Practice problems

1. Let $f : A \rightarrow B$ be a ring homomorphism and $M$, a flat $A$-module. Show that $B \otimes_A M$ is a flat $B$ module.

2. $A$ is reduced if its nilradical is trivial. Show that being reduced is a local property, i.e $A$ is reduced iff $A_m$ is reduced for each maximal ideal $m$ of $A$. Is being an integral domain also a local property ?

3. Find a minimal primary decomposition of ideal $(x^2)$ in $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$.

4. Find a minimal primary decomposition of ideal $(x, y), (y, z)$ in $\mathbb{R}[x, y, z]$. 
Try and try again. And then if needed, look up

1. Read up on what the $\text{Tor}$ functor is. Then if $M$ is an $A$-module, show that $M$ is flat as an $A$ module iff $\text{Tor}^A_n(M, N) = 0$ for all $A$ modules $N$ iff $\text{Tor}^A_1(M, N) = 0$ for all $A$ modules $N$.

2. Problem 16 in Chapter 3 of Atiyah MacDonald (about faithful flatness)

Mandatory problems

1. Let $S = \{1, f, f^2, \ldots \}$ be a multiplicatively closed set of $S$. Show that $\text{Spec}(A_f) = \text{Spec}(S^{-1}A) \subseteq \text{Spec}(A)$ is the complement of $V((f))$.
   (Identify primes in $A_f$ with their restrictions in $A$ to get $\text{Spec}(A_f) \subseteq \text{Spec}(A)$.)

2. Show that $\text{Spec}(A)$ is Hausdorff iff $\text{Spec}(A)$ is $T