)/T$ being finite as $N_G(T)$ is compact, so the open connected identity component must be of finite index (see for example the handout on the Weyl group). Denote temporarily $W = N_G(T)/T$.

Recall the following setup: given a smooth map $M' \xrightarrow{f} M$ between oriented, compact, connected, smooth manifolds of the same dimension $d > 0$, there are pullback maps induced on different cohomologies:
where by Poincaré’ duality \( H^d(M, \mathbb{Z}) \) is a free \( \mathbb{Z} \)-module of rank 1, and the orientation on \( M \) specifies an isomorphism with \( \mathbb{Z} \), and similarly for \( M' \). In particular, \( H^d(M, \mathbb{Z}) \) is a lattice in \( H^d(M, \mathbb{R}) \), so when we apply the maps in the last row we have \( \omega \mapsto f^*\omega \) and integrating both gives us
\[
\int_{M'} f^*\omega = \deg(f) \int_{M} \omega
\]
for a unique \( \deg(f) \in \mathbb{Z} \).

The existence of such an integer is proved in the textbook or it’s an easy consequence of a diagram-chasing argument.

**Claim 75.** Suppose \( \deg(f) \neq 0 \). Then \( f \) is surjective.

**Proof.** Suppose not, then take the open set \( U := M - f(M') \neq \emptyset \). Fix a coordinate ball \( B \) contained in \( U \) and a differential form \( \omega \) supported in the ball so that \( \int_B \omega \neq 0 \). Clearly the pullback gives \( f^*\omega \equiv 0 \) identically, so the previous equality \( \int_{M'} f^*\omega = \deg(f) \int_{M} \omega \) fails, and this is a contradiction. \( \square \)

We will show that \( \deg(q) = |W| \neq 0 \). First of all, we give a hands-on way to describe \( \deg(f) \) in terms of covering spaces.

**Lemma 76.** Let \( M' \xrightarrow{f} M \) be as above, and suppose \( \exists m_0 \in M \) such that the fiber \( f^{-1}(m_0) \) is finite, and around every \( m'_i \in f^{-1}(m_0) \) \( f \) is a local diffeomorphism. Then

1. There is a connected open set \( U \subset M \) containing \( m_0 \) such that \( f^{-1}(U) \xrightarrow{f} U \) is a topological covering map of finite degree.

2. Let \( \{m'_i\}_{i=1}^n = f^{-1}(m_0) \), then
\[
\deg(f) = \sum_{i=1}^n \varepsilon(m'_i)
\]
where \( \varepsilon(m'_i) = \pm 1 \) records if
\[
d f(m'_i) : T_{m'_i}(M') \longrightarrow T_{m_0}(M)
\]
is orientation-preserving or not.

Note that the first claim tells us that \( f^{-1}(U) = \bigcup_{i=1}^n U'_i \) is a finite union of open sets in \( M' \), each diffeomorphic to \( U \) through \( f: U'_i \xrightarrow{f} U \). So when we integrate \( \int_U \omega \) and \( \int_{U'_i} f^*\omega \), the only issue that can avoid the two integrals to be equal is a switch in the orientation. This explains the second claim.

**Proof.** As \( M' \) is Hausdorff, we can choose pairwise disjoints open sets \( m'_i \in V'_i \subset M' \) and by the inverse function theorem for each fixed \( i \) there exists an open subset \( V''_i \subset V'_i \) containing \( m'_i \) where \( f \) is a local diffeomorphism onto an open neighborhood \( U \) of \( m_0 \). Note that we already shrunk the \( V''_i \) to make sure each one is diffeomorphic to the same open neighborhood \( U \subset M \), nonetheless we could have \( f^{-1}(U) \not\subset \bigcup_i V''_i \).

To fix this, we use compactness: \( M' - (\bigcup V''_i) \) is a closed set in \( M' \), hence compact, and it misses \( f^{-1}(m_0) \). Thus its image \( Z \subset M \) is compact, hence closed (\( M \) is Hausdorff) and does not contain
$m_0$. Pick then $M - Z$: this is an open neighborhood of $m_0$ and its preimage is contained in $\sqcup V''_i$, thus replacing $U$ with $\tilde{U} = U \cap (M - Z)$ gives $V''_i \not\supset \tilde{U}$ for every $i$, and hence proves the first claim.

Now we prove the second claim: choose $\omega$ supported in a coordinate ball inside $\tilde{U}$ with
\[
\int_M \omega = \int_{\tilde{U}} \omega = 0.
\]
We also have
\[
\int_{M'} f^* \omega = \sum_{i=1}^n \int_{V''_i} f^* \omega
\]
where the orientation on $V''_i$ is induced by the orientation on $M'$. By the first claim, $f$ realizes a diffeomorphism $U \cong V''_i$, hence (using local coordinates)
\[
\int_{V''_i} f^* \omega = \varepsilon_i \int_{\tilde{U}} \omega
\]
which proves the claim.

\[\square\]

Remark. It is important that the degree of the map $f$ has this property
\[
\int_{M'} f^* \omega = \deg(f) \int_M \omega
\]
globally, on any differential form.

Now we compute $\deg(q)$, where we have
\[
q : M' = (G/T) \times T \longrightarrow G = M.
\]
Recall that we oriented $M'$ and $M$ by choosing $dt$ and $dg$ nonzero, left-invariant, top-degree, smooth differential forms on $T$ and $G$, and hence getting a canonical $d\tilde{g}$ on $G/T$ which is again left-invariant, top-degree, nonzero and smooth and make sure that the Fubini-like theorem seen before holds. Given those differential forms, an orientation is locally given by the coordinates of $dg$, $dt$ and $d\tilde{g}$. Recall that we chose $d\tilde{g}$ using the canonical isomorphism
\[
\Omega_T^{top}(e) \cong \Omega_T^{top}(e) \otimes \Omega_{G/T}^{top}(\bar{e})
\]
and the relation
\[
dg(e) = dt(e) \otimes d\tilde{g}(e).
\]
Concretely, at the identity element $e$ we have the exact sequence of vector spaces
\[
0 \longrightarrow t \longrightarrow g \longrightarrow \bar{g} \longrightarrow 0.
\]
The orientations given by $dg$ and $dt$ specify two oriented basis on $g$ and $t$: the extra vectors present in the basis of $g$ but not in the basis of $t$ give (canonically) a basis for the quotient $\bar{g}$.

Now $M$ and $M'$ are oriented manifolds. We need to choose $m_0 \in M = G$ over which we have a finite fiber.

Claim 77. Choose $t_0 \in T$ that generates a dense subgroup, then $q^{-1}(t_0)$ is finite, of size $|W|$, with $q$ a local diffeomorphism at each $m'_i \in q^{-1}(t_0)$. Moreover, $\varepsilon(m'_i) = 1$ for every $m'_i \in q^{-1}(t_0)$.

By the previous lemma and the considerations on $\deg(q)$ and the surjectivity of $q$, proving the claim will conclude the proof of the conjugacy theorem.
Proof. Let $t_0 = q(\bar{g}, t) = gtg^{-1}$: we have $N_G(T)/T \subset G/T$ and $t_0, t \in T$, thus $g(t)g^{-1} = (t_0)$ which is a dense subgroup of $T$. Taking closures, we get

$$g(\bar{t})g^{-1} = (t_0) = T \Rightarrow \{t\} \text{ is dense in } T,$$

because $\{t\}$ is a closed connected subgroup of $T$ of the same dimension, hence it must be $T$. Therefore, $gTg^{-1} = T$ which means $g \in N_G(T)$.

Thus $q(\bar{g}, t) = t_0$ implies $\bar{g} \in N_G(T)/T = W$, denote then $w = \bar{g} \in W$ and $t = w^{-1}.t_0 = n^{-1}t_0n$ for any representative $n \in N_G(T)$ of $w$. We get

$$q^{-1}(t_0) = \{(w, w^{-1}.t_0) \mid w \in W\}$$

which is a finite set!

We need to check that $q$ is a local diffeomorphism and that $\varepsilon(m'_i) = 1$ for every $i$. To prove the first, we want $dg(w, w^{-1}.t_0)$ to be an isomorphism of tangent spaces preserving orientation, and we compute it using differential forms. We have

$$q^*(dg) = F \cdot d\bar{g} \wedge dt$$

for some smooth function $F$,

and clearly $F(\bar{g}, t) \neq 0$ if and only if $dq(\bar{g}, t)$ is an isomorphism on tangent spaces, because $F(\bar{g}, t)$ is the determinant of the map between tangent spaces, i.e. $\det(dq) = F$.

It thus remains to show the following

Lemma 78. For every $t \in T$ and every $\bar{g} \in G/T$, we have

1. $\det(dq(\bar{g}, t)) = \det\left(\text{Ad}_{G/T}(t^{-1}) - 1\right)$ where

$$\text{Ad}_{G/T} : T \to \text{GL}(T_e(G/T))$$

is the action by $T$ through left-translation, which preserves the identity coset.

2. If $\{t\} = T$, then the right hand side $\det\left(\text{Ad}_{G/T}(t^{-1}) - 1\right)$ is nonzero.

Remark. This does the job, as we are using $d\bar{g}$ which preserves orientation.

14 May 1

Today we will study some applications of the conjugacy theorem.

Recall the setup from last time: let $G$ be a compact, connected Lie group, $T \subset G$ a maximal torus, denote $W = N_G(T)/T$, which is a finite group as $T = N_G(T)^0$, and consider the map

$$q : (G/T) \times T \to G \quad (\bar{g}, t) \mapsto gtg^{-1}.$$

We saw that $q$ is a degree $|W|$ covering map on 'most' of $G$, or more precisely, $q$ has finite fiber $q^{-1}(t_0)$ of size $|W|$ for $t_0 \in T$ such that $\{t_0\}$ is dense in $T$. The existence of such a $t_0$ is proved in section 5 of the handout on Weyl groups.

Remark. The finiteness of the fiber reflects the fact that, on the level of Lie algebras, the real Lie algebra $\mathfrak{g}$ cannot be the countable union of proper subspaces.
We reduced the proof of the conjugacy theorem to the following lemma: choose $d\,g$ on $G$, $d\,t$ on $T$ and get uniquely $d\,\bar{g}$ on $G/T$, a top-degree, left-invariant, smooth differential form.

**Lemma 79.** Consider the following equality at any point of $(G/T) \times T$:

$$q^*(d\,g) = (\det(d\,q)) \, d\,\bar{g} \wedge d\,t,$$

where we denote with $\det(d\,q)$ the $C^\infty$-coefficient of the pullback $d\,g$ in the basis $d\,\bar{g} \wedge d\,t$. Then

1. we have the Weyl integral formula:

$$\det(d\,q(\bar{g},t)) = \det(\text{Ad}_{G/T}(t^{-1}) - 1)$$

where

$$\text{Ad}_{G/T} : T \rightarrow \text{GL}(T_e(G/T)) \quad t \mapsto d\,l_t(e).$$

2. If $(t) = T$, then the determinant $\det(d\,q(\bar{g},t)) > 0$.

The fiber is

$$q^{-1}(t_0) = \{(w, w^{-1}.t_0) \mid w \in W\},$$

thus if $t_0$ generates a dense subgroup we will obtain that $q$ is a local diffeomorphism, orientation-preserving, at each $(w, w^{-1}.t_0)$.

**Remark.** We'll see that $\dim(G/T)$ is even, so that $d\,\bar{g} \wedge d\,t$ and $d\,t \wedge d\,\bar{g}$ amount to the same thing.

**Proof.** 1. The proof is given in the Weyl-Jacobian handout. The key idea is to use enough left-translations and left-invariant properties to reduce the proof to a computation of some map at $(\bar{e}, e)$. Then use the specific definition of $d\,\bar{g}$ as

$$d\,g(e) = d\,t(e) \otimes d\,\bar{g}(\bar{e}).$$

While doing the computations, we will use the facts that $d\,m(\bar{e}, e)$ is just addition on the vector spaces, and similarly $d\,(\text{inv})(e)$ is the negation map. Moreover, notice that for every closed subgroup $H$ we have

$$\text{Ad}_{G/H}(h) = d\,(c_h)(\bar{e})$$

where $c_h$ is conjugation by $h$, because multiplying by $h^{-1}$ on the right on $G/H$ is harmless, thus

$$\text{Ad}_{G/H}(h) = \text{Ad}_G(h) \mod \mathfrak{h}.$$

Later, we will work out explicitly the case of unitary groups.

2. We claim that the matrix

$$d\,q(\bar{g},t) = \text{Ad}_{G/T}(t^{-1}) - 1$$

has no real eigenvalues as operator on $\mathfrak{g}/\mathfrak{t}$. Then, being a matrix with real coefficients, its eigenvalues will be complex conjugates in pairs, hence their products will be a product of complex norms, which is positive. This will also show that $\dim G/T$ is even.

The claim above is equivalent to showing that $\text{Ad}_{G/T}(t^{-1})$ has no real eigenvalues. Now

$$\text{Ad}_{G/T} : T \rightarrow \text{GL}(T_e(G/T))$$
is a representation of the compact group $T$, hence its eigenvalues are in $S^1$, so it is enough to show that $\text{Ad}_{G/T}(t^{-1})$ has no eigenvalues $\pm 1$, or equivalently that its square $\text{Ad}_{G/T}(t^{-2})$ has no eigenvalue 1, that is, $\text{Ad}_{G/T}(t^{-2})$ admits no nonzero fixed vector in $\mathfrak{g}/t$.

Now if $t$ generates a dense subgroup of $T$, so does $-t^2$, because the map

$$S^1 \longrightarrow S^1 \quad x \mapsto x^2$$

is surjective, hence we rename $t^{-2}$ as $t$ and we want to show that

$$(\mathfrak{g}/t)^{t=1} = 0.$$

Suppose now $v$ is a $t$-fixed vector, as $\{t\} = T$ by continuity $v$ is $T$-fixed, so in fact we need to show that

$$(\mathfrak{g}/t)^T = 0.$$

Complexifying the above quotient (so we can use the theory of complete reducibility of representations) we have

$$( (\mathfrak{g}/t)^T )_\mathbb{C} = ( (\mathfrak{g}/t)^T )_\mathbb{C} = \left( (\mathfrak{g}/t)^T \right)_\mathbb{C} = (\mathfrak{g}/t)^T_\mathbb{C} = (\mathfrak{g}/t)^T / \mathfrak{t}_\mathbb{C} = (\mathfrak{g}/t)^T / \mathfrak{t}_\mathbb{C} = (\mathfrak{g}/t)^T_\mathbb{C}$$

when we are using complete reducibility of any $T$-representation over $\mathbb{C}$, and the fact that the whole $\mathfrak{t}_\mathbb{C}$ is $T$-invariant. It is then enough to show that

$$\mathfrak{g}^T = t = \text{Lie}(T).$$

Now for every closed connected subgroup $H \subset G$, taking invariants we always have

$$\mathfrak{g}^H = \text{Lie}(Z_G(H)),$$

so in our case ($H = T$) we get $\mathfrak{g}^T = \text{Lie}(Z_G(T)) = \text{Lie}(Z_G(T)^0)$ because any Lie algebra only depend on the connected component of the identity. But as $T$ is a maximal torus, we have already proved that $T = N_G(T)^0$, hence

$$\mathfrak{g}^T = \text{Lie}(Z_G(T)^0) \subset \text{Lie}(N_G(T)^0) = \text{Lie}(T) = t.$$

As the opposite inclusion $t \subset \mathfrak{g}^T$ is obvious, this completes the proof.

\[\square\]

**Example 37.** Let $G = U(n) \subset \text{GL}_n(\mathbb{C})$ and take the diagonal torus $T = (S^1)^n = (\mathbb{R}/\mathbb{Z})^n$. Next time we’ll show what follows: for $t = (e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n}$ we have

$$\det \left( \text{Ad}_{G/T}(t^{-1}) - 1 \right) = \prod_{1 \leq j < k \leq n} |e^{2\pi i \theta_j} - e^{2\pi i \theta_j}|^2.$$

The key to proving this fact is the understanding of the $\text{Ad}_G$-action of $T$ on $\mathfrak{g}_\mathbb{C}$ (in fact, on $(\mathfrak{g}/t)_\mathbb{C}$), when we already know that

$$\text{Ad}_{G/T}(t) = \text{Ad}_G(t) \mod t.$$  

This will lead us to the theory of root systems.

Now we discuss some applications of the conjugacy theorem in the direction of representation theory. We want to construct representations by finding characters, thus we need to understand class functions and to do so, we study conjugacy classes.

Let $(G, T)$ be as above.
Proposition 80. Let
\[ W(G, T) = N_G(T)/Z_G(T) = N_G(T)/T \]
be the Weyl group. Then we have a bijection
\[ T/W \leftrightarrow \{ \text{conjugacy classes of } G \}. \]

Proof. Surjectivity is clear by the conjugacy theorem: every \( g \in G \) lies in a maximal torus \( T' \), which can be conjugated to \( T \) by an element of \( W \).

Now let’s prove injectivity. Choose \( t, t' \in T \) that are conjugate: \( t' = gtg^{-1} \) for some \( g \in G \). Consider the connected compact Lie group \( H = Z_G(t')^0 \): inside \( Z_G(t') \) we have two tori: clearly \( T \subset Z_G(t') \) because \( t' \in T \) and \( T \) is commutative, but also \( gTg^{-1} \subset Z_G(t') \) is a torus in \( Z_G(t') \), because \( t' \in gTg^{-1} \).

By the conjugacy theorem applied to \( H \), there is \( h \in H \) such that
\[ h(gTg^{-1})h^{-1} = T, \]
that is, \( hg \in N_G(T) \). Moreover \( h \in H \) implies that
\[ c_{hg}(t) = c_h(c_g(t)) = c_h(t') = t', \]
so we found \([hg] \in W\) which moves \( t \) to \( t' \), i.e.
\[ t \equiv t' \mod W \]
which implies \([t] = [t'] \in W\) and ultimately proves injectivity. \( \square \)

Notice that we can view the conjugacy classes \( \text{Conj}(G) \) of \( G \) as a quotient of \( X = G \) under the (continuous) conjugaction action of the compact group \( G \).

Exercise 7. Let \( X \) be a locally compact, Hausdorff space and \( G \) a compact, Hausdorff topological group. Suppose we have a continuous action \( G \simeq X \). Then \( X/G \) with the quotient topology is Hausdorff.

In view of the proposition and the exercise,
\[ T/W \rightarrow \text{Conj}(G) \]
is a continuous bijection if we endow \( \text{Conj}(G) \) with the quotient topology of the exercise. Being a continuous bijection between Hausdorff spaces, it is an homeomorphism.

Now, class functions are just continuous functions on \( \text{Conj}(G) \), thus
\[ \{ \text{continuous class functions on } G \} \quad \xrightarrow{\cong} \quad C^0(T)^W \]
where the right hand side denotes continuous functions on \( T \) that are \( W \)-invariant.

Thus every character of \( G \) will be determined by its restriction on \( G \). How do we distinguish if such a restriction was the character of an irreducible representation?

Definition 15. The representation ring \( R(G) \) is the subring of \( C^0(\text{Conj}(G)) \) generated by the characters of irreducible representations, when the product is given by
\[ \xi_\rho \cdot \xi_{\rho'} = \xi_{\rho \otimes \rho'}. \]
Thus
\[ R(G) = \bigoplus_{\xi \text{ irreducible}} \mathbb{Z}[\xi] \]
and if \( \xi \otimes \xi' = \sum_i n_i \xi_i \) we get
\[ [\xi] \cdot [\xi'] = \bigoplus_i n_i [\xi_i]. \]

**Example 38.** Let \( G = (S^1)^n \), thus \( R(G) = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) where
\[ x_i(t_1, \ldots, t_n) = t_i \]
is the projection character. Note that this is a Laurent polynomial ring, as we need to add inverses to the projection characters, because we have dual representations.

In general,
\[ R(G) \twoheadrightarrow R(T)^W \text{ as } \xi \mapsto \xi|T. \]
We can think of \( R(G) \) as virtual representations, when we allow characters to have negative integer coefficients. We’ll see later that the inclusion above is in fact surjective.

Our basic problem is: how to identify the irreducible representations inside \( R(T)^W \)? The Weyl character formula will be useful. Next time, we’ll work out in detail the case of unitary groups.

### 14.1 Weyl integration formula

How do we integrate continuous class functions on \( G \)?

**Proposition 81** (Weyl integration formula). Let \( f \in C^0(\text{Conj}(G)) \) and take normalized Haar measures \( dg \) on \( G \) and \( dt \) on \( T \), so that also \( \int_{G/T} \bar{g} = 1 \). Then
\[ \int_G f(g) \, dg = \frac{1}{|W|} \int_T f(t) \det(\text{Ad}_{G/T}(t^{-1}) - 1) \, dt. \]

**Proof.** Consider again the map
\[ (G/T) \times T \longrightarrow G. \]
Choose the differential form \( \omega = f \, dg \) on \( G \), thus
\[ \int_G f(g) \, dg = \int_G \omega = \frac{1}{\deg(q)} \int_{(G/T) \times T} q^* \omega. \]
Now the map \( q \) is given by \( q(\bar{g}, t) = gtg^{-1} \) and \( f \) is a class function, thus \( f(gtg^{-1}) = f(t) \), hence
\[ \int_{(G/T) \times T} q^* \omega = \int_{(G/T) \times T} f(t) \frac{q^*(dg)}{J(t)} \, d\bar{g} \land dt \]
and the Jacobian determinant takes value 1. The fact that \( \frac{1}{\deg(q)} = \frac{1}{|W|} \) concludes the proof. \( \square \)

**Example 39.** Let \( G = U(n) \), \( T = (S^1)^n \), then \( W = S_n \), so it is possible to compute explicitly the integral on the left hand side of the formula.
15 May 3

Using the Weyl integration formula, we want to understand the action $\text{Ad}_{G/T}(t) \sim \mathfrak{g}/t$ for $t \in T$. More specifically, $T$ acts on $\mathfrak{g}/t$ via $\text{Ad}_{G/T}$, and this is in fact the same as the induced action from $(\text{Ad}_G)|_T$ on $\mathfrak{g}$. The torus $T$ preserves $\bar{e} \in G/T$, thus acts on $T_\bar{e}(G/T) = \mathfrak{g}/t$. The advantage of identifying this with the action by conjugation of $T$ via $(\text{Ad}_G)|_T$ on $\mathfrak{g}/(\mathfrak{t} \mod \mathfrak{t})$ is that such an action preserves the identity $e \in G$ (as $\text{Ad}_G(t) \sim G$ preserves $e$), unlike the action of $l_t$ on $G$.

We want to decompose $\mathfrak{g}_C$ with respect to the $T$-action, then pass to the quotient. Recall that we consider the complexification because over $\mathbb{C}$ every irreducible representation of any torus is 1-dimensional, while over $\mathbb{R}$ we also have rotations.

Once we pass to the quotient, we will have

$$(\mathfrak{g}_C)^T = (\mathfrak{g}^T)_C = (\text{Lie}(Z_G(T)))_C = t_C$$

because in the handout on the Frobenius theorem we proved the general fact that $\mathfrak{g}^H = \text{Lie}(Z_G(H))$ for any closed subgroup $H \subset G$, and for a maximal torus $T$ we have $T = Z_G(T)$.

Therefore $(\mathfrak{g}_C)^T = t_C$ hence as $T$-representations we have

$$\mathfrak{g}_C = t_C \oplus \left( \bigoplus_{a \in \Phi} (\mathfrak{g}_C)_a \right) \text{ where } \Phi \subset X(T) - \{0\} \text{ is a finite set of roots}$$

and

$$(\mathfrak{g}_C)_a = \{v \mid t.v = t^a v \forall t \in T\} \text{ is called } a\text{-weight space.}$$

Notice that we are using the exponential notation for characters: $t^a$ denotes simply $a(t)$ for any $a \in X(T)$ and any $t \in T$. In particular we obtain

$$(\mathfrak{g}/t)_C = \mathfrak{g}_C/t_C \cong \bigoplus_{a \in \Phi} (\mathfrak{g}_C)_a.$$

We conclude that

$$\det \left( \text{Ad}_{G/T}(t^{-1}) - 1 \right) = \prod_{a \in \Phi} (t^a - 1)^{\text{dim}_C(\mathfrak{g}_C)_a},$$

in fact we will see that $\text{dim}_C(\mathfrak{g}_C)_a = 1$ for every $a \in \Phi$, so every eigenvalue of the action of $T$ on $\mathfrak{g}$ (i.e., every root) has multiplicity one.

Remark. The Weyl group $W = N_G(T)/Z_G(T)$ acts on $T$ by conjugation, thus it acts on $X(T)$, permuting the roots $\Phi$ since it shuffles around the isotypic pieces of this representation (as the $T$-action on $\mathfrak{g}$ extends to an $N_G(T)$-action and even to a $G$-action).

The structure outlined above has a combinatorial flavour and leads to the theory of root systems, which we will see next week.

Example 40. Let $G = U(n) \subset \text{GL}_n(\mathbb{C})$ and $T = (S^1)^n$ the diagonal maximal torus. On the level of Lie algebras we have $\mathfrak{u}(n) \subset \mathfrak{gl}_n(\mathbb{C})$ and we get

$$\text{dim}_\mathbb{R} \mathfrak{u}(n) = \frac{\text{dim}_\mathbb{R} \mathfrak{gl}_n(\mathbb{C})}{2}.$$

By direct calculation one checks that $\mathfrak{u}(n) \cap i\mathfrak{u}(n) = \{0\}$, thus

$$\mathfrak{u}(n)_C \cong \mathfrak{gl}_n(\mathbb{C}),$$
so that the $T$-action on $u(n)_\mathbb{C}$ under $\text{Ad}_{U(n)}$ is in fact the $T$-action under $\text{Ad}_{\text{GL}_n(\mathbb{C})}$, which is the usual conjugation action of $\text{GL}_n(\mathbb{C})$ on $\mathfrak{gl}_n(\mathbb{C}) = \text{Mat}_n(\mathbb{C})$.

We are looking to decompose $u(n)_\mathbb{C}$ into weight spaces for the $T$-action via conjugation: by the reasoning above, the diagonal matrices are $T$-fixed, thus

$$(u(n)_\mathbb{C})^T = t_\mathbb{C} = \{\text{diagonal matrices}\} \subset \mathfrak{gl}_n(\mathbb{C}).$$

Consider now the elementary matrix $E_{ij}$ for $i \neq j$, and let $t = \text{diag}(t_1, \ldots, t_n) \in T$, then

$$\text{Ad}_G(t).E_{ij} = tE_{ij}t^{-1} = \frac{t_i}{t_j}E_{ij}$$

hence the elementary matrices $E_{ij}$ are eigenvectors for the $T$-action, and we also found the eigenvalues.

If we denote $X(T) \cong \mathbb{Z}^n$ via the isomorphism given by the choice of the following basis:

$$t_i \mapsto \frac{a_i}{t_i} \in S^1 \quad \forall 1 \leq i \leq n,$$

then we say that $CE_{ij}$ is a $T$-stable line with eigencharacter

$$t \mapsto t^{a_i - a_j}.$$

The set of weights showing up is then

$$\{a_i - a_j \in X(T) | i \neq j\}$$

and we have seen above that each weight corresponds to a 1-dimensional weight space (the line $CE_{ij}$).

Notice that

$$\Phi \subset \{x \in \mathbb{Z}^n | \sum x_j = 0\} = H,$$

that is, the set of roots is contained in an hyperplane.

For $n = 2$, $H_\mathbb{Q}$ is a line in $\mathbb{Q}^2$, spanned by $(a_1 - a_2)$, and the only other root is $(a_2 - a_1)$.

For $n = 3$ we have six roots:

$$\{a_1 - a_3, a_1 - a_2, a_2 - a_3, a_2 - a_1, a_3 - a_1, a_3 - a_2\},$$

which are placed on the plane $H_\mathbb{Q} = \{x + y + z = 0\} \subset \mathbb{Q}^3$ to form a regular hexagon centered at 0. In general, every $(a_i - a_j)$ has length $\sqrt{2}$, and for any $a \in \Phi$ we have $\mathbb{Q}a \cap \Phi = \{a, -a\}$, that is: the only roots in the line $\mathbb{Q}a$ are $a$ and its opposite $-a$. In particular the above is true for $U(n)$ for every $n \geq 2$.

Remark. The hyperplane $H \subset X(T) = \mathbb{Z}^n$ is exactly given by $H = X(T/Z)$ where

$$Z = S^1 \overset{\text{diag}}{\hookrightarrow} (S^1)^n = T$$

is the center of the unitary group. In fact, the condition $\sum x_i = 0$ which defines $H$ makes sure that the exponents of the character

$$t = (t_1, \ldots, t_n) \mapsto \prod_i t_i^{x_i}$$

sums to zero and thus the character is trivial on $t$ if $t_1 = \ldots = t_n$, that is, if $t \in Z$. 

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We consider now an application. For $G = U(n)$ we have
\[
\det \left( \text{Ad}_{G/T}(t^{-1}) - 1 \right) = \prod_{1 \leq j < j' \leq n} (t^{a_j-a_{j'}} - 1)(t^{a_{j'}-a_j} - 1).
\]

As $t^{a_i-a_j} \in S^1$ for every $i, j$, we have
\[
(t^{a_i-a_j} - 1)(t^{a_j-a_i} - 1) = |t^{a_i-a_j} - 1|^2 = |t^{a_i} - t^{a_j}|^2 = |t_i - t_j|^2 = |e^{2\pi i \theta_i} - e^{2\pi i \theta_j}|^2
\]
if $t = (e^{2\pi i \theta})_i \in (S^1)^n$. This computation makes it possible to apply concretely Weyl’s integral formula.

Remark. In section 2 of the Weyl group handout, one finds that $N_{U(n)}(T) = T \times S_n$, where $S_n \subset U(n)$ is the subgroup of permutation matrices. The Weyl group $W(G, T) = S_n$ is acting on $T$ by permuting the circle factors, so $S_n \sim X(T) = \mathbb{Z}^n$ is the usual action by permutation.

The case of the unitary group $U(n)$ is then completely determined. What about $SO(n)$? $Sp(2n)$? Later we will work out the root systems. In the handout about Weyl computation, we work out the Weyl groups of $SU(n)$ for $n \geq 2$, of $SO(n)$ for $n \geq 3$ and of $Sp(n)$ for $n \geq 1$. For each of these cases we calculated in Homework 3 an ”almost diagonal” subgroup as a maximal torus.

Recall the following table of Weyl groups which can also be found in chapter IV, sections 3.3-3.8 of the textbook:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$W(G, T)$ for the standard maximal torus $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(n)$</td>
<td>$S_n$</td>
</tr>
<tr>
<td>$SO(2m)$</td>
<td>index $2$ in $(\mathbb{Z}/2\mathbb{Z})^m \times S_m$</td>
</tr>
<tr>
<td>$SO(2m+1)$</td>
<td>$(\mathbb{Z}/2\mathbb{Z})^m \times S_m$</td>
</tr>
<tr>
<td>$Sp(n)$</td>
<td>$(\mathbb{Z}/2\mathbb{Z})^n \times S_n$</td>
</tr>
</tbody>
</table>

The obvious analogy between the last two cases is no coincidence (as we will see later via duality for root systems)!

Once we know the Weyl groups, the following natural question arises: what are the roots $\Phi \subset X(T)$? As remarked above, the representation theory of compact groups will be strongly analyzed through the theory of root systems. The weight spaces will always be 1-dimensional and the only $\mathbb{Q}$-relations among roots will be the obvious ones between $a$ and $-a$.

### 15.1 Spin groups and Clifford algebras

We have shown in Homework 5 that $\pi_1(SU(n)) = 1$, $\pi_1(Sp(n)) = 1$, but $|\pi_1(SO(n))| \leq 2$ when $n \geq 3$ and in fact we already know that $|\pi_1(SO(3))| = 2$.

**Claim 82.** In fact, $|\pi_1(SO(n))| = 2$ for every $n \geq 3$.

**Fact 83.** When $G$ is a connected Lie group and $\Gamma \subset G$ is a discrete normal (hence central!) subgroup, we have an exact sequence

\[
1 \rightarrow \pi_1(G) \rightarrow \pi_1(G/\Gamma) \rightarrow \Gamma \rightarrow 1
\]

where the surjective map is constructed in Homework 5. See also the handout on the fundamental groups for the details.
To prove the claim, we want then to construct a sequence
\[ 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Spin}(n) \longrightarrow \text{SO}(n) \longrightarrow 1 \]
where Spin(\(n\)) is some connected Lie group. Then we will get an exact sequence
\[ 1 \longrightarrow \pi_1(\text{Spin}(n)) \longrightarrow \pi_1(\text{SO}(n)) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \]
thus \(|\pi_1(\text{SO}(n))| \geq 2\) and having already the opposite inequality we will get the claim. Moreover, this will prove that \(|\pi_1(\text{Spin}(n))| = 1\), that is, Spin(\(n\)) is simply connected. In fact, Spin(\(n\)) will turn out to be a double cover of SO(\(n\)) we already saw the case for \(n = 3\) when we have
\[ 1 \longrightarrow \{\pm 1\} \longrightarrow \text{SU}(2) \xrightarrow{\pi_1} \text{SO}(3) \longrightarrow 1 \]
and SU(2) are the norm-1 quaternions.

We want then to construct a double cover of SO(\(n\)).

We start constructing the setup for Clifford algebras. Let \((V, q)\) be a finite-dimensional, quadratic space over \(k\) (i.e., \(q : V \longrightarrow k\) is a quadratic form) and we can write
\[ q(v, w) = q(v) + q(w) + B_q(v, w) \quad \forall v, w \in V \]
for some bilinear symmetric form \(B_q\). In particular we will be interested in \(q = -\sum_i x_i^2\) when \(V = \mathbb{R}^n\).

**Question:** can we embed \(V\) into some associative \(k\)-algebra \(A\) such that \(q\) is the square of some linear form? We want
\[ V \hookrightarrow A \text{ such that } j(v)^2 = q(v) \quad \forall v \in V \]
where \(j(v)^2 = j(v) \cdot j(v)\) is the associative product in \(A\).

**Definition 16.** The Clifford algebra is defined as
\[ C(V, q) = C(q) = T(V) / \langle v \otimes v - q(v) \cdot 1 \rangle \]
where \(T(V)\) is the tensor algebra on \(V\) and we quotient by the two-sided ideal \(I\) defined as above.

We are then forcing every square of an element of \(V\) in \(C(q)\) to be equal to the value of the quadratic form on that vector in \(k\).

**Example 41.** \(C(V, 0) = \bigwedge^+ V\) the exterior algebra.

If \(\{e_i\}\) is a \(k\)-basis for \(V\), then
\[ q(e_i + e_j) = q(e_i) + q(e_j) + (e_i e_j + e_j e_i) \]
hence
\[ e_i e_j = -e_j e_i + (q(e_i + e_j) - q(e_i) - q(e_j)) \]
which is an important property, since \((q(e_i + e_j) - q(e_i) - q(e_j)) \in k\). Notice moreover that \(C(V, q)\) is \(\mathbb{Z}/2\mathbb{Z}\)-graded, since the ideal \(I\) has a \(\mathbb{Z}/2\mathbb{Z}\)-gradation, because \(v \otimes v\) has degree 2, and \(q(v) \cdot 1 \in k\) has degree 0. In particular, we have the positive and negative part of the Clifford algebra
\[ C(V, q) = C(q)^+ \oplus C(q)^- \]
and \(C(q)^+\) is a subalgebra.

The upshot of the above formulas is that \(C(q)\) is spanned over \(k\) by the products
\[ e_{i_1} \cdots e_{i_r} \text{ for } i_1 < \ldots < i_r \text{ and any } r \geq 0. \]
Example 42. Define the following Clifford algebra for every \( n \geq 1 \):

\[
C_n = C\left(\mathbb{R}^n, - \sum_i x_i^2\right).
\]

Then \( C_1 = \mathbb{C} = \mathbb{R} \oplus \mathbb{R} e = C_1^+ \oplus C_1^- \) where \( e^2 = -1 \).

Also, \( C_2 = \mathbb{H} = (\mathbb{R} \oplus \mathbb{R} e_1 e_2) \oplus (\mathbb{R} e_1 \oplus \mathbb{R} e_2) = C_2^+ \oplus C_2^- \).

Fact 84. By the observation above about a basis for the Clifford algebras, one gets

\[
\dim_k C(V, q) = 2^{\dim V} \quad \text{(see chapter I, 6.11-6.17 of the textbook)}.
\]

From now on, we will focus on \( C_n = C(n, - \sum_i x_i^2) \). In particular we have the embedding

\[
\mathbb{R}^n = V \hookrightarrow C_n.
\]

Now clearly the units \( C_n^\times \) are a Lie group, thus we can introduce the Clifford group

\[
\Gamma_n := \{ u \in C_n^\times | \alpha(u) V u^{-1} = V \}
\]

where

\[
\alpha : C_n \longrightarrow C_n \text{ is the automorphism of } C_n \text{ extending the } \mathbb{R}^n\text{-map } x \mapsto -x.
\]

Notice that this definition is well-posed as the quadratic form does not 'feel' the minus sign in the \( \mathbb{R}^n\)-map.

We get a map

\[
\rho : \Gamma_n \longrightarrow \text{GL}(V) \quad u \mapsto \left(v \mapsto \alpha(u) v u^{-1}\right)
\]

and can show its kernel is \( \mathbb{R}^\times \). There is a "norm" \( N : C_n \longrightarrow C_n \) on the Clifford algebra that is generally not multiplicative but whose restriction to \( \Gamma_n \) satisfies \( N(\Gamma_n) \subset \ker \rho = \mathbb{R}^\times \) and is multiplicative, extending squaring on \( \mathbb{R}^\times \). Using this, we can show \( \rho(\Gamma_n) = O(n) \), and

\[
\text{Pin}(n) := \ker (N|_{\Gamma_n})
\]

is a subgroup that maps onto \( O(n) \) with kernel \( \{ \pm 1 \} \). Moreover:

Theorem 85. The preimage \( \text{Spin}(n) \) of \( \text{SO}(n) \) in the above map is a connected group and

\[
\ker (\text{Spin}(n) \longrightarrow \text{SO}(n)) \text{ has order } 2.
\]

Proof. See the textbook (Chapter I, section 6) and the handout on Clifford algebras.

16 May 6

Last time we discussed the roots for unitary groups. Today we will revise that and check some general properties.

\( U(n) \), like \( \text{GL}_n(\mathbb{C}) \), has many nice properties, for example its center \( Z(U(n)) = S^1 \) is connected, like \( Z(\text{GL}_n(\mathbb{C})) = \mathbb{C}^\times \) but unlike \( Z(\text{SL}_n(\mathbb{C})) = \mu_n \).

Let then \( G = U(n) \) and \( T = (S^1)^n \) the diagonal maximal torus. Then the character lattice is \( X(T) \cong \mathbb{Z}^n = \oplus \mathbb{Z} t_i \) where \( t_{a_i} = t_i \) is the projection on the \( i \)-th coordinate for every \( 1 \leq i \leq n \) and every \( t = \text{diag}(t_1, \ldots, t_n) \in T \). The center is the diagonally embedded torus

\[
Z(U(n)) = S^1 \overset{\text{diag}}{\hookrightarrow} T,
\]

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as one can check by hand. The easiest way to check that is to compute the $\text{Ad}_{U(n)}$-action of $T$

$$\text{Ad}_{U(n)} \sim U(n) \subset \text{GL}_n(\mathbb{C}) \sim \text{Ad}_{\text{GL}_n(\mathbb{C})}$$

and they induce an isomorphisms on the Lie algebras, once we complexify:

$$u(n) \subset \mathfrak{gl}_n(\mathbb{C}) \rightarrow \mathfrak{gl}_n(\mathbb{C}).$$

We have $T \sim u(n)$ while the Ad-action on $\mathfrak{gl}_n(\mathbb{C}) = \text{Mat}_n(\mathbb{C})$ is by conjugation, thus $T$ acts on $u(n)$ via conjugation, and we found the weight spaces of the decomposition for the $T$-action of $u(n) = \mathfrak{gl}_n(\mathbb{C})$:

$$u(n) = t \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{gl}_n(\mathbb{C})_{\alpha}$$

where $\Phi = \{a_i - a_j \mid i \neq j\}$ and if $a = a_i - a_j \in \Phi$, then $\mathfrak{gl}_n(\mathbb{C})_{\alpha}$ is spanned by the $E_{ij}$ elementary matrix, and the character $a$ acts as

$$t \mapsto t^{a_i - a_j} = \frac{t_i}{t_j}.$$

Recall that we proved the above by direct computation.

Every time a torus acts on a finite-dimensional vector space, the characters appearing in the decomposition are called roots and the subrepresentations are the weight spaces. The roots are the non trivial weights in this decomposition, and we denote them by $\Phi = \Phi(G,T)$.

**Remark.** The whole discussion of real and imaginary roots that the textbook brings on is irrelevant, to our purposes. The book talks about real roots in analogy to the question: when is a complex representation defined over the reals? But we want to understand the complex case first.

**Remark.** Be careful! For $T = S^1 \sim \text{GL}_2(\mathbb{C})$ we obtain weight spaces in $\mathfrak{gl}_2(\mathbb{C})$ and not in $u(2)$, even if the diagonally embedded torus $T$ is contained in $U(2)$. This reflects the fact that $S^1 \sim \mathbb{R}^2$ has eigenvalues in $\mathbb{C}$ and not in $\mathbb{R}$.

Recall the case of $G = U(n)$, when we obtained the roots

$$\Phi = \{a_i - a_j \mid i \neq j\} \subset X(T) \cong \mathbb{Z}^n$$

and under this identification we get

$$\Phi \subset X(T/Z) \text{ the character lattice of the quotient } T/Z$$

where $Z = S^1 \text{ diag } U(n)$ is the maximal central torus embedded diagonally. This is clear, because each such character $a_i - a_j$ obviously kills the center, as on the level of Lie algebras these characters are conjugations.

In general, for any character $\sum n_i a_i \in X(T)$ and any $\text{diag}(t) \in Z$ we get

$$\left(\sum n_i a_i\right).t = t^{\sum n_i}$$

hence

$$X(T/Z) = \{\bar{x} \in \mathbb{Z}^n \mid \sum x_i = 0\}$$
which turns out to be the $\mathbb{Z}$-span of $\Phi$.

We also saw that the Weyl group $W = S_n$ acted via the usual permutation action on $\mathbb{Z}^n$: we computed for this torus that $N_G(T) = T \times S_n$, where $S_n$ is the embedded subgroup of permutation matrices. Then the action $W \sim X(T)$ induces a permutation on the set of roots $\Phi$, because the $T$-action on $\mathfrak{g}_C$ extends to a $N_G(T)$-action, and in fact to the $G$-action given by conjugation. Thus in particular $N_G(T)$ moves around the isotypic pieces of $\mathfrak{g}_C$ as a $T$-representations, which means that $W$ permutes them as well.

Remark. $\mathfrak{g}_C = \mathfrak{u}(n)_C$ has all 1-dimensional weight spaces and $\mathbb{Q}a \cap \Phi = \{ \pm a \}$ inside $X(T)_\mathbb{Q} \subset X(T)_\mathbb{R} = \text{Lie}(T)$, as we will see later.

In fact, all the informations above were already "encoded" in $\text{SU}(n)$, that is we have the multiplication map

$$\mathbb{Z} \times \text{SU}(n) \rightarrow^m \text{U}(n)$$

which is a homomorphism because $\mathbb{Z}$ is central. Obviously it is a surjective map, as if $A \in \text{U}(n)$, pick an $n$-th root $t$ of $\det A$ and take $\text{diag}(t) \in \mathbb{Z}$. In fact we get

$$1 \rightarrow \mu_n \rightarrow \mathbb{Z} \times \text{SU}(n) \rightarrow \text{U}(n) \rightarrow 1$$

and we say that $\text{U}(n)$ is an almost direct product of $\mathbb{Z}$ and $\text{SU}(n)$, up to the $\mu_n$-discrepancy. With "almost" we mean that we’re having a surjective map with finite kernel. We’ll see later that $\text{SU}(n)$ is the commutator subgroup of $\text{U}(n)$ and this is not a coincidence. In general, $T$ is an almost direct product $T_\mathbb{Z} \times G'$ for $T_\mathbb{Z}$ a maximal central torus and $G'$ the commutator subgroup, which moreover will always turn out to be closed.

Remark. In the handout on the Weyl group, we saw that $W(\text{SU}(n)) = S_n$, but we cannot represent $S_n$ inside $\text{SU}(n)$: this is one more reason to prefer $\text{U}(n)$.

The reasoning above is a template for what we will do in the general case. Notice that since $G = \text{U}(n) = \mathbb{Z} \cdot \text{SU}(n)$ and denoting $G' = \text{SU}(n)$ the commutator subgroup, we have that

$$Z(G') = Z \cap G' = \mu_n$$

because $Z(G')$ must commute with the whole $G$, and $Z \cap \text{SU}(n) = \mu_n$ by direct check.

A maximal torus of $G'$ is also given by

$$T' = T \cap G'\{\text{diagonal matrices } A \text{ with } \det A = 1\}$$

and in fact it will be a general feature, that a maximal torus of $G'$ is given by $T \cap G'$. Consider then the isomorphism

$$G'/Z' \xrightarrow{\cong} G/Z,$$

as the roots $\Phi$ are in fact characters of $T/Z$, we focus on this last quotient: we have $T/Z \subset G/Z$ and similarly $T'/Z' \subset G'/Z'$ but in fact the two are also isomorphic via the same map, so we have the commutative diagram

$$\begin{array}{ccc}
T'/Z' & \xrightarrow{\cong} & T/Z \\
\downarrow & & \downarrow \\
G'/Z' & \xrightarrow{\cong} & G/Z
\end{array}$$

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Now, we want to relate the $W'$-action on $\Phi(G',T') \subset X(T'/Z')$ with the $W$-action on $\Phi(G,T) \subset X(T/Z)$. Let’s first relate the character groups: the natural embedding

$$T' \hookrightarrow T$$

induces $\mathbb{Z}^n \cong X(T) \twoheadrightarrow X(T')$, but what quotient is the last surjective map representing? We are looking for character in $X(T)$ which are trivial on $T'$, and this corresponds to killing the diagonal. Hence $(a_1, \ldots, a_n) \in X(T)$ kills $T'$ if and only if all $a_i$’s are equal, because $T'$ is given by the diagonal matrices with determinant $1$, so the product $\prod t_i^{a_i}$ is $1$ on every such matrix if and only if $a_1 = \ldots = a_n$. Therefore

$$\mathbb{Z}^n \cong X(T) \twoheadrightarrow X(T/Z) \cong \mathbb{Z}^n/\Delta.$$ 

Note that this map does not split!

Similarly we have the center

$$S^1 = Z \hookrightarrow T + (S^1)^n$$

inducing $\mathbb{Z}^n = X(T) \twoheadrightarrow X(Z) \cong \mathbb{Z}$ given by

$$\xi \mapsto \xi|_\Delta$$

corresponding via the canonical isomorphisms to $\bar{x} \mapsto \sum_i x_i$

hence the kernel of the surjective map is the hyperplane $H = \{\sum_i x_i = 0\}$.

**Definition 17** (Isogeny). A map $f : H_1 \longrightarrow H_2$ between connected Lie groups is called an *isogeny* if any of the following holds:

- $f$ is surjective with discrete kernel;
- $\dim H_1 = \dim H_2$ and the kernel is discrete;
- $\dim H_1 = \dim H_2$ and $f$ is surjective,
- $\text{Lie}(H_1)^{\text{Lie}(f)} \cong \text{Lie}(H_2)$ is an isomorphism.

**Remark.** In the compact situation, any isogeny has finite kernel.

Now we saw above that

$$Z \times T' \longrightarrow T$$

is an isogeny, in particular surjective, so the map

$$\mathbb{Z} \oplus \mathbb{Z}^n/\Delta \cong X(Z) \oplus X(T') \hookrightarrow X(T) \cong \mathbb{Z}^n$$

given by $\left(\sum_i x_i, \bar{x} \mod \Delta\right) \mapsto \bar{x}$

embeds $X(T)$ as an index $n$-subgroup of $X(Z) \oplus X(T')$. It is clearly not a coincidence that $n = |Z'|$! Note that rationally we have an isomorphism:

$$X(Z)_Q \oplus X(T')_Q \cong X(T)_Q$$

which on the roots give $\left(\{0\} \cup \Phi'\right) \mapsto \Phi$ and in particular splits, being an isomorphism. But on the integral level it does not split. Notice that roots go to roots because $\mathfrak{g}'/\mathfrak{t}' \cong \mathfrak{g}/\mathfrak{t}$, as $Z \subset T$, thus quotienting by $\mathfrak{t}$ kills $Z$. Extending scalars to $\mathbb{C}$, we get weights and roots for the action of $T/Z \cong T'/Z'$, so even if the character lattices $X(T)$ and $X(T')$ don’t match perfectly, the roots do match perfectly!

$$X(T) \supset \Phi \leftrightarrow \Phi' \subset X(T').$$
Claim 86. The Weyl groups also match.

The following actions of the Weyl groups \( W' = N_{G'}(T')/T' \) and \( W = N_G(T)/T \) match

\[ W' \sim (\{0\} \cup \Phi') \subset X(Z) \oplus X(T')_\mathbb{Q} \cong X(T)_\mathbb{Q} \supset \Phi \rhd W, \]

because \( G = Z \cdot G' \), with \( Z \subset T \), and \( T = Z \cdot T' \) for \( T' = T \cap G' \). These properties imply that the natural map

\[ W \longrightarrow W', \]

which arises because any \( r \in N_G(T) \) normalizes \( T' = G \cap T \), is an isomorphism, and in fact is also compatible with the actions, so works out perfectly. We checked the details for \( SU(n) \) in the handout on Weyl groups.

The moral of the story is that a central torus is a nuisance, because all the interesting stuff happens on the level of \( T' \). Now

\[ T' \longrightarrow T/Z \]

is an isogeny hence the induced map on character lattices

\[ X(T') \rhd X(T/Z) \]

is an inclusion of finite free \( \mathbb{Z} \)-modules of the same dimension, which means that tensoring with \( \otimes \mathbb{Z} \mathbb{Q} \) give an isomorphism

\[ X(T')_\mathbb{Q} \cong X(T/Z)_\mathbb{Q}. \]

The conclusion is that \( \Phi \) spans \( X(T')_\mathbb{Q} \) over \( \mathbb{Q} \) but does not span \( X(T') \) over \( \mathbb{Z} \! \! \! . \) This is a reason to prefer \( SU(n) \) over \( U(n) \), for the former the roots span \( X(T')_\mathbb{Q} \).

The above gives rise to the theory of root systems, and an instance is \( W' \sim (X(T')_\mathbb{Q}, \Phi') \).

Remark. Sometimes having a nontrivial central torus is crucial when we want to apply an induction argument on the dimension.

We will later consider for a while the following setup: let \( G \) be a compact connected Lie group, \( T \) a maximal torus, \( Z = Z(G) \) the center (which can be disconnected) and \( Z^0 \) its connected component, which is the maximal central torus.

Example 43. If \( S \subset G \) is any torus, then \( Z_G(S) \) is again a connected compact Lie group, with \( S \) in its center. Sometimes \( Z_G(S) \) has a non-commutative part. We will prove

1. the commutator subgroup \( G' \) is closed and is its own commutator subgroup (i.e. \( [G', G'] = G' \)), so is not commutative;

2. \( T \cup G' = T' \) is a maximal torus of \( G' \), in particular \( Z' \subset T' \) where \( Z' = Z(G') \) is the center of the commutator subgroup. Moreover

\[ Z^0 \times G' \longrightarrow G \]

is an isogeny, just as

\[ Z^0 \times T' \longrightarrow T \]

which has kernel isomorphic to \( Z' \). Furthermore, the full preimage of \( T \) in the first isogeny is exactly \( Z^0 \times T' \), and being an isogeny in the compact situation, its kernel is finite of index \( |Z'| \);
3. \[(1 \times W') \sim (X(Z^0) \oplus X(T')) \leftrightarrow X(T) \sim W\]

see the actions of the Weyl groups matching. Via the identification \(\mathfrak{g}/\mathfrak{t} \cong \mathfrak{g'}/\mathfrak{t'}\) we have that the roots correspond in

\[\Phi \subset X(T/Z) = X(T'/Z') \supset \Phi'\]

and moreover each set of roots \(Z\)-spans the respective character lattice;

4. We will have the root system

\[W' \sim (X(T)_\mathbb{Q}, \Phi').\]

The root system will encode everything we need to study the representation theory of \(G'\) (or \(G = Z \cdot G'\), because \(Z\) is forced to act by scalars thanks to Schur lemma).

Next time we will see why SU(2) is ubiquitous in the general theory.

17 May 8

Today we will show why SU(2) is so important for the general theory, and more generally why it is important to think about groups with a large central torus.

Recall the punchline from last time: given a compact connected Lie group \(G\) with center \(Z\), \(Z^0\) is a maximal central torus, we will prove that \(G' = [G, G]\) is closed and connected and \(G' = (G')'\) thus it is not solvable. In homework 7 we will see that SU(2) = \(\mathbb{H}^\times\) is perfect, that is, is equal to its own commutator subgroup.

More specifically, we will show that in the general case

\[Z^0 \times G' \xrightarrow{m} G\]

is a homomorphism and an isogeny.

**Remark.** In general, it is hard to predict how many commutators one needs to write an element of \(G'\).

Notice that the existence of the isogeny above implies that also

\[G' \longrightarrow G/Z\]

is an isogeny, so we already have some sort of structure on \(G'\), even before proving it is a Lie group.

In which interesting situations do we encounter a \(G\) with large central torus \(Z^0\)? Pick a maximal torus \(T \subset G\) and suppose \(T \neq G\) (that is, \(G\) is not commutative), then \(\Phi(G, T) \neq 0\) because we have some roots. Pick one such root \(a \in \Phi\) and consider

\[T \xrightarrow{a} S^1,\]

the kernel has codimension 1 in \(T\) but may not be connected, hence set \(T_a = (\ker a)^0\) which is a codimension-1 torus killed by \(a\). We will study \(H = Z_G(T_a)\), and we already know for sure that \(H\) is connected and \(H \supset T\).

**Claim 87.** \(T_a\) is a maximal central torus of \(H\).
Proof. Obviously $T$ is a maximal torus in $H$, moreover $T_a$ is central in $H$ and has codimension 1 in $T$. Suppose by contradiction that there is a larger central torus $T_c$ in $H$: it has to have $\dim T_c = \dim T$, in particular $T_c$ is a maximal torus of $H$. By the conjugacy theorem applied to $H$ we get $T_c = T$ as all maximal tori are conjugate and $T_c$ is central, so $T$ is the maximal central torus in $H$. In particular $\text{Ad}_G/\text{div} = 1$ as an action on $\mathfrak{h} = \text{Lie}(H)$. But

$$\text{Lie}(H)_C = \text{Lie}(Z_G(T_a))_C = \mathfrak{g}^T_{C}$$

as we saw in the Frobenius handout, so in the decomposition

$$\mathfrak{g}_C = t_C \oplus \bigoplus_{b \in \Phi} (\mathfrak{g}_C)_b$$

given by the $T$-action, we have that $T_a$ also fixes $t_C$ and acts by restriction on each $(\mathfrak{g}_C)_b$. We then get a decomposition

$$\mathfrak{g}^T_{C} = t_C \oplus \bigoplus_{b \in \Phi, b_{T_a} = 1} (\mathfrak{g}_C)_b$$

so we should understand which roots are trivial on $T_a$. By definition, $a$ is one such, hence the weight space $(\mathfrak{g}_C)^a_a$ appears in the decomposition of $\mathfrak{g}^T_{C}$, thus $T$ does not act trivially on $\mathfrak{g}^T_{C}$ (or else we would have $\mathfrak{g}^T_{C} = t_C$, but above we said that $T$ acts trivially on $\text{Lie}(H) = \mathfrak{g}^T_{C}$). This contradiction concludes the proof.

Remark. This process and the whole construction is very interesting and quite general.

Claim 88. Inside $X(T)_Q$ we have

$$\Phi \cap Qa = \{ b \in \Phi \mid b|_{T_a} \equiv 1 \}.$$ 

Proof. When does $b : T \rightarrow S^1$ have trivial restriction to $(\ker a)^0$? We have

$$b|_{\ker a} \equiv 1 \iff b = na \text{ for some } n \in \mathbb{Z},$$

because $\ker a$ pass to the quotient for $b$ if and only if we can factor through $a$, and any automorphism of $S^1$ is of the type $t \mapsto t^n$:

$$T \xrightarrow{a} S^1 \xleftarrow{b} t^n \xrightarrow{\xi} S^1$$

but taking $(\ker a)^0$ we weaken the condition to $b = \frac{n}{m} a$ for some rational $\frac{n}{m} \in \mathbb{Q}$. More generally, choose a $\mathbb{Q}$-subspace $0 \rightarrow V \rightarrow X(T)_Q$, for example $V = Qa$, then completing the short exact sequence to

$$0 \rightarrow V \rightarrow X(T)_Q \rightarrow V' \rightarrow 0$$

induces on the integral level a torsion free quotient of the character group $X(T)$:

$$X(T) \rightarrow X(S) \rightarrow 0$$

which always turns out to be a character lattice $X(S)$ for some subtorus $S \subset T$ (see the handout on character lattices).

Given a subtorus $S \subset T$, we have a map

$$X(T) \rightarrow X(S) \quad \xi \mapsto \xi|_S,$$
so the subtorus $S$ associated to the subspace $V$ is the unique subtorus such that

$$\xi \in X(T) \text{ is trivial on } S \iff \xi \in V.$$  

**Example 44.** $T \to S^1$ gives rise to

$$\mathbb{Q} \cong X(S^1)_Q \to X(T)_Q$$

whose image is the line $\mathbb{Q}a$. Thus picking as a subspace $X(S^1)_Q$ the associated subtorus will have to be $S = (\ker a)^0$, as this is the only possibility that can work (recall that whether or not $a$ kills $S$ is unaffected by taking power $a^n$ of the character). Hence the centralizer process as above picks out exactly the rational eigenspaces which are $\mathbb{Q}$-linearly dependent on $a$.

The upshot is that we can identify the weight spaces appearing in the decomposition of the Lie algebra of the centralizer:

$$\text{Lie}(Z_G(T_a)) \cong t_C \oplus \left( \bigoplus_{b \in \mathbb{Q} a \cap \Phi} (g_C)_b \right).$$

\[ \square \]

**Remark.** All we know now is that $a \in \mathbb{Q} a \cap \Phi$, we don’t even know that $(-a) \in \mathbb{Q} a \cap \Phi$.

The group operation $Z_G$ picks then some weight spaces of interest and we restrict our attention on these:

$$Z_G(T_a) \supset T \supset T_a$$

where $Z_G(T_a) \supset T_a$ is a central inclusion, while $T \supset T_a$ is a codimension-1 inclusion. Therefore we quotient and get

$$Z_G(T_a)/T_a \cong T/T_a$$

where $T/T_a$ is a 1-dimensional maximal torus and hence $\dim Z_G(T_a)/T_a > 1$ and in particular this quotient is not commutative.

**Theorem 89.** A non-commutative compact Lie group with a 1-dimensional maximal torus is isomorphic to either $SU(2)$ or $SO(3)$.

We will see the proof later.

**Example 45.** Let $G = U(n)$, $a = a_i - a_j \in X(T) \cong \mathbb{Z}^n$ for $i < j$. Let $T$ be the maximal diagonal torus so that

$$t = \text{diag}(t_i) \mapsto t^a = \frac{t_i}{t_j}.$$  

We have

$$T_a = \{ t \in (S^1)^n | t_i = t_j \};$$

as $U(n) \subset \text{GL}_n(\mathbb{C})$ we get

$$Z_G(T_a) = \{ \text{diagonal matrices with zeros } i\text{-th and } j\text{-th entries} \} \bigoplus e(SU(2))$$
where we embed SU(2) as the \( \{i,j\} \times \{i,j\} \) minor in Mat\(_n(\mathbb{C})\), that is

\[
e : SU(2) \hookrightarrow \text{Mat}_n(\mathbb{C}) \quad e(A)_{hh} = \begin{cases} A_{1,1} & \text{if } h = i, k = i \\ A_{1,2} & \text{if } h = i, k = j \\ A_{2,1} & \text{if } h = j, k = i \\ A_{2,2} & \text{if } h = j, k = j \\ 0 & \text{otherwise} \end{cases}
\]

Said in one more different way, \( Z_G(T_a) \) is given by almost-diagonal matrices, where away from \( (j,i) \) and \( (i,j) \) we only have diagonal elements, and the minor \( \{i,j\} \times \{i,j\} \) is in SU(2).

We have then an isogeny

\[
Z_G(T_a) \longleftrightarrow T_a \times SU(\mathbb{C} e_i \oplus \mathbb{C} e_j)
\]

which restrict to another isogeny

\[
T \longleftrightarrow T_a \times \{\text{diagonal matrices in } SU(2)\}.
\]

Once we will have proved that SU(2) is perfect, we will obtain that

\[
(Z_G(T_a))' = SU(\mathbb{C} e_i \oplus \mathbb{C} e_j).
\]

**Theorem 90.** Let \( G \neq T \) its maximal torus and \( \dim T = 1 \), then either \( G \cong SO(3) \) or \( G \cong SU(2) \).

We are almost ready to prove this big theorem, but first we introduce a corollary which applies the theorem to the case \( G = Z_G(T_a)/T_a \).

**Corollary 91.** For a general \( G \) and a root \( a \in \Phi \) it is always the case that

\[
\mathbb{Q} a \cap \Phi = \{ \pm a \} \text{ and } \dim(\mathfrak{g}_C)_a = 1.
\]

**Proof.** We use the representation theory of \( \mathfrak{sl}_2(\mathbb{C}) \). Apply the theorem when \( G = Z_G(T_a)/T_a \) and its maximal torus \( T/T_a \) has dimension 1. We have seen that

\[
\text{Lie}(Z_G(T_a)) = \mathfrak{g}^T_a = \mathfrak{t}_C \oplus \bigoplus_{Q_{a} \cap \Phi} (\mathfrak{g}_C)_a
\]

thus

\[
\text{Lie}(Z_G(T_a)/T_a) = \mathfrak{g}^T_a = \bigoplus_{Q_{a} \cap \Phi} (\mathfrak{g}_C)_a
\]

with the same weight spaces as \( \mathfrak{g}_C \). Then we just stare at SO(3) and SU(2) and verify the two claims by using any maximal torus. \( \Box \)

We are finally ready to prove the theorem:

**Proof.** To control the weights, we will construct a representation of \( \mathfrak{sl}_2(\mathbb{C}) \). We know that \( \dim G/T \) is even, thus it is at least 2, so that \( \dim G \geq 3 \).

Suppose first \( \dim G = 3 \), so that \( \dim T = 1 \), and take the adjoint representation

\[
\text{Ad}_G : G \longrightarrow GL(\mathfrak{g}) \cong GL_3(\mathbb{R}).
\]

Choose a \( G \)-invariant inner product so that we can view \( \text{Ad}_G : G \longrightarrow O(3) \) and by connectedness of \( G \) we have in fact

\[
G \xrightarrow{\text{Ad}_G} SO(3),
\]
notice that \( \dim_{\mathbb{R}} \SO(3) = 3 \).
By connectedness of \( G \), the kernel of \( \text{Ad}_G \) is exactly the center \( Z_G \), as \( g \in G \) acts trivially on \( \mathfrak{g} \) if and only if the conjugation \( c_g \) acts trivially on \( G \). Obviously \( Z_G \subset T \) for every maximal torus \( T \), but \( \dim T = 1 \), and the center is a proper subgroup of the maximal torus, hence it must be finite. This proves that
\[
\text{Ad}_G : G \to \SO(3) \text{ is an isogeny.}
\]
But \( \SO(3) \) has very few connected covers: as \( G/Z_G \cong \SO(3) \) we get an induced map
\[
\pi_1(\SO(3)) \to Z_G
\]
which means that either \( Z_G = 1 \) and \( G \cong \SO(3) \), or \( Z_G = \mathbb{Z}/2\mathbb{Z} \) and
\[
\text{Ad}_G : G \xrightarrow{2:1} \SO(3) \text{ is a double cover.}
\]
We want to show this double cover must be \( \SU(2) \).
Now, the pullback of an isogeny is again an isogeny and pullback also preserves the degrees of covers, hence using the universal cover \( \SU(2) \to \SO(3) \) we get a commutative diagram
\[
\begin{array}{ccc}
H & \xrightarrow{2:1} & \SU(2) \\
\downarrow{2:1} & & \downarrow{2:1} \\
G & \xrightarrow{2:1} & \SO(3)
\end{array}
\]
Now \( \pi_1(\SU(2)) = 1 \), thus \( H \) is disconnected and we must have in fact \( [H : H^0] = 2 \) so that
\[
\begin{array}{ccc}
H^0 & \xrightarrow{1:1} & \SU(2) \\
\downarrow{1:1} & & \downarrow{2:1} \\
G & \xrightarrow{2:1} & \SO(3)
\end{array}
\]
and the map \( H^0 \to G \) must also be an isomorphism, so that
\[
\SU(2) \cong H^0 \cong G
\]
as we wanted to prove.
Next time, we will show how to reduce the proof to the case where \( \dim G = 3 \). We will find a copy of \( \mathfrak{sl}_2(\mathbb{C}) \) inside \( \mathfrak{g}_C \), where the diagonal matrices in \( \mathfrak{sl}_2(\mathbb{C}) \) correspond to the Lie algebra of the torus \( t_C \), and we will use our knowledge about the representations of \( \mathfrak{sl}_2(\mathbb{C}) \). \( \square \)

18 May 10

Last time we had the following

**Theorem 92.** Let \( G \) be compact, connected, of rank 1 (the rank is the dimension of a maximal torus) and suppose \( \dim G > 1 \) (which is equivalent to \( G \) not being commutative). Then
\[
G \cong \SO(3) \text{ or } G \cong \SU(2).
\]
Consider a central extensions of Lie groups

Proposition 93. Consider a central extensions of Lie groups

\[ 1 \rightarrow S \rightarrow G \rightarrow \bar{G} \rightarrow 1 \]

where \( S \) is a torus and \( \bar{G} = SU(2) \) or \( \bar{G} = SO(3) \) (this forces \( G \) to be compact and connected, being an \( S \)-fiber bundle on \( \bar{G} \); in general if the base space and the fiber are compact (resp. connected) then so is the total space). Then the commutator subgroup \( G' = [G, G] \) is closed, connected, and

\[ S \times G' \rightarrow G \]

is an isogeny,

or equivalently

\[ G' \rightarrow \bar{G} \]

is an isogeny.

Moreover
1. If $\bar{G} = SU(2)$ or $G' = SO(3)$, then $G' \to G$ must be an isomorphism, thus there exists a unique splitting $G = S \times \bar{G}$. In general when the kernel of an exact sequence (like the one we are given) is in the center, any two splittings are related by an isomorphism $\bar{G} \to S$, but in our case $G \neq S$ because $G$ is not a torus! This proves why the splitting is unique.

2. Otherwise, $\bar{G} = SO(3)$, $G' = SU(2)$ and we have almost a direct product. In fact, there exists a unique isomorphism (as central extensions of $G$ by $S$)

$$G \cong S \times G'/\{\pm 1\}$$

because the kernel of the isogeny $S \times G' \to G$ is exactly $S \cup G' \subset Z_{G'} = Z(SU(2)) = \{\pm 1\}$ so if the kernel is not trivial, it must be exactly $\{\pm 1\}$. 

Remark. In Homework 7 we will show that $SU(2)$ is its own commutator subgroup.

Proof. We give the idea of the proof, the details are in section 2 of the handout on $SU(2)$. We proceed by cases.

Suppose $\bar{G} = SU(2)$: we want to split the sequence of Lie algebras

$$0 \to s \to g \to su(2) \to 0$$

and then apply the exponential map to the splitting $su(2) \to g$ in order to recover an embedding $SU(2) \to G$ (here we use that $SU(2)$ is simply connected).

As $S$ is central $S \subset \ker(ad_g)$ so the adjoint representation factors through the quotient

$$\begin{array}{ccc}
\mathfrak{g}^c & \xrightarrow{ad} & \text{End}_k(\mathfrak{g}) \\
\downarrow & & \\
\mathfrak{g}/s & \cong & su(2)
\end{array}$$

hence complexifying give us

$$sl_2(\mathbb{C}) \cong su(2)_c \to \text{End}_\mathbb{C}(\mathfrak{g}_c)$$

and we can now apply the theory of $sl_2(\mathbb{C})$-representations.

Suppose now $G = SO(3)$, then we consider the universal cover $SU(2) \to SO(3)$ and construct the pullback $E$:

$$\begin{array}{ccc}
1 & \to & S \\
\downarrow & & \downarrow \\
1 & \to & S \\
\downarrow & & \downarrow \\
& & \\
1 & \to & G & \to & SU(2) & \to & 1 \\
& & \downarrow & & \downarrow & \downarrow \\
& & 1 & \to & G & \to & SO(3) & \to & 1
\end{array}$$

By the previous case, we know how to split the exact sequence in the top row, which gives us either a splitting of the bottom row, or a map $SU(2) \to G$ which embeds $SU(2)$ as a closed subgroup which is the commutator.

Remark. It is an important principle to use commutators in order to get splittings.

We can get a refinement of the above result. Consider the following setup:

$$S \times G' \to \bar{G}$$

is an isogeny,
the kernel is discrete and normal, hence central, hence contained in any maximal torus. Now in any isogeny every maximal torus maps onto a maximal torus, and vice versa any maximal torus of \( \bar{G} \) has full preimage containing a maximal torus.

The upshot is that under central extensions of compact connected Lie groups, the set operations of 'image' and 'preimage' give inverse bijection between sets of maximal tori:

\[
\begin{align*}
S \times G' \xrightarrow{\text{isogeny}} G \\
S \times S' \xleftarrow{\pi^{-1}} T
\end{align*}
\]

where \( S' = T \cap G' \) is a 1-dimensional maximal torus of \( G' \) and gives an isogeny complement \( S \times S' \xrightarrow{\text{isogeny}} T \). The consequences of the theorems still hold true even at the level of the maximal tori.

Let's come back to the setup where we had \( Z_G(T_a) = Z_G(T_a') \cdot T_a \) where now \( \cdot \) means that the two factors commute and they have finite intersection. We also get \( T = T_a' \cdot T_a \) with \( T_a' \) a 1-dimensional maximal torus of \( Z_G(T_a') \).

We then have a pair \((Z_G(T_a'), T_a)\)
that is either

\[
\begin{cases}
\text{SU}(2), \begin{pmatrix} \frac{t}{t^{-1}} & 0 \\ 0 & 1 \end{pmatrix} & \text{or} \\
\text{SO}(3), \begin{pmatrix} \frac{t}{0} & 0 \\ 0 & 1 \end{pmatrix}
\end{cases}
\]

At the rational level we have isomorphisms for the character lattices:

\[
X(T)_{\mathbb{Q}} \xrightarrow{\cong} X(T'_{\mathbb{Q}}) \oplus X(T_a)_{\mathbb{Q}}
\]

where the line \( \mathbb{Q}a \in X(T)_{\mathbb{Q}} \) maps completely onto the line \( \mathbb{Q}a = \mathbb{Q}a|_{T_a'} \in X(T'_{\mathbb{Q}}) \).

Inside SU(2) and SO(3), where \( T_a \cong S^1 \) is a maximal torus, we can find explicit elements in the Weyl group \( W \), because

\[
W \cong \text{GL} \left( X(T_a) \right) \cong \mathbb{Z}^\times = \{ \pm 1 \}.
\]

We are then looking for a root whose action on the torus is inversion, so in particular the root does not centralize the torus.

This inversion of the Weyl group acts as

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ on SU}(2)
\]

and as

\[
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ on SO}(3).
\]

**Example 46.** We have the tori

\[
T_0 = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \right\} \subset \text{SU}(2)
\]
and
\[ T_0 = \begin{pmatrix} [z] & 0 \\ 0 & 1 \end{pmatrix} \in \text{SO}(3) \text{ where } [z] \in S^1 \text{ acts by rotation on } \mathbb{R}^2. \]

In both cases the Weyl group is \{1, r\} where the inversion \( r \) can be represented by the matrices we specified above: \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) for \( \text{SU}(2) \) or \( \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \) for \( \text{SO}(3) \).

For each root \( a \) we found then an explicit order 2 element \( r_a \in W(G, T) \) which negates \( a \) and is trivial on \( X(T_a)_{\mathbb{Q}} \). In the usual two-dimensional pictures for a root system, where \( a \) and \( -a \) semi-lines of the same straight line, we can view \( r_a \) as the symmetry with respect to the axis of this straight line passing from the center.

It turns out that \( W(G, T) \) is generated by the reflections \( \{r_a\} \). What is the nature of this reflection \( r_a : X(T)_{\mathbb{Q}} \rightarrow X(T)_{\mathbb{Q}} \)? It must be given by
\[ x \mapsto x - l_a(x)a \text{ for some linear form } l_a : X(T)_{\mathbb{Q}} \rightarrow \mathbb{Q}. \]
Thus \( l_a \in (X(T)_{\mathbb{Q}})^* = X_*(T)_{\mathbb{Q}} \) via the usual perfect pairing.

The miracle is that
\[ \exists a^\vee \in X_*(T) \text{ such that } l_a(x) = \langle x, a^\vee \rangle \quad \forall x \in X(T) \]

where \( \langle , \rangle : X(T) \times X_*(T) \rightarrow \text{End}(S^1) = \mathbb{Z} \) is the standard perfect pairing.

The linear form \( l_a \) has then good integrality property, and the discussion above will give the complete root system.

19 May 13

Let’s recap what we saw last time. Let \( G \) be compact, connected, \( T \) a maximal torus. Take \( a \in \Phi(G, T) \) and \( T_a = (\ker a)^0 \subset T \) for \( a : T \rightarrow S^1 \) a weight.

We had \( T_a \subset T \subset Z_G(T_a) \) for \( T_a \subset Z_G(T_a) \) a maximal central extension which has as isogeny complement the derived group \( Z_G(T_a)' \). We proved this latter one is closed and is either \( \text{SO}(3) \) or \( \text{SU}(2) \) and
\[ T_a \times Z_G(T_a)' \rightarrow Z_G(T_a) \text{ is an isogeny.} \]

This setup will help us in reducing question about \( Z_G(T_a) \) to question about \( Z_G(T_a)' \).

**Proposition 94.** Given \( H' \rightarrow H = H'/Z \) for \( Z \subset H' \) a central subgroup, we have a correspondence between maximal tori:

\[ \{ Z \subset T' \subset H' \text{ where } T' \text{ is a maximal torus} \} \leftrightarrow \{ T'/Z \subset H'/Z \text{ a maximal torus} \} \]

and in fact this bijection is given via image and preimage, that is
\[ T' = \pi^{-1}(\pi(T')) \text{ for any maximal torus } T' \subset H' \]

and
\[ T'/Z = \pi\left(\pi^{-1}(T'/Z)\right) \text{ for any maximal torus } T'/Z \subset H'/Z. \]
See homework 7 for the details.

We apply the result above to the setup introduced before: $T_a \subset T \subset Z_G(T_a)$ so using the isogeny $m : T_a \times Z_G(T_a) \rightarrow W$ we find a maximal torus of the domain as

$$m^{-1}(T) = T_a \times (T \cap Z_G(T_a))',$$

in particular $T \cap Z_G(T_a)'$ must be connected, and a 1-dimensional torus. Denote it by $T_a' = T \cap Z_G(T_a)'$, even if it is not the commutator of any subgroup. We will use it to cook up particular elements of the Weyl group for any given root.

Let’s compare the Weyl groups

$$W (Z_G(T_a)', T_a') \text{ and } W (Z_G(T_a), T) \subset W(G, T) \sim X(T),$$

clearly $W (Z_G(T_a)', T_a')$ is much simpler because $Z_G(T_a)'$ is either SU(2) or SO(3).

**Claim 95.** We have a natural embedding

$$W (Z_G(T_a)', T_a') \rightarrow W (Z_G(T_a), T)$$

**Proof.** Let $x \in W (Z_G(T_a)', T_a') = N_{Z_G(T_a)'}(T_a')/T_a' = N_{Z_G(T_a)}(T_a')/(T \cup Z_G(T_a))$. Elements in $N_{Z_G(T_a)'}(T_a')$ normalize $T_a'$ and centralize $T_a$, thus they also normalize $T_a \cdot T_a' = T$, so this gives an injection $N_{Z_G(T_a)}(T_a') \hookrightarrow N_{Z_G(T_a)}(T)$. But now if $x$ dies in $W (Z_G(T_a), T) \subset N_{Z_G(T_a)}(T)/T$, then $x \in T$ and also $x \in Z_G(T_a)$ hence $x \in T \cup Z_G(T_a) = T_a'$, hence the map

$$N_{Z_G(T_a)'}(T_a')/T_a' \rightarrow N_{Z_G(T_a)}(T)/T$$

is well-defined and still injective. \hfill \Box

Now we have

$$W (Z_G(T_a), T) \subset \text{GL} (X(T)_\mathbb{Q}) \text{ where } X(T)_\mathbb{Q} = X(T_a)_\mathbb{Q} \oplus X(T_a')_\mathbb{Q}.$$ By definition any element of $Z_G(T_a)$ acts trivially on $X(T_a)$, and if it normalizes $T$ then it also normalizes $T_a' = T \cup Z_G(T_a)'$ (because the derived subgroup $Z_G(T_a)' \subset Z_G(T_a)$ is always normal), hence

$$W (Z_G(T_a), T) \text{ preserves the decomposition of } X(T) \text{ and acts as } \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \in \text{GL} \left( X(T_a)_\mathbb{Q} \oplus X(T_a')_\mathbb{Q} \right)$$

where $*$ is the action on $X(T_a')_\mathbb{Q}$. The upshot is that the Weyl groups embedding is compatible with the action on $X(T_a')_\mathbb{Q}$:

$$\begin{array}{ccc}
W (Z_G(T_a'), T_a') & \rightarrow & W (Z_G(T_a), T) \\
\text{GL} (X(T_a')_\mathbb{Q}) & = & \text{GL} (\mathbb{Z}) = \mathbb{Z}^\times = \{ \pm 1 \}
\end{array}$$

hence we only need to produce a non-trivial element $r_a \in W (Z_G(T_a'), T_a')$ in order to squeeze $W (Z_G(T_a), T)$ in between the two groups of order 2 and prove that

$$W (Z_G(T_a'), T_a') = W (Z_G(T_a), T).$$
The element $r_a$ will clearly have order 2 and will act on $T'_a \cong S^1$ in the only possible way, that is by inversion. Then we’ll get

$$W(Z_G(T_a), T) = \{1, r_a\}.$$  

We need then to produce an element of $W(Z_G(T'_a), T'_a)$ acting on $T$ that centralizes $T_a$ and normalizes $T'_a$, we define it explicitly in the two possible cases for $Z_G(T'_a)$. 

<table>
<thead>
<tr>
<th>$Z_G(T'_a)$</th>
<th>SU(2)</th>
<th>SO(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T'_a$</td>
<td>${\begin{pmatrix} z &amp; 0 \ 0 &amp; \bar{z} \end{pmatrix}}$</td>
<td>$\begin{pmatrix} r_\theta &amp; 0 \ 0 &amp; 1 \end{pmatrix}$ with $r_\theta \in SO(2)$</td>
</tr>
<tr>
<td>reflection $r$</td>
<td>$\begin{pmatrix} 0 &amp; 1 \ -1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 1 &amp; 0 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; -1 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

**Remark.** For SU(2) the reflection $r$ we chose has $r^2 = -\text{id}$, thus $r$ has order 2 in the Weyl group but not as an element of SU(2)!

This eventually shows that

$$W(Z_G(T_a), T) = \{1, r_a\}$$

where the action $r_a \cdot X(T)_Q$ in the decomposition $X(T)_Q = X(T_a)_Q \oplus X(T'_a)_Q$ is given by

$$r_a = \begin{pmatrix} 1 & \in \text{GL}(X(T_a)_Q) & 0 \\ 0 & -1 & \in \text{GL}(X(T'_a)_Q) \end{pmatrix}.$$  

**Definition 18** (Algebraic definition of reflection). We call reflection a linear transformation $r$ of a finite-dimensional $\mathbb{R}$-vector space $V$ which is an element of order 2 of $\text{GL}(V)$ and such that the eigenvalue $-1$ has multiplicity one.

**Example 47.** We want to compute $r_a$ for $G = SU(3)$. Take the maximal torus

$$T = \left\{ \begin{pmatrix} t_1 \\ t_2 \\ \frac{1}{t_1 t_2} \end{pmatrix} \mid t_1, t_2 \in S^1 \right\},$$

we have the roots

$$\Phi = \{a_i - a_j \mid 1 \leq i, j \leq 3\}.$$  

Fix $a = a_1 - a_2 : \begin{pmatrix} t_1 \\ t_2 \\ \frac{1}{t_1 t_2} \end{pmatrix} \mapsto \frac{t_1}{t_2}$, then we have

$$T_a = (\text{ker} \ a)^0 = \text{ker} \ a = \left\{ \begin{pmatrix} t \\ t \\ \frac{1}{t^2} \end{pmatrix} \mid t \in S^1 \right\}$$

because clearly $\text{ker} \ a$ is already connected. Hence

$$Z_G(T_a) = \left\{ \begin{pmatrix} g \\ 0 \\ \frac{1}{\det g} \end{pmatrix} \mid g \in U(2) = U(C_{c_1} \oplus C_{c_2}) \right\}$$
and taking commutators we obtain

\[ Z_G(T_a)' = \left\{ \begin{pmatrix} \text{SU}(\mathbb{C}e_1 \oplus \mathbb{C}e_2) & 0 \\ 0 & 1 \end{pmatrix} \right\} . \]

Then

\[ T_a' = T \cup Z_G(T_a)' = \left\{ \begin{pmatrix} z & z^{-1} \\ 1 & 1 \end{pmatrix} \mid z \in S^1 \right\} \]

and \( r_a \), contained in \( Z_G(T_a)' \), is represented by \( \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). The isogeny

\[ T_a \times T_a' \rightarrow T \]

has kernel

\[ T_a \cap T_a' = \left\{ \begin{pmatrix} \varepsilon & \varepsilon \\ 1 & 1 \end{pmatrix} \mid \varepsilon = \pm 1 \right\} \]

and hence we get the short exact sequence

\[ 1 \rightarrow \left\{ \begin{pmatrix} \varepsilon & \varepsilon \\ 1 & 1 \end{pmatrix} \mid \varepsilon = \pm 1 \right\} \rightarrow T_a \times T_a' \rightarrow T \rightarrow 1. \]

At the level of the character lattices we get

\[ X(T_a) \oplus X(T_a') \twoheadrightarrow X(T); \]

if we fix the isomorphism \( X(T) \overset{a_1, a_2}{\rightarrow} \mathbb{Z}^2 \), the induced isomorphisms on the left hand side are \( X(T_a) \overset{a}{\rightarrow} \mathbb{Z} \) and \( X(T_a') \overset{z}{\rightarrow} \mathbb{Z} \) where \( z : X(T_a') \rightarrow \mathbb{Z} \) is just projection on one of the factor. Take then \( (n, m) \in X(T) \), which works as

\[ (n, m) : \left( \begin{array}{ll} t_1 & t_2 \\ \frac{1}{t_1 t_2} & 1 \end{array} \right) \mapsto t_1^n t_2^m, \]

hence plugging the general element of \( T_a \) gives

\[ \left( \begin{array}{ll} t & t \\ t & 1 \end{array} \right)^{(n, m)} t^{n+m} \text{ an element of } X(T_a), \]

while applying \( (n, m) \) to \( T_a' \) gives

\[ \left( \begin{array}{ll} z & z^{-1} \\ z & 1 \end{array} \right)^{(n, m)} t^{n-m}, \]
The map between character lattices is then
\[ X(T) \leftrightarrow X(T_a) \oplus X(T'_a) \quad (n,m) \leftrightarrow (n+m,n-m) \]
which has index-2 image on the integral level, as \( \text{Im} = \{ (x,y) \in X(T_a) \oplus X(T'_a) \mid x \equiv y \mod 2 \} \). We also get
\[ r_a \mapsto X(T)_{\mathbb{Q}} \text{ is valued } \begin{cases} \text{id} & \text{on } X(T_a)_{\mathbb{Q}} \\ -\text{id} & \text{on } X(T'_a)_{\mathbb{Q}} \end{cases} \]

Remark. Each \( X(T_a), X(T'_a) \) is in general a quotient of \( X(T) \), only when we find a complement piece we can view \( X(T_a)_{\mathbb{Q}} \) as a subspace of \( X(T)_{\mathbb{Q}} \).

We now want to find how \( r_a \) acts on the roots \( a_1 - a_2, a_1 - a_3, a_2 - a_3 \). By our choice of maximal torus, \( a_3 = -(a_1 + a_2) \). In the isomorphism we chose \( X(T) \cong \mathbb{Z}^2 \) we have
\[ a_1 - a_2 \leftrightarrow (1,-1), \quad a_1 - a_3 = 2a_1 + a_2 \leftrightarrow (2,1), \quad a_2 - a_3 = a_1 + 2a_2 \leftrightarrow (1,2). \]

We know how \( r_a \) acts on the decomposition of \( X(T)_{\mathbb{Q}} \), so we get
\[ r_a(n,m) \leftrightarrow r_a(n+m,n-m) = (\text{id}(n+m),-\text{id}(n-m)) = (n+m,m-n) \leftrightarrow (m,n) \]
hence \( r_a \) swaps the two coefficients, in particular
\[ r_a((1,-1)) = (-1,1) \leftrightarrow a_2 - a_1, \quad r_a((2,1)) = (1,2) \leftrightarrow a_1 - a_3, \quad r_a((1,2)) = (2,1) \leftrightarrow a_2 - a_3. \]

Thus \( r_a \) sends \( a_1 - a_2 \) to its inverse, and swaps the other two roots.

ROOT SYSTEM PICTURE

Reflection along the line denoted as \( r_a \) gives exactly the action of \( r_a \). This will be a general situation: we will be able to put an inner product on \( X(T)_{\mathbb{R}} \) so that \( r_a \) always acts as a reflection. Note that the reflections \( \{ r_a \} \in \Phi \) generates the Weyl group \( W = S_3 \), which again will be true in the most general situation.

Definition 19. Let \( V \) be a finite-dimensional \( k \)-vector space, with \( k \) a field of characteristic zero. A reflection \( r : V \to V \) is an order 2 automorphism such that \(-1\) has multiplicity one as an eigenvalue, so that \( V = L \oplus H \) with \( L \) a line where \( r \) acts as \(-\text{id}\) and \( H \) an hyperplane where \( r \) acts as \( \text{id}\).

Lemma 96. If \( \Gamma \subseteq \text{GL}(V) \) is a finite group, and \( r, r' \in \Gamma \) are two reflections that negates the same line, then they are equal.

Proof. Assume \( k = \mathbb{Q} \) or \( k = \mathbb{R} \). Choose a non-degenerate bilinear form \( B : V \times V \to k \) that is \( \Gamma \)-invariant, if necessary by averaging (if \( k \) is not as above, then while averaging we could lose non-degeneracy, if the bilinear form we start with is not positive definite).

Consider then \( V = L \oplus L^\perp \) where \( L \) is the line where both reflections acts as negation. By the \( \Gamma \)-invariance of the inner product, both \( r \) and \( r' \) preserves \( L^\perp \), so consider their restrictions to \( L^\perp \).
Both have finite order and are unipotent (as the eigenvalue \(-1\) with multiplicity one is busy on \( L \)), hence in characteristic zero this implies both are trivial, i.e. equal to \( \text{id} \), on \( L^\perp \).
Hence \( r \) and \( r' \) coincides both on \( L \) and on \( L^\perp \), which concludes the proof.

Remark. It is fundamental for the lemma to be true that \( r \) and \( r' \) belongs, a priori, to the same finite group. Notice moreover that this is a completely algebraic result, as there's no inner product defined on \( V \).
Let’s describe two extra properties of the setup above.

**Proposition 97.**  
1. $Z_G$ is finite if and only if $\mathbb{Q}\Phi = X(T)_{\mathbb{Q}}$ (that is, the root generates the rational character lattice), which is also equivalent to $\mathbb{Z}\Phi \subset X(T)$ being a finite-index subgroup (in fact, of index $|Z_G|$). Every condition is equivalent to the non-existence of a non-trivial central torus.

For example, this is the situation for the SU(3)-example above. In such cases, $G = G'$: we will later prove that in fact this last condition (the group being equal to its commutator subgroup) is equivalent to all the above ones.

The reflections $\{r_a\} \subset W(G, T)$ acts on $X(T)_{\mathbb{Q}}$ and permutes the roots:

$$r_a(\Phi) = \Phi,$$

because $a \in N_G(T)/T$ is shuffling around the weight spaces, as it clearly preserves the $T$-action decomposition.

2. Consider the decomposition $V = L \oplus L^\perp$ for a reflection. Then the action of the reflection on $V$ consist of changing every vector by an additive multiple of its $L$-component. Let $a \in X(T_{\alpha})_{\mathbb{Q}}$, as $a|_{T_{\alpha}} = 1$, then in the embedding

$$a \in X(T)_{\mathbb{Q}} = X(T_{\alpha})_{\mathbb{Q}} \oplus X(T'_\alpha)_{\mathbb{Q}}$$

$a$ must have 0 component on the hyperplane $X(T_{\alpha})_{\mathbb{Q}}$. Thus

$$r_a(x) = x + l_a(x)a$$

for some linear form $l_a : X(T)_{\mathbb{Q}} \rightarrow \mathbb{Q}$.

The miracle is that in fact

$$l_a(X(T)) \subset \mathbb{Z},$$

that is, $r_a(x) \in x + Za$.

See the handout on coroots for a proof.

**Definition 20** (Root system). Let $k$ be a field of characteristic zero, $V$ a finite-dimensional $k$-vector space. A **root system** in $V$ is a finite set

$$\Phi \subset V - \{0\}$$

such that

1. $k\Phi = V$, that is, $\Phi$ spans $V$.
2. $ka \cap \Phi = \{\pm a\}$ for every $a \in \Phi$. This condition makes the root system **reduced**.
3. For every $a \in \Phi$ it is given a reflection $r_a : V \rightarrow V$ such that $r_a(a) = -a$, $r_a(\Phi) = \Phi$ and $r_a(x) \in x + Za$.

The last condition on the reflection $r_a$ is in fact the most important.

Using this combinatorial structure we can completely classify the root systems. Then we can get an isogeny decomposition of $G$ in terms of ”simple” factors by using this decomposition.
20  May 15

Today we’ll analyze $G_2$. We will recall reflections from last time, then draw some root systems.

The setup is: $G$ a compact, connected, Lie group with finite center $Z_G$ (that is, $G$ has no nontrivial central torus).

**Example 48.** Starting with any $H$ compact, connected Lie group we can take

$$G = K/Z_H^0$$

where $Z_H^0$ is the maximal central torus. Then $Z_H/Z_H^0$ is the finite center of $G$ and we have a setup like the one above.

In fact, we’ll show on Homework 7 that for any central closed subgroup $Z \subset G$, the quotient $G/Z$ has center $Z_G/Z$.

**Example 49.** $G/Z_G$ has trivial center.

*Remark.* In ordinary group theory the above is not true! That is, $G/Z(G)$ can have non-trivial center, for example when $G$ is solvable.

**Example 50.** In homework 7 we work out the relation between the center $Z_G$ and the root system.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$Z_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU($n$)</td>
<td>$\mu_n$</td>
</tr>
<tr>
<td>Sp($n$)</td>
<td>$\mu_2$</td>
</tr>
<tr>
<td>SO($2m+1$)</td>
<td>1</td>
</tr>
<tr>
<td>SO($2m$)</td>
<td>$\mu_2$</td>
</tr>
</tbody>
</table>

Choose a maximal torus $T \subset G$ and get the roots $\Phi = \Phi(G,T) \subset X(T)$. The roots $\Phi$ spans $X(T)_\mathbb{Q}$ over $\mathbb{Q}$ when $G$ has no central torus, but this is false if $G$ has a non-trivial central torus!

Let $N_G(T)/T = W = W(G,T) \ltimes X(T)_\mathbb{Q}$ be the Weyl group. Clearly it shuffles around the isotypic pieces of any $T$-representation, in particular it permutes the roots because they label the weights of the $T$-action on $g$.

We built then $\{r_a\}_{a \in \Phi} \subset W(G,T)$, which through the identification

$$X(T)_z \xrightarrow{\cong} X(T_a)_\mathbb{Q} \oplus X(T_a')_\mathbb{Q}$$

behaves as $a \mapsto (0, a|_{T_a'})$, thus $X(T_a')_\mathbb{Q} = \mathbb{Q}a$. So $r_a$ acts as $\text{id}$ on $X(T_a)_\mathbb{Q}$ and as $-\text{id}$ on $\mathbb{Q}a = X(T_a')_\mathbb{Q}$.

*Remark.* We defined reflections using linear algebra, but we did not specify any inner product (yet).

We have the decomposition $X(T)_\mathbb{Q} = X(T_a)_\mathbb{Q} \oplus \mathbb{Q}a$, thus we get for any $x \in X(T)_\mathbb{Q}$ a decomposition $x = (x', l_a'(x)a)$ for some linear form $l_a': X(T)_\mathbb{Q} \rightarrow \mathbb{Q}$.

Apply this to $r_a$, we obtain

$$r_a(x)r_a((x', l_a'(x)a)) = (x', -l_a'(x)a) = x - 2l_a'(x)a,$$

thus in general

$$r_a(x) = x - l_a(x)a$$

where $l_a = 2l'_a$, $l_a \in X(T)^*_\mathbb{Q}$.

This map has some integrality properties, that is $l_a(\Phi) \in \mathbb{Z}$, i.e. $r_a(x) - x \in \mathbb{Z}a$. The geometric meaning is that extending scalars to $\mathbb{R}$ one obtains $X(T)_\mathbb{R} = t^*$ (see the handout on character
Choose an inner product $(\cdot | \cdot)$ on $X(T)_{\mathbb{R}}$ that is $W$-invariant. Then $r_a$ preserves the inner product $(\cdot | \cdot)$ and in fact it preserves the line $\mathbb{R}a$ where it acts as $-\text{id}$. Hence $r_a$ also preserves the complementary hyperplane $(\mathbb{R}a)^\perp$ and the only eigenvalues left for the $r_a$-action are $+1$'s. We obtain
\[
r_a(x) = x - 2 \left( x | \frac{a}{\|a\|} \right) \frac{a}{\|a\|} = x - \frac{(x|2a)}{(a|a)}a = x - \left( x | \frac{2a}{(a|a)} \right) a,
\]
hence
\[
l_a(x) = \left( x | \frac{2a}{(a|a)} \right)
\]
and the coroot
\[
a^\wedge = \frac{2a}{(a|a)} \in X(T)_{\mathbb{Q}}
\]
belongs in fact to the $\mathbb{Z}$-dual of $X(T)$.

As the element that we most care about are the roots, denote $r_{a,b} = \frac{2(a|b)}{(a|a)} \in \mathbb{Z}$ for $a, b \in \Phi$. These are called the Cartan integers. For example we have $r_{a,a} = 2$, $r_{a,-a} = -2$.

Given $\Phi \subset X(T)_{\mathbb{R}}$, we found coroots $a^\wedge \in X(T)_{\mathbb{Q}}$ which give reflections preserving $\Phi$.

**Example 51.** We list the 2-dimensional root systems. In general, we had the abstract notion of a root system, with some properties imposed via the definition. But for root systems coming from Lie groups we also get $l_a(X(T)) \subset \mathbb{Z}$, while for a generic root system we only get $r_a(X) - x \in \mathbb{Z}a$.

1. $A_1 \times A_1$.

![Diagram of A1 x A1 root system]

\[
\{r_a\}_{a \in \Phi} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]

This is related to the group $SU(2) \times SU(2) = SO(4)$.

2. $A_2$
\[ \langle r_a \rangle_{\alpha \in \Phi} = S_3. \]

For example this is the root system of SU(3).

3. \(B_2\)

\[ \langle r_a \rangle_{\alpha \in \Phi} = S_2 \times (\mathbb{Z}/2\mathbb{Z})^2 \] where \(S_2\) acts by swapping factors.

An example of a Lie group having this root system is SO(5). Notice that half of the roots have length 1 and half have length \(\sqrt{2}\). In general what will turn out to be significative is the ratio of root lengths.

4. \(G_2\)
This is the root system of the exceptional group $G_2$, which is not a classical group.

Note that in every case, every root is either a non-negative or non-positive linear combinations of some "irreducible" roots. More, every line through the origin splits $\Phi$ in two.

These four above are the only root systems in rank 2. This is more useful than it could appear at a first glance, because often we want to prove claims about root systems which only involves two roots, so we can restrict ourselves to the plane they generate.

**Remark.** The central torus is never seen by the root system.

Let's now study the group interpretation of $a^\wedge$. Consider the perfect pairing

$$X(T) \times X_*(T) - \text{Hom}(T, S^1) \times \text{Hom}(S^1, T) \rightarrow \text{End}(S^1) = \mathbb{Z} \quad (\chi, \lambda) \mapsto \chi \circ \lambda.$$ 

Once we extend coefficients to $\mathbb{Q}$ we still have a perfect pairing

$$X(T)_\mathbb{Q} \times X_*(T)_\mathbb{Q} \rightarrow \mathbb{Q}$$

thus any linear form $(X(T)_\mathbb{Q})^*$ comes from a cocharacter $X_*(T)_\mathbb{Q}$. In fact a form $l \in (X(T)_\mathbb{Q})^*$ comes from $X_*(T)$ precisely when $l(X(T)) \in \mathbb{Z}$.

The integrality of the pairing says exactly that

$$\exists! a^\wedge : S^1 \rightarrow T$$

such that $r_a(X) = x - (x, a^\wedge)a$, or equivalently the reflection $r_a$ is given exactly by this formula. Then

$$r(a) = -a \iff (a, a^\wedge) = 2.$$ 

Then $a^\wedge$ is called the coroot associated to $a$.

**Corollary 98.**

$$(-a)^\wedge = -(a^\wedge)$$
Remark. In general the operation of "taking coroot" is not a linear operation, because
\[ a^\wedge = \frac{2a}{(a|a)} \]
and the length (a|a) could vary among roots.

Let us now write some coroots for SU(2) and SO(3). In the handout on coroots, we reduce integrality properties to the case \( T'_a \subset Z_G(T_a)' \) with \( Z_G(T_a)' \cong \text{SU}(2) \) or \( Z_G(T_a)' \cong \text{SO}(3) \).

**Example 52.** Let \( G = \text{SU}(2) \) and take the torus
\[ T = \left\{ \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \right\} \subset \text{SU}(2). \]
We have \( \mathfrak{su}(2)_C = \mathfrak{sl}_2(C) \) with \( T \) acting by conjugation. On the root spaces
\[ \left\{ \begin{pmatrix} 0 & \ast \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 & 0 \\ \ast & 0 \end{pmatrix} \right\} \subset \mathfrak{sl}_2(C) \]
\( T \) has roots (respectively) \( t^2 = a_+ \) and \( t^{-2} = a_- \).

As \( T \) is 1-dimensional, to check if the formula \( r_a(x) = x - (x, a^\wedge)a \) holds, we only have to check it on one single \( x \neq 0 \). Taking \( x = a \), this reduces to checking that \((a, a^\wedge) = 2\). Now we have
\[ a^\wedge_+ = \frac{2a_+}{(a_+|a_+)} = \frac{2 \cdot 2}{2 \cdot 2} = 1 = \text{id} \]
hence
\[ a^\wedge_+: S^1 \to T \quad t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \]
in particular
\[ (a_+, a^\wedge_+) = a_+ \circ a^\wedge_+ = a_+ \circ a^\wedge_+: t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} a_+ \mapsto t^2 \]
thus
\[(a_+, a^\wedge_+) = 2. \]

A similar calculation works for \( a_- \).

**Example 53.** Consider now \( G = \text{SO}(3) \). Let
\[ T = \left\{ \begin{pmatrix} [t]_{\mathbb{R}^2} & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset \text{GL}_3(\mathbb{R}) \text{ where } t \in S^1 \subset \mathbb{C}^\times \text{ acts on } \mathbb{C} \cong \mathbb{R}^2. \]

At the level of the Lie algebras, we have
\[ \mathfrak{so}(3) \xrightarrow{\gamma} \text{Mat}_3(\mathbb{R}) \]
\[ \xrightarrow{\gamma} \mathfrak{so}(3)_C \cong \text{Mat}_3(\mathbb{C}) \]
and we want to compute the weights.

If we denote by \( \{e_1, e_2\} \) the basis of \( \mathbb{R}^2 \) that is acted on by \([t] \) as above, the eigenlines in \( \mathbb{C}^2 \) for
this $T$-action are $e_1 \pm e_2$ with eigenvalues $t = \pm 1 = a_\pm \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$; we want to find the cocharacters. By a calculation like the one above, one gets the coroots:

$$a_\pm^\wedge : t \mapsto \begin{pmatrix} t^{\pm 2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Next time we will present non-obvious properties of the coroots, then start classifying the root systems.

21 May 17

Today we will prove that the Weyl group is generated by reflections.

Recall that right now we only know that $G = \text{SU}(n)$ has root system

$$V = \{ \tilde{x} \in \mathbb{Q}^n \mid \sum_i x_i = 0 \} \quad \text{and} \quad \Phi = \{ e_i - e_j \} \subset V - \{0\}$$

where we view $V$ as a direct factor of

$$\mathbb{Q}^n \cong X(T)_{\mathbb{Q}} \cong V \cong \mathbb{Q}^n / \text{diag}(\mathbb{Q})$$

and $T$ is the diagonal torus in $U(n)$.

Strictly speaking, the character lattice $X(T')$ for a maximal torus $T' \subset \text{SU}(n)$ is a quotient of $X(T)$ but we have seen in fact the first splits as a direct factor once we rationalize. The coroots are

$$\Phi^\wedge = \{ e_i^\ast - e_j^\ast = a_\pm^\wedge \} \subset V^\ast - \{0\}.$$

On Monday we will work out the cases of $\text{SO}(n)$ and $\text{Sp}(n)$.

Let us now introduce a general setup. Take $(V, \Phi)$ with $V$ a finite-dimensional vector space over $k$, $\text{ch}k = 0$ (in fact we only care if $k = \mathbb{Q}$ or $k = \mathbb{R}$), $\Phi \subset V - \{0\}$ is a finite subset that $k$-spans $V$ and satisfies

1. $ka \cap \Phi = \{ \pm a \}$. This condition makes $(V, \Phi)$ reduced.

2. For every $a \in \Phi$, there is a reflection $V : \overset{r_a}{\rightarrow} V$ such that $r_a(a) = -a$, $r_a(\Phi) = \Phi$ and

$$r_a(b) - b \in \mathbb{Z}a \quad \forall b \in \Phi. \quad (1)$$

Remark. We also have a theory of root systems for non-compact Lie groups, but condition 1 is weakened.

We know that

$$r_a(v) = v - l_a(v)a$$

for some linear form $l_a \in V^\ast - \{0\}$

hence condition 1 is equivalent to asking that

$$l_a(\Phi) \subset \mathbb{Z}.$$

Now we proceed by defining an abstract Weyl group $W(\Phi)$ for a general root system. If $G$ is a compact connected Lie group with finite center, it will turn out that $W(G, T) = W(\Phi(G, T))$.
**Definition 21.** Given a root system \((V, \Phi)\), we define the Weyl group associated to it as

\[ W(\Phi) = \langle r_a \rangle_{a \in \Phi} \subset \text{GL}(V). \]

**Remark.** This is a finite group, because \( W(\Phi) \subset \{ T \in \text{GL}(V) \mid T(\Phi) = \Phi \} \) and \( \Phi \) is a finite spanning set for \( V \).

As this Weyl group \( W(\Phi) \) is finite, every reflection \( r_a \in \{ r_c \}_{c \in \Phi} \) is uniquely determined by the condition \( r_a(a) = -a \), because we have seen that, if \( \text{char} k = 0 \), two reflections in a finite subgroup of \( \text{GL}_n(k) \) which negate the same line must coincide.

**Remark.** If \( r_c = r_a \) then the \((-1)\)-eigenlines of the two reflections coincide, thus \( kc = ka \) hence

\[ \{ \pm c \} = kc \cap \Phi = ka \cap \Phi = \{ \pm a \} \implies c = \pm a. \]

The viceversa \( c = \pm a \implies r_c = r_a \) is obvious.

Then using the formula \( r - a(v) = v - l_a(v)a \) we get

\[ l_{-a} = -l_a. \]

**Claim 99.** \( l_a \) determines \( a \), in the sense that

\[ l_a = l_c \iff a = c. \]

**Proof.** We have that \( \ker l_a \) is the \((+1)\)-eigenspace of \( r_a \), an hyperplane. Thus a linear form \( l_a \) identifies the hyperplane which is the \((+1)\)-eigenspace of the respective reflection \( r_a \), i.e.

\[ l_a = l_c \implies r_a, r_c \text{ have the same pointwise-fixed hyperplane } H \subset V. \]

Now we repeat the same argument we used to prove that reflections in a finite group negating the same line must in fact coincide. Consider

\[ 0 \to H \to V \to V/H \to 0 \]

then \( r_a = r_c = \text{id} \) on \( H \) thus \( r_a \) and \( r_c \) must both be \(-\text{id}\) on \( V/H \). Hence \( r_c^{-1} \circ r_a \) acts on \( V \) as the identity on both \( H \) and \( V/H \), thus is unipotent of finite order, and as \( \text{char} k = 0 \) this means \( r_c^{-1} \circ r_a = \text{id} \).

Therefore \( r_a = r_c \) which implies \( c = \pm a \). But \( l_{-a} = -l_a \), hence \( c \neq a \), so we get the claim.

We thus call the forms \( \{ l_a \} \) coroots.

Define a map

\[ \Phi \to \Phi^\wedge = \{ a^\wedge : l_a \subset V^* - \{0\} \} \quad a \mapsto a^\wedge. \]

What we proved above is that this map, associating to each root the correspondent coroot, is bijective.

**Theorem 100.** \((V^*, \Phi^\wedge)\) with the dual reflections

\[ r_a^\wedge := r_a^* : V^* \to k \text{ acting like } x^* \mapsto x^* - \langle a, x^* \rangle a^\wedge \]

form a root system, called the dual root system.

More precisely,
1. $\Phi$ spans $V^*$,

2. $r_a^\vee(b^\vee) = (r_a(b))^\vee$ for all roots $a, b \in \Phi$.

**Proof.** It’s hard to prove this “by hand”, we want to use the nice properties of reflections we proved before. For the details, see the handout on the dual root systems: the key trick is to show that the space spanned by the roots have a *canonical* rational structure, i.e. letting $V_0 = \mathbb{Q}\Phi \subset V$, this satisfies

$$V_0 \otimes_{\mathbb{Q}} k \stackrel{\sim}{\longrightarrow} V.$$  

Notice that $r_a(\Phi) = \Phi$ implies $r_a(V_0) = V_0$ for every reflection $r_a$. Thus $W(\Phi)$ acts on $V_0$ and since $l_a(\Phi) \subset \mathbb{Z} \subset \mathbb{Q}$, also $l_a(V_0) \subset \mathbb{Q}$, which means that $l_a|_{V_0}$ is in fact in the $\mathbb{Q}$-dual of $V_0$. Again, see the handout for the details.

**Example 54.** $SU(n)$ and $SO(2m)$ have self-dual root systems, but $SO(2m+1)$ and $Sp(n)$ do not. We’ll see more on Monday.

Notice that

$$W(\Phi) \cong (W(\Phi^\vee))^\text{opp}$$

using the isomorphism

$$\text{GL}(V) \xrightarrow{\cong} \text{GL}(V^*) \quad T \mapsto (T^*)^{-1} \text{ which works as } r_a \mapsto r_a^\vee(-r_a^\vee).$$

In particular for a usual pair $(G, T)$ where $G$ is compact, connected, with finite center and $T$ a maximal torus, the property

$$W(G, T) \cong W(\Phi(G, T))$$

can be checked using the dual root system

$$(X_*(T)_\mathbb{R}, \{a^\vee\}) \text{ where } X_*(T)_\mathbb{R} \cong \mathfrak{t} \text{ with the usual } W\text{-action.}$$

Recall why $X_*(T)_\mathbb{R} \cong \mathfrak{t}$: we have $X_*(T) = \text{Hom}(S^1, T)$ and given $\lambda : S^1 \longrightarrow T$, we get

$$\text{Lie}(\lambda) : \mathbb{R} \longrightarrow \mathfrak{t} \text{ which is just an element } \text{Lie}(\lambda)(1) \text{ of } \mathfrak{t}.$$  

Hence we obtain a map

$$X_*(T) \longrightarrow \mathfrak{t}$$

which becomes an isomorphism once we extend scalars to the reals.

We want to show that the action $W \acts \mathfrak{t}$ is faithful, and we will check that by using the fact that this action is given by the reflections and the coroots. We will also use euclidean geometry.

As $\{r_a\} \subset W(G, T)$ we obviously have

$$W(\Phi(G, T)) = \langle r_a \rangle_{a \in \Phi} \subset W(G, T).$$

**Theorem 101.** In fact we have the equality

$$W(\Phi(G, T)) = W(G, T).$$
To prove it, we can pass to the dual root system, so it is enough to show that $W(G, T) \subset \text{GL}(t)$ is generated by $r_{a^\lor}$ as $a$ varies in $\Phi$. We make use of the fact that over $\mathbb{R}$ it makes sense to discuss Weyl chambers.

We make a short digression on Weyl chambers. Let $(V, \Phi)$ be a root system over $\mathbb{R}$: for any $a \in \Phi$ we get $H_a = \ker(a^\lor) \subset V$ the orthogonal complement to $Ra$ upon choosing a $W$-invariant inner product $(\cdot, \cdot)$ on $V$.

We define the Weyl chambers of $(V, \Phi)$ as the connected components of $(V - \bigcup_{a \in \Phi} H_a)$. It is important to be over $\mathbb{R}$ in order to make sense of the notion of connectedness.

It will turn out that $W$ acts simply transitively on the Weyl chamber, and this will be very important for us. We now draw some Weyl chambers for the dimension-two root systems. Weyl chambers are alternately colored white and green.
Let's go back to the proof. Choose a Weyl chamber $K$, we call a wall of $K$ a root hyperplane $H_a$ such that its intersection with the boundary of $\overline{K}$ has nonempty interior in $H_a$. In homework 8 we prove the following

**Lemma 102.** Choose a Weyl chamber $K$. Then for every $x$ in the boundary of $\overline{K} \subset V$, the walls of $K$ through $x$ are mutually independent and their union is an open neighborhood of $x$ in this boundary. In fact, this is a general fact for any locally finite collection of affine hyperplanes of $V$, but we don't need such generality.

Moreover, we have

$$K = \bigcap_{H \text{ walls of } K} H^*$$

where $H^\pm$ is the open half-space identified by $H$ which lies on the same side of $K$, with respect to $H$.

**Remark.** For any root system, every wall pass through the origin.

The following theorem will conclude the proof:

**Theorem 103.** For any chamber $K$,

$$W(G, T) = \langle r_a \rangle_{H_a \in \{\text{walls of } K\}}.$$

In fact, we will have that $\langle r_a \rangle_{H_a \in \{\text{walls of } K\}} = W(\Phi)$ for any real root system $(V, \Phi)_\mathbb{R}$, as we'll prove later.

Moreover, $W(\Phi)$ acts simply transitively on the set of Wall chambers.

The theorem above is not obvious in dimensions higher than two!

**22 May 20**

Let's recall some definitions from last time. Let $(V, \Phi)$ be a root system over $\mathbb{R}$. A chamber $K$ is a connected component of $V - \bigcup_{a \in \Phi} H_a$ where $H_a = (\ker a^\vee)^0 = (\mathbb{R} a)^\perp$ where the orthogonal is taken with respect to $B_q$. We will see later that a $W$-invariant inner product is invariant up to scaling.

A wall of $K$ is an $H_a$ such that the interior of $H_a \cap \overline{K}$ in $H_a$ is nonempty.
Remark. For any hyperplane $H \subset V$, a nonempty open subset $U \subset H$ is not contained in any other hyperplane! So if $H$ is a wall of $K$, it is in fact determined by the open set $H \cap \overline{K}$.

On homework 8 we will show that every $x$ in the boundary of $\overline{K}$ lies, in fact, in a wall, i.e. 
\[
\partial \nu \overline{K} = \bigcup_{\text{walls } H \text{ of } K} (H \cap \overline{K}).
\]

**Theorem 104.** Fix a chamber $K$, then 
\[W(\Phi) = \langle r_c \rangle_{H_c, \text{walls}(K)}\]
and in fact $W$ acts simply transitively on the set of Weyl chambers.

We will prove this later. The claim on the transitivity of the $W$-action is clear in the 2-dimensional examples but not at all obvious in higher dimension: roughly, the idea of the proof is that conjugating one reflection $r_c$ by another gives the whole Weyl group.

Recall that $W(G,T) \supset W(\Phi(G,T))$, by construction, using $r_a(x) = x - (x, a^\wedge) a$ for $a^\wedge : S^1 \to T$ a cocharacter.

**Theorem 105.** Given the usual pair $(G,T)$ and a chamber $K$ of $(X(T)_R, \Phi(G,T))$, we have $W(G,T) = \langle r_c \rangle_{H_c, \text{walls}(K)}$.

In fact, we will prove a dual statement:

**Theorem 106.** Given $(G,T)$, consider the dual root system 
\[(X_*(T)_R, \Phi(G,T)^\wedge)\]
and choose a chamber $K$ in it. Then the Weyl group $W(G,T)$ acts on this dual root system and 
\[W(G,T) = \langle r_{c^\wedge} = r_{c^*} \rangle_{H_c, \text{walls}(K)}\] where the equality between groups comes from duality.

**Remark.** In the dual situation, 
\[H_{a^\wedge} = \ker(X(a)_R) = X_*(T_a)_R = \text{Lie}(T_a) \subset t.\]

The method of the proof of the last theorem is to show that $W(G,T)$ acts simply transitively on these dual chambers. Later we will show that in fact every root system (over $\mathbb{R}$) comes from a pair $(G,T)$. We will prove the first theorem we stated today for this particular case, which is hence not a special case.

**Proof.** (of the last theorem stated). The claim is in fact a group-theoretic statement, but in order to prove it we have to extend scalars to $\mathbb{R}$. Let $W' = \langle r_{c^\wedge} \rangle_{H_c, \text{walls}(K)} \subset W(G,T)$, the proof consists of two steps:

1. Show that $W'$ acts transitively on the set of chambers using Euclidean geometry;

2. Show that $W(G,T)$ acts freely (simply transitively) on the set of chambers, i.e. $w(K) = K$ implies $w = \text{id}$. 

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Then the first step forces $W$ to be already transitive, thus if it is simply transitive we must have $W' = W$.

Let’s start proving the first step: this comes from considerations on the root system. Choose $x \in K$ and pick another chamber $K'$ and $y \in K'$. As these are connected components and $W(\Phi) = \Phi$, then $W$ preserves the hyperplanes $H_a$’s and thus $W$ shuffles around the chambers.

Consider then the $W'$-orbit of $y$, we want to prove this orbit meets $K$, as obviously two chambers are either disjoint or the same (hence if two chambers have non-trivial intersection they actually coincide).

In the $W'$-orbit of $y$ there is a point that is closest to $x$, so replace the pair $(y, K')$ with this closest point and the chamber it belongs to. Hence without loss of generality $|y - x|$ is minimized among all $y$’s in the $W'$-orbit, where the norm $|\cdot|$ comes from a $W$-invariant inner product.

**Claim 107.** This minimizing $y$ belongs in fact to $K$.

Suppose not, since $y$ does not belong to one of the hyperplanes $H_a$’s, we know that $y \notin \overline{K} = K \cup \{$some walls$\}$ hence $y$ must be on the opposite side (from $x$) of some wall $H_a$ of $K$ (on the homework we show that $K$ is in fact the intersection of the open half-spaces defined by the walls to which $K$ belongs).

Let $H_a$ be a wall of $K$ such that $x$ and $y$ are on the opposite side of it, i.e. $a(x) > 0$ and $a(y) < 0$. Reflect $y$ on the other side of $H_a$ by using $r_a \in W'$, a general fact about hyperplanes in $\mathbb{R}^n$ is that if $H \subset \mathbb{R}^n$ is one such hyperplane through 0 and $x, y$ lies on opposite sides of $H$, then reflecting $y$ across $H$ is closer to $x$ than $y$ is. The proof of this simple fact comes from computing distances using the decomposition $\mathbb{R}^n = H \oplus H^\perp$.

But now $H_a$ is a wall of $K$ and applying $r_a \in W'$ to $y$ brought $y$ to a point strictly closer to $x$. This is a contradiction by the choice of $y$, so the proof of the first step is complete.

Now we proceed to the proof of step 2: we already know that $W(G, T)$ acts transitively on the set of chambers, we just need to prove that it acts simply transitively, then the transitivity of $W'$ will imply $W = W'$.

We use features of the dual root system. Suppose then that $w(K) = K$ for some $w \in W$, pick $x \in K$ so that $w(x) \in K$ and hence the whole $(w)$-orbit of $x$ is in $K$. Now

$$t = X_*(T)_\mathbb{R} \overset{\text{dense}}{\geq} X_*(T)_{\mathbb{Q}},$$

thus by (harmlessly) wiggling $x$ around we can assume $x \in X_*(T)_{\mathbb{Q}}$.

Now we average $x$: as $K$ is a cone (geometrically) the average of $x$ over $(w) \subset X_*(T)_{\mathbb{Q}}$ is still in $K$! So we take as $x$ this new average, and now we obtain

$$x \in K, x \in X_*(T)_{\mathbb{Q}} \text{ such that } w(x) = x.$$

Now multiply out the denominators so that we can assume $x \in X_*(T)$ is integral, i.e. $x \in \text{Hom}(S^1, T)$. We get then a cocharacter

$$S^1 \overset{x}{\rightarrow} T$$

which is centralized by a representation $g \in N_G(T)$ and $x$ does not belong to any root hyperplane of the dual root system. What we want to show is that $g \in T$, so that $w = \text{id}$.

Look at

$$Z_G(x) := Z_G\left(x(S^1)\right),$$
this is compact connected and clearly containing $T$ and $g$, so it is enough to show that $Z_G(x) = T$. Suppose not, then $Z_G(T) \not= T$, hence we must have some non-trivial $T$-weights on $\text{Lie}(Z_G(x))_C$. This weight correspond to some $a \in \Phi(G,T)$ so that

$$(g_C)_a \in \text{Lie}(Z_G(x))_C = g^{(S^1)}$$

by the usual fact about Lie algebras of centralizer of tori.

We know that

$$x(S^1) \subset Z(Z_G(x))$$

the center of $Z_G(x)$,

hence $x(S^1)$ must be killed by all the roots of $Z_G(x)$, so that

$$x(S_1) \subset \ker a \Rightarrow a \circ x \equiv 1$$

which means that the cocharacter $x : S^1 \rightarrow T$ lands inside $(\ker a)^0 = T_a$ and thus by definition

$$x \in X_*(T_a)_\mathbb{R} = H_{a^\vee} \subset X_*(T)$$

which is a contradiction as we are assuming that $x$ does not belong to any of the root hyperplanes of the dual root system. This concludes the proof of the second step and therefore of the theorem. □

Next time we will compute the root system for $\text{SO}(2m)$.

23 May 22

Last time we saw that $W(G,T) = W(\Phi(G,T))$ for $G$ with finite center $Z_G$. This also implies $G = G'$, as we’ll see in the next homework (this condition says that $G$ is semisimple). We’d like to show things about the original root system, and not the dual one, so we resort to proving a general fact about any root system.

**Theorem 108.** Let $(V, \Phi)$ be a root system over $\mathbb{R}$ and $K \subset V - (\bigcup_{\alpha \in \Phi} H_{\alpha})$ be a chamber, where $H_{a} = \ker a^\vee$. Recall that we proved that $H_c = H_c'$ if and only if $c = \pm c'$, thanks to the reducedness condition. Then

1. $W(\Phi) = \langle r_c \rangle_{H_c \subset \text{Walls}(K)}$.

2. $W(\Phi)$ acts simply transitively on the set of chambers.

**Remark.** This underlies a presentation for this Weyl group as a Coxeter group. In general the geometry behind the root system helps in giving this presentation.

**Proof.** Let $W' = \langle r_c \rangle_{H_c \subset \text{Walls}(K)}$; in the homework we will see that the boundary of $K$ is covered by the walls.

The argument from last time (using distance with respect to a $W$-invariant inner product $(\cdot | \cdot)$) shows that $W'$ acts transitively on the set of chambers; let’s recall this argument briefly: taking $x \in K$, $y \in K'$ we look at the $W'$-orbit of $y$ and pick the element closest to $x$. Then we show such an element must in fact lie in $K$.

What remains to be shown is a ”faithfulness” property, that is, that $W(\Phi)$ acts simply transitively, i.e. if $w \in W(\Phi)$ is such that $w(K) = K$ then $w = \text{id}$. We will not, however, follow this strategy.
We need to prove that $r_a \in W'$ for any $a \in \Phi$. Certainly $H_a$ is a wall for some chamber $K'$ and we know that $K' = w'(K)$ for some $w' \in W'$. Therefore

$$H_a \in \text{Walls}(K') = \text{Walls}(w'(K)) = w'(\text{Walls}(K)),$$

hence $H_a = w'(H_c)$ for some $H_c \in \text{Walls}(K)$. We thus obtain

$$r_a = w'r_cw'^{-1} \in W'$$

because both sides fixes the same hyperplane $H_a$ and this property characterize a single reflection in the finite group $W(\Phi)$. Hence $r_a \in W'$ which concludes the proof of the first claim.

We now wants to show that $W'$ acts simply transitively (i.e. freely) on the set of chambers, i.e. given $w' \in W'$ and $K$ a chamber with $w'(K) = K$, we must show that $w' = \text{id}$. Our strategy will be the following: we will write $w'$ as a word in $\{r_a\}_{a \in \text{Walls}(K)}$ and show that it must have length zero. Let

$$w' = r_{a_1} \ldots r_{a_n}$$

for a sequence of (possibly non distinct) walls $H_{a_1}, \ldots H_{a_n}$ of $K$ and without loss of generality we assume $n \geq 1$, or the proof is complete.

Now we argue by induction on $n$. Clearly we can’t have $n = 1$ or else $w'(K) = r_a(K) \neq K$ for any $H_a \in \text{Walls}(K)$, so we assume that $n \geq 2$ and the result is known for words of size less than $n$, i.e. any word of size less than $n$ which fixes $K$ is the identity, by induction hypothesis.

We consider the sequence

$$K, r_{a_1}(K), \ldots, r_{a_i} \ldots r_{a_n}(K) = w'(K) = K,$$

as $r_{a_1}(K)$ is a Weyl chamber lying on the opposite side of $K$ with respect to $H_{a_1}$, there must be a least $i$ ($n \geq i > 1$) such that $r_{a_1} \ldots r_{a_i}(K)$ is on the same side of $K$ with respect to $H_{a_1}$: we fix such an $i$ and consider $w'' = r_{a_1} \circ \ldots \circ r_{a_{i-1}}$.

Now $w''(K)$ is on the opposite side of $K$ with respect to $H_{a_1}$ while $(w'' \circ r_{a_i})(K)$ is on the same side of $H_{a_1}$ as $K$. Thus, $K$ and $r_{a_i}(K)$ are on the opposite sides of $(w'')^{-1}(H_{a_1})$ and we look at

$$K \cap r_{a_i}(K) \subset (w'')^{-1}(H_{a_1}).$$

But $K \cap r_{a_i}(K)$ has nonempty interior where it meets $H_{a_1}$, so if an hyperplane $(w'')^{-1}(H_{a_1})$ contains an open subset of another hyperplane, the two must coincide:

$$H_{a_i} = (w'')^{-1}(H_{a_1}) \Rightarrow r_{a_i} = (w'')^{-1} \cdot r_{a_1} \cdot w''.$$ 

And thus we get

$$r_{a_1} \cdot \ldots \cdot r_{a_{i-1}} \cdot r_{a_i} = w'' \cdot r_{a_i} = r_{a_1} \cdot w'' = r_{a_1} \cdot r_{a_1} \cdot r_{a_2} \cdot \ldots \cdot r_{a_{i-1}} = r_{a_2} \cdot \ldots \cdot r_{a_{i-1}}$$

because $r_{a_1}$ is an involution. So the length of $w = r_{a_1} \cdot \ldots \cdot r_{a_n}$ gets also reduced by two, and this concludes the proof. 

**Remark.** In classifying root systems via Dynkin diagrams, it is very important that $W$ acts transitively on the set of chambers, and moreover $W$ is determined by reflections through the walls of a single chamber.
**Definition 22 (Basis of a root system).** Let \((V, \Phi)\) be a root system over \(\mathbb{R}\) and fix a chamber \(K\). Label the set of walls

\[ \text{Walls}(K) = \{ H_{a_1}, \ldots, H_{a_n} \} \]

where \(a_i^\wedge|_K > 0\), in order to distinguish between \(\pm a_i\). Once we choose a \(W\)-invariant inner product, this condition is in fact equivalent to \((a_i|v) > 0\) for all \(v \in K\).

We call a basis of the root system

\[ B(K) = \{ a_1, \ldots, a_n \} \]

with the choice made as above.

**Theorem 109.** Let \(B(K) = \{ a_1, \ldots, a_n \}\) be as above. Then

1. \(B(K)\) is a \(\mathbb{R}\)-basis for \(V\) and for every root \(c \in \Phi\) we have
   \[ c = \sum n_i a_i \text{ for some } n_i \in \mathbb{Z} \]
   and the \(n_i\)'s are either all positive or all negative.
   Thus it makes sense to talk about positive and negative roots with respect to the chamber \(K\). The roots in \(B(K)\) are called **indecomposable**.

2. The set of positive roots with respect to \(K\), denoted by \(\Phi^+(K)\) is such that
   \[ \Phi^+(K) = \{ c \in \Phi | (v, c^\wedge) > 0 \forall v \in K \} \]
   where once we choose a \(W\)-invariant inner product \((\cdot, \cdot)\), the condition \((v, c^\wedge) > 0 \forall v \in K\) is equivalent to \((v|c) > 0 \forall v \in K\).
   We call \(B \subset \Phi\) a basis if \(B\) is a linearly independent set in \(V\) and satisfies the condition which defines positive and negative roots.

3. We have a bijective correspondence
   \[ \{ \text{chambers} \} \leftrightarrow \{ \text{basis of } \Phi \} \]
   given by
   \[ K \mapsto B(K) \text{ defined as above} \]
   and whose inverse is
   \[ B \mapsto K(B) = \{ v \in V | (v, c^\wedge) > 0 \forall c \in B \} \]

Read the handout on computation of root systems for more details.

**Proposition 110.** For any chamber \(K\) we have

\[ \Phi = W(\Phi)(B(K)) \]

**Proof.** For every root \(a \in \Phi\), \(H_a\) is the wall of some chamber \(K'\) so up to flipping across \(H_a\) if needed, we can arrange \(a \in B(K')\). But now \(W(\Phi)\) is transitive on the set of chambers, thus on basis given by chambers (by the bijection of the theorem), so there exists an element \(w \in W\) such that \(w(B(K)) = B(K') \Leftrightarrow a\) and this proves the claim. \(\square\)

As \(B(K)\) are the simple roots with respect to the chamber \(K\), we call \(\{ r_c \}_{c \in B(K)}\) the simple reflections with respect to \(K\). In particular we have the following

**Corollary 111.** Every root is the \(W\)-translate of some simple root.

Next time we will show how to use the results above for the study of Dynkin diagrams.
We consider on one last general fact before starting the classification of the root systems.

Remark. If $B$ is a basis of the root system $(V, \Phi)$, then

$$B^\wedge = \{ a^\wedge \}_{a \in B}$$

is a basis for the dual root system $(V^*, \Phi^*)$. This is proposition 4.13, chapter V, from the textbook, whose proof uses the facts that $\Phi = W(B)$ and $r_{a^\wedge}(b^\wedge) = r_a(b)^\wedge$.

The remark above gives one more instance of how given a basis we can reconstruct a lot of the original informations encoded in the root system.

24.1 Classification of low-rank root systems

In dimension one, the only root system is $\Phi = \{ \pm a \}$ because the roots must span $V$ and be linearly independent, up to a sign. $A_1$ is the root system for $SO(3)$ and $SU(2)$

\[
\begin{array}{c|c|c}
-a & 0 & a \\
\end{array}
\]

The length of the root can be anything, but the standard convention (over $\mathbb{R}$) is $\sqrt{2}$, because in this case

$$a^\wedge = \frac{2a}{|a|^2} = a.$$ 

This is nice, but not so important after all.

Remark. We have in general the notion of product of root systems: given $(V_1, \Phi_1)$ and $(V_2, \Phi_2)$, we consider the direct sum

$$(V_1 \oplus V_2, \Phi_1 \bigsqcup \Phi_2)$$

which is also a root system (check!) with coroots given by

$$\Phi_1^\wedge \bigsqcup \Phi_2^\wedge \subset (V_1 \oplus V_2)^*.$$ 

We denote this root system by $\Phi_1 \times \Phi_2$.

We consider now rank 2 root systems. Let’s start with $A_1 \times A_1$.

Then we have $A_2$, the root system of (for example) $SU(3)$:
Among the groups whose root system is $B_2 = C_2$ we have $SO(5)$ and $Sp(2)$.

There are four classical types of root systems, see the handout on the computation of root systems for the details:

<table>
<thead>
<tr>
<th>System</th>
<th>Group (e.g.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$SU(n)$</td>
</tr>
<tr>
<td>B</td>
<td>$SO(2m+1)$ for $m \geq 2$</td>
</tr>
<tr>
<td>C</td>
<td>$Sp(n)$ for $n \geq 2$</td>
</tr>
<tr>
<td>D</td>
<td>$SO(2m)$ for $m \geq 3$</td>
</tr>
</tbody>
</table>

Remark. $SO(4) = \frac{SU(2) \times SU(2)}{\mu_2}$ is a particular case.

We also have exceptional root systems: $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$ (as in the figure). The index is the dimension of the maximal torus.
Definition 23. We call simple a compact, connected Lie group such that every closed normal subgroup is in the center. This turns out to be equivalent to the irreducibility of the root system.

Theorem 112. The above are the only rank 2 root systems.

It is very useful that the rank 2-cases are so few and so explicit: in general given some set of roots and the spanning subspace, this pair forms a root system. Thus proving something about 2 roots only reduces (often) to some question about the rank 2 root system they span.

Proof. Choose a basis \( \{a, b\} \) of \( \Phi \). We showed last time that a basis can be chosen by fixing a chamber \( K \), considering the walls \( H_a \) of \( K \) and picking, between \( a \) and \( -a \), the root that is on the same side of \( K \) with respect to \( H_a \).

Recall that we have the Cartan integers
\[
n_{a,b} = (b, a^\wedge) = \frac{2(a|b)}{(a|a)} = \frac{2|b|}{|a|} \cos \langle a, b \rangle
\]

once we choose a Weyl-invariant inner product \( \langle \cdot, \cdot \rangle \), where \( \cos \langle a, b \rangle \in (0, \pi) \) as \( a \) and \( b \) are linearly independent.

Remark. An isomorphism of root systems can be checked over \( \mathbb{R} \)! This fact is important as it allows us to use geometry.

The other Cartan integer is \( n_{b,a} = \frac{2|a|}{|b|} \cos \langle b, a \rangle \) thus we have
\[
\mathbb{Z} \ni n_{a,b} \cdot n_{b,a} = 4 (\cos \langle a, b \rangle)^2 < 4
\]
as \( \cos \langle a, b \rangle < 1 \). Hence the product of the two Cartan integers must be an integer between 0 and 3.

Up to swap the two roots if necessary, we can arrange that \( |a| \leq |b| \) and hence \( n_{a,b} \geq n_{b,a} \), which leaves us with very few possibilities.

<table>
<thead>
<tr>
<th>( n_{a,b} \cdot n_{b,a} )</th>
<th>( \cos \langle a, b \rangle )</th>
<th>( \angle \langle a, b \rangle )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( \frac{\pi}{2} )</td>
</tr>
<tr>
<td>1</td>
<td>( \pm \frac{1}{2} )</td>
<td>( \frac{\pi}{3} ) or ( \frac{2\pi}{3} )</td>
</tr>
<tr>
<td>2</td>
<td>( \pm \frac{1}{\sqrt{2}} )</td>
<td>( \frac{\pi}{4} ) or ( \frac{3\pi}{4} )</td>
</tr>
<tr>
<td>3</td>
<td>( \pm \frac{1}{2} )</td>
<td>( \frac{\pi}{6} ) or ( \frac{5\pi}{6} )</td>
</tr>
</tbody>
</table>
Lemma 113. The angle between the two roots in the basis cannot be acute, i.e.
\[ \alpha(a, b) \geq \frac{\pi}{2}. \]
Equivalently, \( \cos \alpha(a, b) \leq 0 \) or (again equivalently) \( n_{a,b} \leq 0 \).

Proof. Basically, the lemma is true because \( a \) and \( b \) form a basis for the root system, which is a stronger condition than being a basis for the vector space.
In details, we argue by contradiction and suppose \( n_{a,b}, n_{b,a} > 0 \). As they are integers whose product is strictly less than 4, at least one of them must be equal to 1 and we are assuming that \( n_{a,b} \geq n_{b,a} \), thus \( n_{b,a} = 1 \). Then we get
\[ \Phi \ni r_b(a) = a - n_{b,a}b = a - b \]
which is a contradiction, as every root is a positive (or negative) combination of the basis \( \{a, b\} \).

Remark. Even if \( n_{a,b} \neq n_{b,a} \), the vanishing of the Cartan integers is a symmetric condition, i.e.
\[ n_{a,b} = 0 \iff (a|b) = 0 \iff (b|a) = 0 \iff n_{b,a} = 0 \]
as the orthogonality of two vectors is independent of which \( W \)-invariant inner product we choose. This explains why in the proof of the lemma we can assume, when arguing by contradiction, that both Cartan integers are (strictly) positive.

Thanks to the lemma we conclude that
\[ \alpha(a, b) \in \left\{ \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6} \right\}. \]

Now we consider the lengths: recall that we had the following two pieces of information:
\[ \Phi = W(\Phi)(\{a, b\}) \text{ and } W(\Phi) = \langle r_a, r_b \rangle, \]
we can then take cases for the Cartan integers.

1. \( n_{a,b} = 0 = n_{b,a} \). Then the roots are orthogonal but we don’t know if they have the same length.

\[ \Phi = \langle r_a, r_b \rangle \cdot \{a, b\} = \{\pm a, \pm b\} \] as there are no other roots. The Weyl group is \( W = \mathbb{Z}/2 \times \mathbb{Z}/2 \).
2. \( n_{a,b} = -1 \). Then \( \varpi(a, b) = \frac{2\pi}{3} \) and \( |a| = |b| \) because \( n_{a,b} = \frac{2|b|}{|a|} \cos \varpi(a, b) \). Once we start reflecting the roots, we get \( A_2 \):

\[
\begin{array}{c}
\begin{array}{c}
\text{This is the general way to handle the cases: from the roots in the basis we get the reflections. We use these to generate the full Weyl group and then use } \Phi = W(B) \text{ to get all the roots.}
\end{array}
\end{array}
\]

3. \( n_{a,b} = -2 \), then \( \varpi(a, b) = \frac{3\pi}{4} \) and thus \( \frac{|b|}{|a|} = \sqrt{2} \), so once we start reflecting the roots in the basis we get \( B_2 = C_2 \):

4. \( n_{a,b} = -3 \), we get \( \varpi(a, b) = \frac{5\pi}{6} \) and thus \( \frac{|b|}{|a|} = \sqrt{3} \), so once we start reflecting the roots in the basis we get \( G_2 \), because this is the only possibility left and in fact works.
There are no more cases left to analyze and we found exactly all the rank 2 root systems, so this concludes the proof.

Remark. In the rank 2, non-orthogonal case, the $W(\Phi)$-action on $V$ is absolutely irreducible (by inspection) and thus the $W$-invariant inner product is unique up to a sign. This happens because a $W$-invariant inner product $(\cdot | \cdot)$ is an isomorphism $V \rightarrow V^*$ of $W$-modules, so by Schur lemma if it exists, it must unique up to scaling.

By this result we just found, it is clear that the ratio of root lengths is intrinsic, i.e. it does not depend on the choice of the $W$-invariant inner product, and hence it makes sense to speak of short roots and long roots without having to specify which $W$-invariant inner product we are using.

**Corollary 114.** In the rank 2 non-orthogonal case, each case has a different ratio of root lengths, thus this

$$\frac{\text{long root}}{\text{short root}}$$

determines the root system!

What happens in general, that is, for a generic rank? The Dynkin diagrams will help us taking a hold of Cartan numbers.

**Theorem 115.** Let $(V, \Phi)$ be a root system and $B$ a basis. Then $(V, \Phi)$ is determined, up to isomorphism, by the Cartan matrix

$$(n_{a,a'})_{(a,a') \in B \times B}.$$

**Proof.** Suppose given $(V', \Phi')$ another root system with the same Cartan integers, i.e. there is a bijection $B \xrightarrow{f} B'$ between bases which preserves the Cartan integers.

Then $f$ extends to $V \xrightarrow{\pi} V'$ and $f(\Phi) = \Phi'$, which we denote by

$$f : (V, \Phi) \xrightarrow{\pi} (V', \Phi').$$
Let's prove this statement, when in fact such a linear extension of $f$ is unique, because $B$ and $B'$ are basis.

We now claim that

$$f \circ r_a = r_{f(a)} \circ f \quad \forall a \in B,$$

it is enough to prove the two sides behaves in the same way on the base $B$. We have

$$(f \circ r_a)(b) = f(r_a(b)) = f(b - n_{a,b}a) = f(b) - n_{a,b} f(a) = f(b) - n_{f(a),f(b)} f(a) = r_{f(a)}(f(b)) = (r_{f(a)} \circ f)(b)$$

for every $b \in B$, hence the claim is proved. But then

$$f \circ r_c = r_{f(c)} \circ f \quad \forall c \in \Phi,$$

that is

$$\forall w \in W(\Phi) f \circ w = w' \circ f \text{ for some } w' \in W(\Phi') = W'$$

and hence

$$f(\Phi) = f(W(B)) \subset W'(f(B)) = W'(B') = \Phi'.$$

Applying the same reasoning to the inverse bijection we get $f(\Phi) = \Phi'$ which proves the isomorphism.

\[\square\]

**Example 55.** The Cartan matrix for $B_2$ is $\left(\begin{smallmatrix} 2 & -2 \\ -1 & 2 \end{smallmatrix}\right)$ and the Cartan matrix for $G_2$ is $\left(\begin{smallmatrix} 2 & -3 \\ -1 & 2 \end{smallmatrix}\right)$. In both cases, the Cartan matrix is written in terms of the ordered basis \{short root, long root\}.

**Definition 24.** We say that $(V, \Phi)$ is a reducible root system if

$$(V, \Phi) \cong (V_1, \Phi_1) \times (V_2, \Phi_2) \text{ for } V_1 \neq 0 \neq V_2.$$ 

Otherwise, a nonzero $(V, \Phi)$ is called irreducible.

**Fact 116.** Every nonzero root system is uniquely a product of irreducible root systems. See the handout for a proof.

**Remark.** This result is much better than what we have for representations of finite groups: in this latter case, the only intrinsic thing is the isotypic component, but there are many choices of a copy of an irreducible representation inside a reducible representations. For root systems, the irreducible root system is uniquely determined.

The upshot of this discussion is that the Cartan matrix uniquely determines the root systems. Moreover, it is clear that a reducible root system $(V, \Phi)$ has Cartan matrix made of diagonal blocks, each block corresponding to the Cartan matrix of an irreducible component of $(V, \Phi)$.

We can finally introduce the Dynkin diagrams.

**Definition 25.** The Dynkin diagram of a root system $(V, \Phi)$ is a directed weighted graph. The vertices are the elements of a basis $B$ and the edges are defined case by case, using the classification of rank 2 root systems.

Fix two roots $a, b \in B$, then consider the root system they generate and the Cartan integers.

<table>
<thead>
<tr>
<th>$n_{a,b}$</th>
<th>$n_{b,a}$</th>
<th>ratio of root lengths</th>
<th>number of edges</th>
<th>notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$b \quad a$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$b \quad a$</td>
</tr>
<tr>
<td>2</td>
<td>$\sqrt{2}$</td>
<td>2</td>
<td></td>
<td>$b \quad a$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{3}$</td>
<td>3</td>
<td></td>
<td>$b \quad a$</td>
</tr>
</tbody>
</table>
Notice that the arrow always goes from the long root to the short root. Moreover, the Dynkin diagram of any root system is the disjoint union of the diagrams of its irreducible pieces.

Dynkin diagrams are completely classified! See for example section 11 and 12.1 of Humphrey’s book on Lie Algebras for a proof of existence, uniqueness and classifications of Dynkin diagrams.

25 May 29

In this lecture we organize an overview to wrap up the general theory. We want to put together various things we proved and see how they fit together.

1. Let $G$ be a compact connected Lie group and $T$ a maximal torus. We want to decompose it in an ”almost direct product” and show that this decomposition is related to the decomposition of the reducible root system. Suppose $G$ has finite center $Z_G$, then we have the following consequences (check also HW8, ex. 2)

(a) $G = G'$, which we can express by saying that $G$ is very not commutative.

(b) \[ (X(T)_\mathbb{Q}, \Phi = \Phi(G,T)) \]

is a root system (in particular, $\Phi$ spans $X(T)_\mathbb{Q}$) and the Weyl group $W(G,T) = N_G(T)/T$ is generated by the reflections $r_a$ as $H_a$ varies among the walls of $K$ for any Weyl chambers $K$.

(c) $\Phi = \Phi(G, T) \subset X(T)$, with $\mathbb{Z}\Phi \subset X(T)$ is a finite-index inclusion. Moreover

\[ X(T)/\mathbb{Z}\Phi = \text{Hom}(Z_G, S^1) \]

by HW7, ex 1(iii), so if $Z_G = 1$ then $\Phi$ spans $X(T)$ even on the integral level, and in particular $X(T)$ admits a $\mathbb{Z}$-basis of simple roots given by a basis $B$ of the root system. This means

\[ T \xrightarrow{\cong} \prod_{b \in B} S^1 \quad t \mapsto (t^b)_{b \in B} \]

is an isomorphism.

These three results are all consequences of having finite center, which hence turns out to be quite a strong condition!

2. We want to study what happens to the root system as we ‘move’ $G$ in its isogeny class. Let $Z \subset G$ be a finite central subgroup, and consider an isogeny

\[ G \xrightarrow{\pi} \bar{G} = G/Z. \]

Then there’s a bijective correspondence between maximal tori given by

\[ T \xrightarrow{\text{isogeny}} \bar{T} = T/Z \quad \text{and} \quad T = \pi^{-1}(\bar{T}). \]

By composing with the isogeny we have identifications

\[ X(T)_\mathbb{Q} \xleftarrow{\cong} X(\bar{T})_\mathbb{Q} \quad \text{and} \quad g \xrightarrow{\cong} \bar{g}. \]
This happens because the center acts trivially in the conjugation action, hence the adjoint
actions of \( G \) and \( \bar{G} \) are compatible, which means that the roots correspond:
\[
\Phi \leftrightarrow \bar{\Phi} \text{ (see HW8, ex.4(i)) as } a \leftrightarrow \bar{a} \text{ (HW7, ex. 1(i))}.
\]
Every root is trivial on \( Z \) and once we dualize we get
\[
X_\ast(T)_Q \longrightarrow X_\ast(\bar{T})_Q \quad a^\ast \mapsto (\bar{a})^\ast
\]
so we have a correspondence also for the coroots.

The conclusion is that the root system is unaffected by passing to finite central quotients.

We want now to study the opposite situation: starting with \( \bar{G} \) and considering compact
connected isogeneous covers \( G \) of \( \bar{G} \), how big can this \( G \) be? Notice that when looking for
an upper bound on the degree of the isogeneous cover, we don’t want to have a nontrivial
central torus.

3. We want to find a converse to the first result in this list: we will prove that
\[
G = G' \Rightarrow Z_G \text{ is finite.}
\]
Is \( G' \) a closed subgroup? We saw that when \( \text{rk} G = 1 \), then \( G/T \) is isomorphic to either \( \text{SU}(2) \)
or \( \text{SO}(3) \) and we constructed \( G' \): in fact something similar will be true in any rank.
On HW 9 we will show (ex.4):

(a) \( G' \) is always closed and \( (G')' = G' \). Notice that this fact is generally false in group theory,
e.g. any solvable group.

(b) \( Z_G^0 \times G' \rightarrow G \) is an isogeny. For example
\[
S^1 \times \text{SU}(n) \longrightarrow \text{U}(n)
\]
where \( S^1 = \{ \text{diag}(z) \mid z \in S^1 \} \). This is not a direct product, as \( S^1 \cap \text{SU}(n) = \mu_n \).

Let’s see a consequence of the isogeny above. If \( |\pi_1(G)| < \infty \), then \( Z_G^0 = 1 \) implies that \( Z_G \) is
finite, as \( |\pi_1(G)| \) bounds above the degree of finite connected covers of \( G \) and if \( Z_G^0 \neq 1 \) then
\( Z_G^0 = (S^1)^k \) which has covers of arbitrarily large degrees.

We want to show the equivalence of three conditions:
\[
G = G' \iff Z_G \text{ is finite} \iff |\pi_1(G)| < \infty.
\]
Then, given the isogeny above \( Z_G^0 \times G' \rightarrow G \) we will have \( G' \) with the properties above, and
\( Z_G^0 \) a torus, hence easy to study. In particular, the root systems of \( G \) and \( G' \) are the same,
so we split \( G \) and study separately \( G' \) and \( Z_G^0 \).

Exercise 3(v) shows that if \( Z_G \) is finite, then \( \pi_1(G) \) is finite and abelian. The reason behind
this is that the universal cover \( \tilde{G} \) of \( G \), with its unique compatible structure of Lie group,
is compact and the covering \( \tilde{G} \rightarrow G \) has kernel \( \pi_1(G) \), thus one only has to prove that this
covering is of finite degree.

**Example 56.** Consider
\[
\text{Spin}(n) \overset{2:1}{\rightarrow} \text{SO}(n) \quad \text{for } n \geq 3.
\]
We proved in the Homeworks that \( \text{Spin}(n) \) is simply connected.
Remark. Compactness of $G$ is fundamental for all the results we are proving! For example $G = \text{SL}_2(\mathbb{R})$ has finite center but its fundamental group is $\pi_1(G) = \mathbb{Z}$ because topologically $\text{SL}_2(\mathbb{R}) \cong S^1 \times \mathbb{R}^2$.

Given an isogeny $\tilde{G} \rightarrow G$, where could a torus $\tilde{T} \subset \tilde{G}$ go? We know the isogeny induces an isogeny of tori $\tilde{T} \rightarrow T$ and hence we get

$$X(\tilde{T}) \hookrightarrow X(T)$$

which becomes an isomorphism at the rational level:

$$(V^*)^* = X(\tilde{T})_\mathbb{Q} \leftarrow X(T)_\mathbb{Q} = V.$$

In $V^*$ we have coroots and we can consider forms that are integral on the coroots. We get

$$X(\tilde{T}) \subset \text{Hom}_\mathbb{Z}(\mathbb{Z}\Phi^\wedge, \mathbb{Z}) \subset (V^*)^*$$

and thus $\text{Hom}_\mathbb{Z}(\mathbb{Z}\Phi^\wedge, \mathbb{Z})$ is an upper bound for the possibilities for $\tilde{T}$, and we know the above because the root systems for $(G,T)$ and $(\tilde{G},\tilde{T})$ coincide. Thus

$$|\pi_1(G)| \text{ divides } [ (\mathbb{Z}\Phi^\wedge)' : X(T) ]$$

where we denotes by $(\mathbb{Z}\Phi^\wedge)' = \text{Hom}_\mathbb{Z}(\mathbb{Z}\Phi^\wedge, \mathbb{Z})$ the weight lattice, which is also denoted simply by $P$, sometimes.

For the universal cover $\tilde{G}$ we have, in fact, $X(\tilde{T}) = \text{Hom}_\mathbb{Z}(\mathbb{Z}\Phi^\wedge, \mathbb{Z})$.

We can also consider the representation theory point of view. To understand $\text{Rep}_G$, we study $\text{Rep}_{\tilde{G}}$ and then figure out which ones are trivial on the center of $\tilde{G}$ and which ones factor through $G$.

**Example 57.** Consider the Dynkin diagram $B_2 = C_2$ for $\text{Sp}(4)$ (also, for $\text{Spin}(5)$):

```
  b
 o =-+ a =-o
```

We have that

$$X(T) \supset \mathbb{Z}a \oplus \mathbb{Z}b = \mathbb{Z}\Phi$$

is an index-2 inclusion. What is $\text{Hom}_\mathbb{Z}(\mathbb{Z}\Phi^\wedge, \mathbb{Z})$ inside $\mathbb{Q}a \oplus \mathbb{Q}b$?

```
    b +a+b 2a+b
     \downarrow \downarrow \downarrow
-2a+b -a+b -b
```

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We have to understand the pairings \( \langle b, a^\wedge \rangle \) and \( \langle a, b^\wedge \rangle \). By the discussion on \( \langle a, b \rangle \), they must be either 1 or 2, and the larger is the one where we divide by the larger root, hence we get
\[
\langle b, a^\wedge \rangle = -1 \quad \text{and} \quad \langle a, b^\wedge \rangle = -1.
\]

Consider now \( x, y \in \mathbb{R} \) such that
\[
\langle xa + yb, a^\wedge \rangle = n \in \mathbb{Z} \quad \text{and} \quad \langle xa + yb, b^\wedge \rangle = m \in \mathbb{Z},
\]
using bilinearity we get
\[
n = 2x - 2y, \quad m = -x + 2y
\]
hence
\[
\begin{cases}
x = n + m \in \mathbb{Z} \\
y = m + \frac{n}{2} \in \frac{1}{2}\mathbb{Z}
\end{cases}
\]
Then we get \( xa + yb = (n + m)a + \left(m + \frac{n}{2}\right)b \in \mathbb{Z}a \oplus \mathbb{Z}b \) which means
\[
\text{Hom}_\mathbb{Z}(\mathbb{Z}a^\wedge, \mathbb{Z}) = \mathbb{Z}a \oplus \mathbb{Z}\frac{b}{2}
\]
and hence we draw the character lattice for the universal cover \( \tilde{T} \):

**Definition 26.** A group \( G \) which satisfies the three equivalent conditions
\[
G = G' \iff Z_G \text{ is finite} \iff \pi_1(G) \text{ is finite}
\]
is called *semisimple*.

**Example 58.** The groups \( \text{SU}(n) \) (for \( n \geq 2 \)), \( \text{Sp}(n) \) (for \( n \geq 1 \)), \( \text{SO}(n) \) (for \( n \geq 3 \)), \( \text{Spin}(n) \) (for \( n \geq 3 \)), are all semisimple.

**Example 59.** The circle \( S^1 = \text{SU}(1) \) is not semisimple. And also \( \text{U}(n) \) is not semisimple, as \( Z_{\text{U}(n)} \cong S^1 \).
**Fact 117.** We have
\[ G = G' \iff g = [g, g] \]
and thus the right hand side condition turns out to be the definition for a semisimple Lie algebra.

Moving in between an isogeny class we have two invariants: the center \( Z_G \) and the fundamental group \( \pi_1(G) \).

The universal cover \( \tilde{G} \) has \( \pi_1(\tilde{G}) \) trivial and \( Z_{\tilde{G}} \) as large as possible. At the bottom of the isogeny class we find the adjoint group \( G^{ad} \), which has \( \pi(G^{ad}) \) as large as possible and trivial center \( Z_{G^{ad}} \).

Suppose \( G \) is semisimple, we have the diagram
\[
\begin{array}{ccc}
\tilde{T}^c & \xrightarrow{} & \tilde{G} \\
\downarrow & & \downarrow \\
T^c & \xrightarrow{} & G \\
\downarrow & \xrightarrow{deg|Z_G|} & \downarrow \xrightarrow{Ad_G} \\
T^{ad} = T/Z_G^c & \rightleftharpoons & G^{ad} = G/Z_G^c \\
& & \xrightarrow{\text{GL}(g)} \text{GL}(g^{ad})
\end{array}
\]

where the adjoint group \( G^{ad} \) has no center, hence embeds into \( \text{GL}(g) \), and being compact the embedding is a closed embedding.

By dualizing, we get
\[
\begin{array}{ccc}
X(\tilde{T}) & \xrightarrow{} & \text{Hom}_\mathbb{Z}(Z\Phi^\perp, \mathbb{Z}) = P \\
\downarrow |\pi_1(G)| & & \downarrow \\
X(T) & \xrightarrow{|Z_G|} & \\
\downarrow |Z_G| & & \\
X(T^{ad}) = Z\Phi =: Q
\end{array}
\]
where \( P \) is the weight lattice and \( Q \) is the root lattice. We thus obtain
\[ [P : Q] = |\pi_1(G)| \cdot |Z_G|. \]

The punchline is that \( X(T) \) is a \( \mathbb{Z} \)-lattice between \( P \) and \( Q \) and 'keeps track' where \( G \) is positioned between \( \tilde{G} \) and \( G^{ad} \).

**26 Lecture 26 - May 29**

Consider the following setup. Given an isogeny \( G_1 \xrightarrow{f} G_2 \) between connected compact semisimple Lie groups, we get
\[ Z_{G_2} = Z_{G_1}/\ker f \]
as seen in the homework.

**Definition 27.** A connected, compact Lie group \( G \) is said to be of adjoint type if the natural map
\[ G \longrightarrow \text{GL}(g) \]
is faithful.
The definition explains itself with the trivial observation that for any compact connected Lie group \( G \), we have
\[
G^{\text{ad}} = G/Z_G \xrightarrow{\text{Ad}} \text{GL}(g) = \text{GL}(g^{\text{ad}})
\]
and thus the adjoint group \( G^{\text{ad}} \) is always of adjoint type.

**Example 60.** \( SO(2m + 1) \) is of adjoint type for every \( m \geq 2 \).

By the isogeny above, we get an isomorphism between the adjoint groups
\[
G_1^{\text{ad}} \xrightarrow{\cong} G_2^{\text{ad}}
\]
and a commutative diagram
\[
\begin{array}{ccc}
\tilde{G}_1, \tilde{c}_1 & \xrightarrow{\tilde{f}} & \tilde{G}_2, \tilde{c}_2 \\
\text{isogeny} & & \text{isogeny} \\
G_1 & \xrightarrow{f} & G_2
\end{array}
\]
Hence the lift \( \tilde{f} \) is also an isogeny but then \( \pi_1(\tilde{G}_2) = 1 \) implies that \( \tilde{f} \) has degree 1 i.e. is an isomorphism.

Therefore the isogeny class is a finite poset with a *unique* maximal element \( (\tilde{G}) \) and a *unique* minimal element \( (G^{\text{ad}}) \).

**Remark.** Choosing a Weyl chamber gives a basis \( B \) of the root system, and hence a choice of positive roots \( \Phi^+ = \Phi^+(B) \), and from this a \( \mathbb{Z} \)-basis \( B \) of \( \mathbb{Z}\Phi = X(T^{\text{ad}}) \) that works as
\[
T^{\text{ad}} \xrightarrow{\cong} \prod_{b \in B} S^1 \quad t \mapsto (t^b)_{b \in B}.
\]

In the adjoint case, a basis of the character lattice gives a basis of the root system. Likewise if \( B^\wedge \) is a basis of the dual root system \( \Phi^\wedge \), then \( B^\wedge \) is a \( \mathbb{Z} \)-basis of \( \mathbb{Z}\Phi^\wedge \).

We then conclude that the \( \mathbb{Z} \)-dual basis of \( B^\wedge \) is a \( \mathbb{Z} \)-basis of the weight lattice \( P = \text{Hom}_\mathbb{Z}(\mathbb{Z}\Phi^\wedge, \mathbb{Z}) \subset X(T)_\mathbb{Q} \). The elements of such \( \mathbb{Z} \)-dual basis are hence called *fundamental weights*.

The upshot is that by choosing a basis \( B \) of the original root system, we also get a dual basis \( B^\wedge \) and hence the fundamental weights. This fundamental weights correspond, via the Weyl character formula, to *fundamental representations*.

Recall that the fundamental covering \( \tilde{G} \xrightarrow{\pi} G \) gives an isogeny of maximal tori \( \tilde{T} \longrightarrow T \) and has \( \ker \pi = \pi_1(G) \), thus
\[
X(T) \xrightarrow{\text{index} [\pi_1(G)]} X(\tilde{T}) \subset P = \text{Hom}_\mathbb{Z}(\mathbb{Z}\Phi^\wedge, \mathbb{Z})
\]
We then have the following important result, whose proof is in a handout.

**Theorem 118.** \( X(\tilde{T}) \subset P \) is, in fact, always an equality. That is, the weight lattice \( P \) coincides with the character lattice of the maximal torus of the fundamental cover.

The proof does not consist of building a connected covering space of \( G \) of degree \([P : X(T)]\). Instead, we take an open subset of \( \tilde{G} \) with the same fundamental group and build a covering space of this.

**Corollary 119.** The index of the inclusion \( X(T) \subset P \) is then \( [\pi_1(G)] \).
**Example 61.** Take $G = \text{SO}(m)$ for $m \geq 3$. Then $[P : X(T)] = 2$ which corresponds to the degree-2 universal covering $\text{Spin}(m) \rightarrow \text{SO}(m)$.

**Example 62.** Consider $\text{SU}(n)$ and $\text{Sp}(n)$ for $n \geq 2$. We expect the weight lattice $P$ to be equal to $X(T)$ as these groups are simply connected. We have the equivalence

$$P = X(T) \iff Z\Phi^\wedge = X^\ast(T),$$

that is, $\Phi^\wedge$ contains a $\mathbb{Z}$-basis of $X^\ast(T)$.

This fact is easier to check explicitly, e.g.

$$X^\ast(T) = \left\{ \sum_j x_j = 0 \right\} \subset \mathbb{Z}^n \text{ has as integral basis } \{a_i^\ast - a_{i+1}^\ast\}_{i=1}^{n-1} \subset \Phi^\wedge$$

while $X(T) = \mathbb{Z}^n/\mathbb{Z}$ where we quotient with the diagonally embedded copy of $\mathbb{Z}$.

So for the SU($n$)-case we can check this by hand, and it works because $\pi_1(\text{SU}(n)) = 1$.

In which sense can a general group be built up of simple factors? The next thing we want to do is relate the irreducible decomposition of the root system to the 'almost direct product' decomposition of $G$ (a compact, connected, semisimple Lie group).

**Example 63.** An instance of what we want to obtain is the isomorphism

$$\text{SO}(4) \cong (\text{SU}(2) \times \text{SU}(2))/\mu_2.$$

On HW9, ex.5 one shows that the minimal nontrivial closed, connected, normal subgroups $\{G_i\}$ of $G$ are pairwise commuting and $T_i = T \cap G_i$ is a maximal torus in $G_i$ for every $i$. Moreover,

$$\prod G_i \rightarrow G$$

is an isogeny which induces an isogeny between maximal tori

$$\prod T_i \rightarrow T.$$

So we get an isomorphism of rational character lattice

$$\prod X(T_i)_{\mathbb{Q}} \xrightarrow{\cong} X(T)_{\mathbb{Q}}$$

and a correspondence between roots

$$\bigsqcup \Phi_i \leftrightarrow \Phi.$$

We obtain then the irreducible components

$$\{(X(T_i)_{\mathbb{Q}}, \Phi_i)\} \text{ for the root system } (X(T)_{\mathbb{Q}}, \Phi).$$

Furthermore, if we are given a decomposition of the root system, we can reconstruct the normal subgroups it comes from by using the formula

$$(Z_G(T_i))' = G_i$$

where the left hand side is the commutator of the centralizer of the maximal torus $T_i$. 

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Remark. In fact, every closed, connected, normal subgroup \( N \triangleleft G \) is generated by some of the \( G_i \)'s.

What does the discussion above say at the level of the adjoint group? We obtain isomorphisms of direct products

\[
\prod \tilde{G}_i \xrightarrow{\cong} \tilde{G} \quad \text{and} \quad \prod G_{i}^{ad} \xrightarrow{\cong} G^{ad}.
\]

If \( G \) is simply connected, then \( Z_G \) is dual to \( X(T)/Z\Phi = P/Q \).

**Example 64.** Consider \( SU(12) \) and \( Spin(10) \), both simply connected. So we can try to cook up semisimple groups by taking 'almost direct products of those 2, for example

\[
(SU(12) \times Spin(10))/\mu_4 \text{ or } (SU(12) \times E_6)/\mu_3.
\]

In general we will have

\[
\prod \text{simply connected pieces} \div \prod \text{subgroups of the centers} = \text{generic semisimple compact connected Lie group}
\]

and for the roots and coroots

\[
\Phi = \bigsqcup \Phi_i \Rightarrow \Phi^\wedge = \bigsqcup \Phi_i^\wedge.
\]

We also have

\[
\prod \text{Hom}_Z(\mathbb{Z}\Phi^\wedge_i, \mathbb{Z}) = \text{Hom}_Z(\mathbb{Z}\Phi^\wedge, \mathbb{Z}) = X(\tilde{T}) \text{ at the universal cover level}
\]

and

\[
\prod \mathbb{Z}\Phi_i = \mathbb{Z}\Phi = X(T^{ad}) \text{ at the adjoint level}.
\]

We will see that we can read off the structure of the center \( Z_G \) from the Dynkin diagram of \( G \).

**Theorem 120.** The \( \tilde{G}_i \)'s are classified up to isomorphism by the irreducible root systems. Or equivalently, if \( G \) and \( G' \) have isomorphic root systems and are both simply connected, then \( G \cong G' \).

This result is not obvious at all! There is also an existence problem: are there compact connected Lie groups whose root system is a prescribed one? The answer to this question is not trivial, especially for the exceptional root systems.

The theory of semisimple Lie Algebras over \( \mathbb{C} \) allows one to show the following

**Theorem 121** (Existence-Isomorphism-Isogeny theorem). 1. (Existence and Isogeny) The isomorphism classes of semisimple, simply connected Lie groups are classified by the irreducible, reduced root systems. Or equivalently, root systems characterize isogeny classes of semisimple, compact, connected Lie groups.

**Remark.** It is important that \( G \xrightarrow{\cong} G' \) induces an isomorphism \( \tilde{G} \xrightarrow{\cong} \tilde{G}' \).

An equivalent formulation is: the simple, simply connected, semisimple Lie groups are in bijective correspondence with irreducible, reduced, root systems.

It is also possible to show that the automorphisms of the Dynkin diagram is the outer automorphism group of \( \tilde{G} \). Ths we could have different central quotients of \( \tilde{G} \) which are isomorphic.

2. (Isomorphism) The isomorphism classes of compact, connected, Lie groups are in bijective correspondence with isomorphism classes of root data.
Definition 28. Given a pair \((G, T)\), we have a root datum

\[ R(G, T) = (X(T), \Phi, X_*(T), \Phi^\wedge) \]

with reflections \(r_a\) and \(r_{a^\wedge}\) satisfying the same properties as in the definition of a root system. In general a root datum is given by two pairs

\[ (X, \Phi \subset X - \{0\}, X^\wedge, \Phi^\wedge \subset X^\wedge - \{0\}) \]

with some additional integral structure with respect to a usual root system, but without requiring that \(Q\Phi = X(T)_Q\). This 'missing' rational condition allows us to consider cases where there is a nontrivial central torus.

The proof of the theorem is way beyond the scope of this course.

27 May 31

Recall what we proved last time. For every \(G\) compact, conencted, Lie group we have that

1. \(Z_0 \times G' \rightarrow G\)

   is an isogeny, with \(G'\) semisimple and \((G')' = G'\). \(Z_0^\circ\) is the maximal central torus. Hence \(G = G'\) if and only if \(Z_0^\circ\) is finite.

2. If \(G\) is semisimple, then we have

   \(\prod \tilde{G}_i / Z \rightarrow G\)

   where each \(\tilde{G}_i\) is semisimple, simply connected, with irreducible root system \(\Phi_i = \Phi_i(\tilde{G}_i, \tilde{T}_i)\). Those are the irreducible components of \(\Phi(G, T)\) where \(T = (\prod \tilde{T}_i) / Z\) is the maximal torus of \(G\) and \(Z \subset \prod Z_{\tilde{G}_i}\) is a finite subgroup.

Remark. The images of \(\tilde{G}_i\) in \(G\) are what we call simple factors, and they in fact they turn out to be the minimal normal subgroups.

The simply connected case is then the most interesting from the representation-theory point of view, because if \(G = H / Z\) with \(Z\) a central subgroup, then

\[ \text{Rep}(G) = \{ \phi \in \text{Rep}(H) \text{ such that } \phi \text{ is trivial on } Z \} \]

Hence in our previous setup in order to understand representations of \(G\) we just need to understand \(\text{Rep}(\tilde{G}_i)\) and then figure out which ones factors through \(Z\).

From now on, we will denote by \(\text{Irrep}(G)\) the irreducible representations of \(G\). To describe \(\text{Irrep}(G)\), the key case is

\[ G = (\text{torus}) \times \prod (\text{simply connected, simple, semisimple } G_i) \]

Proposition 122. Let \(G_1, G_2\) be compact Lie groups (we’re not assuming they are connected in order to generalize the correspondant result on finite groups), and let \(G = G_1 \times G_2\). Then

1. If \((V_i, \rho_i) \in \text{Irrep}(G_i)\) for \(i \in \{1, 2\}\), then \(V_1 \otimes V_2 \in \text{Irrep}(G)\).
2. If $V \in \text{Irrep}(G)$, then there exist unique (up to isomorphism) $V_i \in \text{Irrep}(G_i)$ for $i \in \{1, 2\}$ such that $V_1 \otimes V_2 \cong V$.

Proof. 1. We want to use character theory. Let $\chi_i$ be the character of $V_i$ on $G_i$, then $V_1 \otimes V_2$ has character $\chi_1 \otimes \chi_2$ which acts as

$$\left(\chi_1 \otimes \chi_2\right)(g_1, g_2) = \chi_1(g_1) \cdot \chi_2(g_2).$$

We just need to check that $(\chi_1 \otimes \chi_2, \chi_1 \otimes \chi_2) = 1$ when we use integration on $G_1 \times G_2$. Now, the product measure on $G_1 \times G_2$ is the volume-1, invariant measure $d\chi_1 \otimes d\chi_2$ on $G = G_1 \times G_2$, where $d\chi_1$ and $d\chi_2$ are the volume-1 Haar measures on $G_1$ and $G_2$ respectively. Denote $\chi = \chi_1 \otimes \chi_2$, we get

$$\int_G |\chi|^2(g) \, d\chi = \int_{G_1} |\chi_1|^2(g) \, d\chi_1 \cdot \int_{G_2} |\chi_2|^2(g) \, d\chi_2 = 1 \cdot 1 = 1$$

by Fubini.

Notice that what made this proof easy is that we have an easy-to-check character-theory criterion for irreducibility.

2. To prove this fact by a general technique, we would need the Peter-Weyl theorem for the group ring, so we will use another approach.

As $G_1$-representation, $V_1 \otimes V_2$ is just given by $\dim V_2$-copies of $V_1$, similarly if we consider $V_1 \otimes V_2$ as a $G_2$-representation. Thus once we prove existence of the $V_i$'s, the uniqueness is clear as $V \cong V_1 \otimes V_2$ is isotypic of type $V_1$ as a $G_1$-representation and isotypic of type $V_2$ as a $G_2$-representation.

Consider now $V \neq 0$ as a $G_1$-representation and choose $0 \neq V_1 \subset V$ a $G_1$-irreducible submodule of $V$. As $G_2 \subset G$ commutes with $G_1$, $G_2 \curvearrowright V$ is a $G_1$-equivariant action and hence preserves the $G_1$-isotypic subspaces.

Take thus the $V_1$-isotypic subspace $W_1 \subset V$: this is both $G_1$- and $G_2$-stable, hence this is a $G = G_1 \times G_2$-submodule of $V$, but as $V$ is irreducible we must have $W_1 = V$.

Remark. We have to consider the $V_1$-isotypic submodule of $V$ because there is no canonical copy of $V_1$ in $V$!

For any $W$ which is a $V_1$-isotypic $G_1$-representation, we have the map

$$V_1 \otimes \text{Hom}_{G_1}(V_1, W) \rightarrowtail W \quad v_1 \otimes T \mapsto T(v_1)$$

but by Schur lemma, $\text{Hom}_{G_1}(V_1, W) \cong \mathbb{C}^n$ where $n$ is the multiplicity of $V_1$ in $W$, hence $W \cong V_1^\otimes n$ by the isomorphism above.

In our situation, taking $W_1 = V$ gives

$$V_1 \otimes_{\mathbb{C}} \text{Hom}_{G_1}(V_1, V) \rightarrowtail V$$

but using the $G_2$-action on $\text{Hom}_{G_1}(V_1, V)$ and on $V$ turns this isomorphism into a $G_1 \times G_2$-equivariant map (the action of $G_1$ is only on the first factor $V_1$).

But then, $\text{Hom}_{G_1}(V_1, V)$ must be an irreducible $G_2$-representation, or $V$ would not be irreducible for $G$. This concludes the proof.

\[\square\]
Now we focus on a compact connected Lie group $G$ that is also simply connected and semisimple. We want to understand the irreducible representations of $G$ via a map

$$\text{Rep}(G) \rightarrow \text{Rep}(T)^W.$$  

We saw in a previous lecture that every class function on $G$ gives rise to a Weyl-invariant class function on $T$, so we want to describe $\chi|_T$ for $\chi$ the character of an irreducible representation of $G$: the answer to this question will be given by Weyl’s character formula.

We will be interested in describing the Weyl character formula for such a $G$, in terms of its root systems (see also section 5 of Chapter VI of the textbook).

We introduce

$$\rho := \frac{1}{2} \sum_{a \in \Phi^+} a \in X(T)_\mathbb{Q},$$

and we will see in fact that $\rho \in X(T)$ when $G$ is simply connected.

**Example 65.** Consider the root system $B_2 = C_2$ for the group $\text{Sp}(2) = \text{Sp}_{2n}(\mathbb{C})$.

![Diagram of root system $B_2 = C_2$](diagram.png)

clearly we have

$$\rho = \frac{1}{2} (a + (b + 2a) + (b + a) + b) = 2a + \frac{3}{2}b.$$  

Last time we calculated the coroot lattice and we got that for $G$ simply connected the weight lattice is

$$P = \text{Hom}_\mathbb{Z}(\mathbb{Z}\Phi^\vee, \mathbb{Z}) = X(T)$$

But we also know from last time that

$$P = \mathbb{Z}a \oplus \mathbb{Z}b \frac{2}{2}$$

in this case, thus

$$\rho \in \mathbb{Z}a \oplus \mathbb{Z}b \frac{2}{2} = P = X(T).$$

**Proposition 123.** If $G$ is simply connected, then $\rho \in X(T)$. Notice that the claim is equivalent to

$$\langle \rho, a_{\text{wedge}} \rangle = 1 \quad \forall a \in B$$
Proof. Recall the formula for a reflection $r_c(x) = x - \langle x, c^\lor \rangle c$ for every $x \in X(T)_\mathbb{Q}$. Then when $c = a \in B$ and $x = \rho$ we get

$$r_a(\rho) = r_a\left(\frac{1}{2} \sum_{c \in \Phi^+} c \right) = \frac{1}{2} \sum_{c \in \Phi^+} r_a(c).$$

We know that $r_a$ permutes the roots $\Phi$, consider then its action on the subset of positive roots $\Phi^+$. Obviously $r_a(a) = -a \notin \Phi^+$, while for any other positive root $c = \sum_{b \in a} n_b b + n_a a$ we have

$$r_a(c) = r_a\left(\sum_{b \in a} n_b b + n_a a \right) = \left(\sum_{b \in a} n_b b + n_a a \right) - ma = \sum_{b \in a} n_b b + (n_a - \mu) a.$$

As $c \neq a$, some of the $n_b$ is strictly positive, and thus $r_a(c)$ must be again a positive root, as every root is either an all-negative or all-positive linear combination of simple roots in $B$ and we just discarded the first possibility. In particular, we showed that

$$r_a(\Phi^+ - \{a\}) = \Phi^+ - \{a\},$$

i.e. $r_a$ permutes the positive roots different from $a$. Therefore

$$r_a(\rho) = \frac{1}{2} \sum_{c \in \Phi^+} r_a(c) = \frac{1}{2} \sum_{c \in \Phi^+} r_a(c) - \frac{1}{2} a = \rho - a.$$

But the generic formula also gives us

$$r_a(\rho) = \rho - \langle \rho, a^\lor \rangle a$$

and thus we must have

$$\langle \rho, a^\lor \rangle = 1$$

which concludes the proof. \hfill \qed

27.1 Weyl character formula

As above, let $G$ be a simply connected, semisimple, compact Lie group and $T \subset G$ a maximal torus. We want to describe representations of $G$ in terms of the highest weight. Denote by $W = W(G, T)$ the Weyl group and define

$$\varepsilon : W \rightarrow \{\pm 1\} \quad w \mapsto \det (w|_{X(T)_\mathbb{Q}})$$

so that

$$\varepsilon(r_c) = -1 \quad \forall c \in \Phi$$

because the only root $r_c$ switches sign to is exactly $c$.

Choose a Weyl chamber $K$ and consider the basis $B = B(K)$ and the positive roots $\Phi^+ = \Phi^+(B) \subset \Phi$. Choose $\lambda \in X(T)$ such that

$$\langle \lambda, a^\lor \rangle \geq 0 \quad \forall a \in B,$$

so that $\lambda$ belongs to the Weyl chamber associated to $B^\lor$.

Theorem 124. 1. (Theorem of highest weight) There exists a unique $V_\lambda \in \text{Irrep}(G)$ such that all the $T$-weights are of the form

$$\lambda - \sum_{b \in B} n_b b \quad \text{for} \quad n_b \in \mathbb{Z}_{\geq 0},$$

and the $\lambda$-weight space is one-dimensional.
2. (Weyl’s character formula) For \( t \in T \) that is regular, that is \( t^a \neq 1 \) for any \( a \in \Phi \), we have

\[
\text{Tr} \left( t|_{V_{\lambda}} \right) = \frac{\sum_{w \in W} \varepsilon(w) t^{w(\lambda + \rho)}}{\sum_{w \in W} \varepsilon(w) t^{w(\rho)}}
\]

which is well-defined, because one can verify that the denominator is always nonzero on the regular locus \( T^{\text{reg}} \).

3. (Dimension formula) We have

\[
\dim V_{\lambda} = \prod_{a \in \Phi^+} \frac{\langle \lambda + \rho, a^{\text{wedge}} \rangle}{\langle \rho, a^{\wedge} \rangle}.
\]

This theorem will strongly help us in decomposing a given representation \( V \), because every irreducible representation is uniquely identified by its highest weight, so if some weight \( \lambda \) is maximal in \( V \), we know that \( V_{\lambda} \) appears in \( V \) with multiplicity equal to the dimension of the \( \lambda \)-weight space in \( V \).

28 June 3

Today we will discuss some low rank cases.

**Example 66.** What follows is the root system for SU(3), where as a basis we take \( B = \{a, b\} \subset \Phi \). We have \( \langle a, b^\wedge \rangle = -1 = \langle b, a^\wedge \rangle \).

In general, let \( G \) be a simply connected, semisimple, compact Lie group and \( T \) a maximal torus. Take \( \Phi^+ \subset \Phi(G, T) \) to be a positive system of roots associated to some Weyl chamber \( K \).
and let \( B = B(K) \) be the simple positive roots in \( \Phi^+ \).

Recall that
\[
K = \{ \xi \in X(T)_\mathbb{R} = t^* | (\xi, a^\vee) > 0 \forall a \in B \}
\]
and the condition \( (\xi, a^\vee) > 0 \) is equivalent to \( (\xi|a) > 0 \) once we fix a Weyl-invariant inner product \( (\cdot | \cdot) \) that identifies \( X(T)_\mathbb{R} \) with its dual.

In the simply connected case we have
\[
X(T) = P = \text{Hom}_\mathbb{Z}(\mathbb{Z}\Phi^+, \mathbb{Z}).
\]

Let \( Q = \mathbb{Z}\Phi = \mathbb{Z}a \oplus \mathbb{Z}b \) be the root lattice. What is the weight lattice? We know that
\[
[P:Q] = |Z_G| \cdot |\pi_1(G)| = |Z_G|
\]
and in our example \( Z_G = \mu_3 \), thus \([P:Q] = 3\). Moreover
\[
P = \{ xa + yb | x,y \in \mathbb{Q} \text{ and } (xa + yb, c^\vee) \in \mathbb{Z} \forall c \in \Phi \},
\]
as the coroot lattice is spanned by \( B^\vee \).

We know that \( (a, a^\vee) = 2 = (b, b^\vee) \) and we also know \( (a, b^\vee) = -1 = (b, a^\vee) \) so the conditions
\[
\begin{align*}
(xa + yb, a^\vee) &= n \in \mathbb{Z} \\
(xa + yb, b^\vee) &= m \in \mathbb{Z}
\end{align*}
\]
eventually give us
\[
P = \mathbb{Z}\left(\frac{2a + b}{3}\right) \oplus \mathbb{Z}\left(\frac{a + 2b}{3}\right).
\]

In fact, we don’t need this whole calculation, we can just boost up the torus
\[
SU(3) \supset T = \left\{ \begin{pmatrix} z & \zeta' \\ \zeta & \zeta' \end{pmatrix} \right\} \in (S^1)^3
\]
which is the standard torus \( T \) in \( SU(3) \).

The induced map on character lattices is
\[
X(T) \leftarrow \mathbb{Z}^3 / \mathbb{Z} = X((S^1)^3 / T_G)
\]
where \( T_G \) is the diagonally embedded torus. Given \( \{e_1, e_2, e_3\} \) a basis of \( X((S^1)^3) \), and denoting by \( \overline{e_i} \) the images in the quotient \( X((S^1)^3 / T_G) \), let’s check that the inclusion
\[
\mathbb{Z}\overline{e_1} \oplus \mathbb{Z}\overline{e_2} \oplus \mathbb{Z}\overline{e_3} \supset P \supset Q = \mathbb{Z}(\overline{e_1} - \overline{e_2}) \oplus \mathbb{Z}(\overline{e_2} - \overline{e_3}) = \mathbb{Z}a \oplus \mathbb{Z}b
\]
has index 3.

We have
\[
3e_1 - 2(e_1 - e_2) + (e_2 - e_3) + (e_1 + e_2 + e_3) \Rightarrow 3\overline{e_1} = 2a + b
\]
and similarly \( 3\overline{e_2} \in \mathbb{Z}a \oplus \mathbb{Z}b \).

Remark. In section 6, Ch. V, \( \rho \) is listed for all the classical types (for a specific choice of basis).
Let's recall the statement of the character formula and see which representations correspond to highest weights.

**Theorem 125.** We have a bijection

\[ \overline{K} \cap P \leftrightarrow \text{Irrep}(G) \quad \lambda \mapsto V_\lambda \]

where \( V_\lambda \) is the unique irreducible representation having highest \( T \)-weight \( \lambda \) with respect to the \( B \)-coefficient expansions in \( X(T)_Q = \bigoplus_{a \in B} Qa \).

By 'highest' we mean that every other nontrivial weight has \( B \)-coefficient not larger than \( \lambda \) in each root in \( B \).

**Remark.** The basis \( B \) is a span of the root lattice, not of the weight lattice!

Moreover, there exists a unique irreducible representation with highest \( T \)-weight \( \lambda \) with respect to the \( B \)-coefficients expansion in \( X(T)_Q = \bigoplus_{a \in B} Qa \) and the \( \lambda \)-weight space is one-dimensional.

Finally, for

\[ t \in T^{reg} = \{ t \in T \mid t^a \neq 1 \quad \forall a \in \Phi \} \]

we have the character formula

\[ \text{Tr}(t|V_\lambda) = \frac{\sum_{w \in W} \varepsilon(w) t^{w(\lambda + \rho)}}{\sum_{w \in W} \varepsilon(w) t^{w(\rho)}} \]

and the dimension formula

\[ \dim V_\lambda = \prod_{a \in \Phi^+} \frac{\langle \lambda + \rho, a^\vee \rangle}{\langle \rho, a^\vee \rangle}. \]

The character formula is in practice very hard to use, even if in fact it can be used to compute multiplicities in decomposition of tensor products.

**Proof.** It can be found in Chapter VI, Thm 1.3 and Prop. 2.6(ii) of the textbook. The proofs are quite heavily computational and rely on a crucial way on Weyl integration formula and Peter-Weyl theorem (see chapter III). We will talk about the latter on Wednesday. \( \square \)

**Example 67.** Let \( G = \text{SU}(2) \) and choose the maximal torus

\[ T = \{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \} \sim \text{su}(2)_\mathbb{C} = \mathfrak{sl}_2(\mathbb{C}) \]

which has as nontrivial weights

\[ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mapsto z^{a^2}. \]

Identify then \( X(T) \cong \mathbb{Z} \) by using the projection on the first entry, then we get

\[ \Phi = \{ \pm 2 \} \subset X(T) = \mathbb{Z} \]

and we choose the positive root(s) to be \( \Phi^+ = \{ 2 \} \) so that \( B = \{ 2 \} \) is the associated basis, and the associated Weyl chamber is just \( K = \mathbb{R}_{>0} \).

\[ \begin{array}{ccc}
-2 & 0 & 2 \\
\end{array} \]

As we should have \( \langle a, a^\vee \rangle = 2 \), we obtain \( \Phi^\vee = \{ \pm 1 \} \), and we’re looking for elements of \( \mathbb{Z} \Phi^\vee \) which have integral pairing against \( \Phi^+ \). As

\[ P = \mathbb{Z} = X(T) \supset 2\mathbb{Z} = \mathbb{Z} \Phi, \]

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we get
\[ \overline{K} \cap P = \mathbb{Z}_{\geq 0}, \]
hence the highest weights are indexed by non-negative integers.

Now we look for a representation of highest weight \( n \). There’s only one positive root \( a = 2 \), hence \( \rho = \frac{a}{2} = 1 \), so we have \( \dim V_n = \frac{n+1}{2} = n + 1 \) by the dimension formula.

We know one representation of this dimension! Take \( \text{Sym}^n(\mathbb{C}^2) \) where by \( \mathbb{C}^2 \) we mean the standard action of \( \text{SU}(2) \) on it. A \( \mathbb{C} \)-basis of this representation is given by
\[ \{ e_1^n e_2^{-j} \}_{j=0}^n \]
and the action as a character of a vector in this basis is
\[ e_1^j e_2^{n-j} \left( \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \right) = z^j \cdot \left( \frac{1}{z} \right)^{n-j} = \frac{z^j}{z^{n-j}} = z^{2j-n} \]
thus \( e_1^j e_2^{n-j} \) has weight \( 2j - n \), so
\[ \text{Sym}^n(\mathbb{C}^2) \text{ has weights } \{ n, n-2, \ldots, -n+2, -n \}. \]

The highest weight \( n \) has weight space of dimension 1 in \( \text{Sym}^n(\mathbb{C}^2) \), as it is spanned by \( e_1^n \), hence \( V_n \) appears with multiplicity 1 in \( \text{Sym}^n(\mathbb{C}^2) \) (we know it appears since \( V_n \) is the only irreducible representation with highest weight \( n \)).

Thus we have
\[ V_n \subset \text{Sym}^n(\mathbb{C}^2) \quad \text{and} \quad \dim V_n = n + 1 = \dim \text{Sym}^n(\mathbb{C}^2) \]
so that the two must coincide! \( V_n \cong \text{Sym}^n(\mathbb{C}^2) \).

**Corollary 126.** The symmetric power of the standard representation \( \text{Sym}^n(\mathbb{C}^2) \) is irreducible.

Sometimes (in more complicated examples) the outline of what we did above is in fact the easiest way to prove that some representation is irreducible: we find its highest weight \( \lambda \) and compare its dimension to the dimension of the unique irreducible representation \( V_\lambda \) having \( \lambda \) as highest weight. If the dimensions match, the representation we are given is in fact an isomorphic copy of \( V_\lambda \) and in particular is irreducible.

Let \( t = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \), then we have
\[ \text{Tr}(t|V_n) = z^n + z^{n-2} + \ldots + z^{-n} = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} \]
as long as \( z \neq \pm 1 : \)
\[ \text{this is exactly what Weyl’s character formula says, because } z \neq \pm 1 \text{ is the condition for } t \text{ being regular, and the rightmost side is what we get when we think about } j \in \mathbb{Z} = X(T) \text{ as } t^j = z^j \in \mathbb{C}^\times. \]

**Example 68.** Let \( G = \text{SU}(3) \). We have the three highest weights \( \frac{2a+b}{3}, \frac{a+2b}{3} \) and \( a + b \). Using the dimension formula we get
\[ \dim V_{\frac{2a+b}{3}} = 3, \quad \dim V_{\frac{a+2b}{3}} = 3, \quad \dim V_{a+b} = 8. \]
It is easy to see that the two 3-dimensional representations correspond to the standard representation and its dual: the standard representation has \( T \)-weights \( \overline{e_1}, \overline{e_2} \) and \( \overline{e_3} \) each with multiplicity 1, where \( \overline{e_i} \) is the image of \( e_i \in X((S^1)^3) \) (element of the standard basis) under the canonical map
\[ X(T) \hookrightarrow X((S^1)^3). \]
By comparison with our choice of basis \( \{a, b\} \), we have
\[
\overline{e_1} = \frac{2a + b}{3}, \quad \overline{e_2} = \overline{e_1} - a, \quad \overline{e_3} = \overline{e_1} - a - b
\]
thus \( \overline{e_1} \) must be the highest weight for the standard representation \( \mathbb{C}^3 \), which means \( V_{\frac{2a+b}{3}} \subset \mathbb{C}^3 \) and as both have dimension 3, this is in fact an equality which proves our claim.

Notice that in dimension 3 the pairing
\[
V \otimes \wedge^2(V) \rightarrow \wedge^3(V) \cong \mathbb{C}
\]
is a perfect pairing, thus \( \wedge^2(V) \cong V^* \). In our situation and with \( V = \mathbb{C}^3 \cong V_{\frac{2a+b}{3}} \), this is in fact a map of SU(3)-representation, and the only possible action on \( \mathbb{C} \) is the trivial action, because SU(3) is semisimple hence has no nontrivial 1-dimensional representation (because these corresponds to maps into \( S^1 \)).

Therefore \( \mathbb{C}^3 \otimes (\mathbb{C}^3)^* \) contains a copy of the trivial representation, and it turns out that
\[
V_{a+b} \cong (\mathbb{C}^3) \otimes (\mathbb{C}^3)^*/1_{\mathbb{C}}.
\]

**Example 69.** Let \( G = \text{Sp}(2) \), and see also HW 10. Last time we calculated \( P = \mathbb{Z}a \oplus \mathbb{Z}\frac{b}{2} \) and got the weight lattice

\[
\begin{array}{c}
\vdots \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \vdots & \vdots & \vdots \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \vdots & \vdots \\
\bullet & \vdots & \vdots & \vdots \\
\vdots \\
\end{array}
\]

Then \( V_{a+b} \) has dimension 4 and turns out to be the standard representation, because \( \text{Sp}(2) = \text{Sp}_4(\mathbb{C}) \cong GL_4(\mathbb{C}) \).

Consider the following question: given a character \( \lambda' \) of \( T \), what multeplicity of this show up in a given \( V_\lambda \)? A theorem by Kostant answer this question with a specific formula, see Chapter VI, theorem 3.2 of the textbook.

A theorem by Steinberg, instead, gives us an answer to the following question: given \( \lambda, \mu \in \mathcal{K} \cap P \), how do we decompose \( V_\lambda \otimes V_\mu \) into other irreducibles? The theorem gives us the multelplicities with which each \( V_\gamma \) appears in the tensor product.

Notice that given \( B = \{a_i\} \) and \( B^\wedge = \{a_i^\wedge\} \) a basis of \( \mathbb{Z}\Phi^\wedge \), we have that \( P = \text{Hom}_\mathbb{Z}(\mathbb{Z}\Phi^\wedge, \mathbb{Z}) \) has a \( \mathbb{Z} \)-basis \( \{\varpi_i\} \) dual to \( \{a_i^\wedge\} \).

These \( \varpi_i \)'s are called the *fundamental weights* and the correspondent representations \( \{V_{\varpi_i}\} \) are called *fundamental representations*. 

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Example 70. For SU(3), the fundamental representations are $V_{\frac{2}{3}}$ and $V_{\frac{4}{3}}$.

Fact 127. Every representation is in fact built up of tensor products of these fundamental representations.

Proof. Consider $\otimes_i V^{n_i}_{\varpi_i}$, this has highest weight $\lambda = \sum_i n_i \varpi_i \in \overline{K} \cap P$. Thus $\otimes_i V^{n_i}_{\varpi_i}$ contains $V_\lambda$ exactly once, so we strip it away and proceed to decompose. In particular, every $V_\lambda$ is contained in some tensor product of copies of the $V_{\varpi_i}$'s.

Example 71. Consider SU($n$), the fundamental representations are $\wedge^i(\mathbb{C}^n)$ for $1 \leq i \leq n-1$, where $\mathbb{C}^n$ denotes the standard representations.

To prove this, find the highest weight $\lambda$ of $\wedge^i(\mathbb{C}^n)$ and show that the correspondent $V_{\lambda}$ has the same dimension (using Weyl dimension formula).

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Let $G$ be a compact, semisimple, simply connected Lie group and $T$ a maximal torus. We get the root system $\Phi = \Phi(G,T) \subset X(T) = P = \text{Hom}_\mathbb{Z}(\mathbb{Z}^\Phi, \mathbb{Z})$ where the weight lattice coincide with the character lattice because $G$ is simply connected.

Choose a Weyl chamber $K$ and get the associated basis $B = B(K) = \{a_1, \ldots, a_r\}$ of $\Phi$. We then obtain the dual basis $B^\vee = \{a_i^\vee\}$ of $\mathbb{Z}^\Phi$ and also the fundamental weights $\{\varpi_i\}$ which are a $\mathbb{Z}$-dual basis of $P = X(T)$.

We saw the bijection

$$\overline{K} \cap P \leftrightarrow \text{Irrep}(G) \quad \lambda \mapsto V_\lambda$$

where $V_\lambda$ is the unique irreducible representation having highest weight $\lambda$. In particular, $\{V_{\varpi_i}\}$ are the fundamental representations.

Example 72. Let $G = \text{SU}(n)$ and $T$ be the diagonal maximal torus. Then a basis is given by

$$B = \{a_i = \overline{e_i} = \overline{e_{i+1}} | 1 \leq i \leq n-1\}$$

so that $t^{a_i} = \overline{t}^{i_{i+1}}$ if $t = \text{diag}(t_i)$. As usual, we denote as $\overline{e_i}$ the projections of $e_i \in X(T_{U(n)})$ in the canonical quotient map

$$X(T_{U(n)}) \twoheadrightarrow X(T_{U(n)})/\mathbb{Z} = X(T)$$

where we quotient by the character lattice of the diagonal central torus of $U(n)$.

Then $B^\vee = \{e_i^\vee - e_{i+1}^\vee\} \subset \{\sum_i x_i = 0\} \subset \mathbb{Z}^n$.

Claim 128. The fundamental weights are $\varpi_i = \overline{e_1} + \ldots + \overline{e_i}$ as $1 \leq i \leq n-1$. In particular, each $\varpi_i$ is the highest weight of the $i$-th wedge power of the standard representation $\wedge^i(\mathbb{C}^n)$.

We can then 'draw' the fundamental weights on the Dynkin diagram.

Example 73. For $G = \text{Spin}(n)$, with root system $D_n$, we can label the fundamental representations on the Dynkin diagram:
The fundamental representations $V_{\varpi_{n-1}}$ and $V_{\varpi_n}$ are the trickiest to construct: they are called *half-spin representation* and one needs, in fact, Clifford construction in order to show a copy of them.

We want now to describe the representations of the group $G$; let $T$ be a maximal torus. Consider then the representation ring

$$R(G) = \bigoplus_{\rho \in \text{Irrep}(G)} \mathbb{Z}[\rho]$$

which uses $\otimes$ as multiplication.

Notice that via the map $\rho \mapsto \xi_\rho$ which sends a representation to its character, we get an embedding

$$R(G) \hookrightarrow \{\text{class functions on } G\},$$

so the question is: which class functions arise as characters of representations?

Notice that the map

$$\{\text{class functions on } G\} \longrightarrow \{\text{Weyl-invariant class functions on } T\} \quad f \mapsto f|_T$$

is injective, because if $f|_T \equiv 1$ by the conjugacy theorems and the invariance of $f$ on conjugacy classes, we get that $f \equiv 1$ on the whole $G$.

This restricts to an injection

$$R(G) \overset{j}{\rightarrow} R(T)^W$$

where the target space is the free abelian group on Weyl-invariant $T$-representations.

**Claim 129.** The map $j$ is in fact an isomorphism when $G$ is a compact, connected, Lie group.

**Remark.** Notice that we are 'forced' to deal with virtual representations, i.e. we allow negative characters.

**Proof.** Let's assume first that $G$ is semisimple and simply connected; a handout will take care of bootstrapping the argument to the general $G$ case, using the fact that such a $G$ is

$$G = \frac{\text{torus } \times \text{ simply connected}}{\text{central subgroup}} \cong \frac{T \times \prod \tilde{G}_i}{Z}. $$

In the general case one uses central characters to deal with central subgroups, but there's an additional obstacle given by the fact that dealing with virtual representations does not allow one to talk about central characters without adding some other argument first. Once the issue of the central subgroup is solved, the isomorphism

$$R() \cong R(T) \otimes \bigotimes_i R(\tilde{G}_i)$$

reduces the problem to the simply connected, semisimple case.

**Proposition 130.** Suppose that $G$ is semisimple and simply connected and let $\{\varpi_i\}_{i=1}^n$ be the fundamental weights. Then the $\mathbb{Z}$-algebra map

$$\mathbb{Z}[Y_{\varpi_1}, \ldots, Y_{\varpi_n}] \longrightarrow R(G)$$

induced by $Y_{\varpi_i} \mapsto [V_{\varpi_i}]$

is an isomorphism.
Proof. Let’s first prove surjectivity: it is enough to show that every \( V_\lambda \) stays in the image for \( \lambda \in \overline{K} \cap P \). Take then \( \lambda = \sum_i n_i \varpi_i \in \overline{K} \cap P \), as \( \lambda \) is in the closure of the Weyl chamber \( K \) we have that
\[
    n_i = \langle \lambda, \varpi_i \rangle \geq 0.
\]
When we consider \( V := \bigotimes_i V_{\varpi_i}^{n_i} \), the weight \( \lambda \) appears in \( V \) and is in fact the highest weight, because every other weight appears with some strictly smaller coefficient with respect to some \( \varpi_i \)'s. Thus
\[
    V \cong V_\lambda \oplus \text{representations with smaller weights},
\]
so that
\[
    \prod_i Y_{\varpi_i}^{j_i} \mapsto [V_\lambda \oplus \text{representations with smaller weights}]
\]
and by induction on the height of the weights we can produce monomials of degree smaller than \( \sum_i n_i \) to take care of the representations with smaller weights, so that once we subtract such monomials to \( \prod_i Y_{\varpi_i}^{j_i} \) we obtain exactly \( [V_\lambda] \) in \( R(G) \).

Now let’s show injectivity. Consider two polynomials \( f, g \in \mathbb{Z}[Y_{\varpi_1}, \ldots, Y_{\varpi_n}] \) such that \( j(f) = j(g) \). As \( j \) is linear on monomials, up to summing some monomials with positive coefficients to both \( f \) and \( g \) we can assume
\[
    f, g \in \mathbb{Z}_{\geq 0}[Y_{\varpi_1}, \ldots, Y_{\varpi_n}].
\]
Consider now a lexicographical ordering on monomials and compare the maximal exponents of \( f \) and \( g \), which correspond to maximal weights of the representations \( j(f) \) and \( j(g) \). But \( j(f) = j(g) \), in particular these representations have the same maximal weights (remember that the weights are only partially ordered!) and thus \( f \) and \( g \) have the same maximal monomials with the same coefficients.

Arguing inductively on the degree of the monomials appearing in \( f \) and \( g \), we get \( f = g \), so injectivity is also proved. \( \square \)

What can we say about the representation ring of the torus? As \( X(T) = \bigoplus_i \mathbb{Z} \chi_i \) we get
\[
    R(T) = \mathbb{Z}[X(T)].
\]
The Weyl group acts as usual on \( X(T) \) and by conjugation on the representation ring \( R(T) \) of \( T \): these two actions match up! The equality above is then an equality of \( W \)-representations, and we have
\[
    \mathbb{Z}[X(T)]^W = \bigoplus_{x \in X(T)/W} \mathbb{Z} \left[ \sum_{w \in W} w(x) \right],
\]
where the right hand side is the free abelian group on the \( W \)-orbits on \( X(T) \).

As \( G \) is simply connected, \( X(T) = P \) so every \( W \)-orbit has a unique representative in \( \overline{K} \), hence
\[
    R(T)^W = \bigoplus_{\lambda \in \overline{K} \cap P} \mathbb{Z} \left[ \sum_{w \in W} w(\lambda) \right].
\]
We can then define the linear map
\[
    \mathbb{Z}[Y_{\varpi_1}, \ldots, Y_{\varpi_n}] \longrightarrow R(T)^W \text{ induced by } Y_{\varpi_i} \mapsto \sum_{w \in W} w(\varpi_i)
\]
and it is possible to check that this is in fact a ring isomorphism.

It only remains to check that this ring isomorphism is compatible with the description of \( R(G) \):
take the character $\chi_{V_{\varpi_i}}|_T$ which contains $\varpi_i$ with multiplicity 1. Since $\xi_{V_{\varpi_i}}$ is a class function on $G$, the entire $W = N_G(T)/T$-orbit shows up with multiplicity 1, and this ultimately shows that the two isomorphisms we defined above are compatible.

The most important consequence of this proposition is that, to reconstruct $R(G)$, it is enough to know $\text{Rep}(T)$ with the $W$-action. In particular if we want to check that some element of the ring $R(G)$ corresponds to an actual representation and not just to a virtual one, we just need to pair it against all the irreducible representations of $G$ and check that the coefficients are non-negative integers.

Let’s finally wrap up two loose ends.

We firstly want to talk about the Peter-Weyl theorem (see chapter III of the textbook): suppose we have a compact Lie group $G$ and we want to relate its finite-dimensional representation theory to $L^2(G)$.

**Theorem 131.** Every irreducible Hilbert space representation of $G$ is in fact finite-dimensional. An Hilbert space representation is a representation on a vector space with an Hilbert space structure. It is irreducible when there are no nontrivial closed stable subspaces.

This is a hard result in functional analysis!

**Theorem 132 (Peter-Weyl theorem).** We have

$$L^2(G) = \bigoplus_{V \in \text{Irrep}(G)} (V^\oplus \dim V)$$

where the right-hand side is a Hilbert space direct sum.

**Example 74.** When $G = S^1$, we obtain the usual Fourier series result:

$$L^2(S^1) = \bigoplus_{n \in \mathbb{Z}} \mathbb{R} e^{i n \theta}$$

**Corollary 133.** The irreducible characters $\{\chi_V\}_{V \in \text{Irrep}(G)}$ span a dense subspace of the class functions $C^0(G)^G$.

An useful consequence is that, to check if two class functions $f_1, f_2$ are equal, it is enough to integrate each against every $\chi_V$ and check the two integrations give the same result. This is for example used (by the textbook) in the proof of Weyl character formula.

Suppose we want to reconstruct the whole space $C^0(G)$ of continuous functions on $G$.

**Theorem 134.** The matrix entries functions $\{a_{i,j}(g)\}$ of irreducible representations of $G$ span a dense subspace of $C^0(G)$. They are called $G$-finite functions or representative functions.

This result is quite surprising as it is not at all obvious what is so special about the matrix entries, so let’s try to dig deeper.

Consider $(V, \rho)$ a finite-dimensional $G$-representation and denote $\rho(g) = (a_{i,j}(g))$. As $\rho(gh) = \rho(g)\rho(h)$ is a product of matrices, we obtain

$$a_{i,j}(gh) = \sum_k a_{i,k}(g)a_{k,j}(h).$$

On $C^0(G)$ we have a left action given by the right regular representation:

$$(h.f)(g) := f(gh) \quad \forall f \in C^0(G), \forall g, h \in G.$$
Suppose then $h \in G$ is fixed and $g \in G$ is varying, the identity of the matrix coefficients becomes

$$h.a_{i,j} = \sum_k a_{k,j}(h) \cdot a_{i,k}$$

which proves that the finite-dimensional span of $\{a_{i,j}\}$ is stable under the right regular representation! Therefore each such collection of matrix coefficients $\{a_{i,j}\}$ generates a finite-dimensional subspace of $C^0(G)$ under the right regular representation. In fact, the union of these stable subspaces generate a dense subspace in the space of all $G$-finite functions.

Finally, we want to show that we can reconstruct $G$ from its representative functions: this is a construction due to Chevalley that link compact Lie groups with matrix groups over $\mathbb{R}$.

Let $G$ be a compact Lie group and consider $A(G)$, the $\mathbb{R}$-algebra of $G$-finite functions: this is a finitely generated $\mathbb{R}$-algebra as we only need the matrix entries of the fundamental representations to produce all the matrix entries, which are dense in the space of $G$-finite functions.

Then Spec$(A(G))$ reconstruct $G$ as an affine variety over $\mathbb{R}$, in the sense that the mapping

$$G \mapsto \text{Spec}(A(G))$$

is an equivalence of categories! This is called Tannaka-Krein duality.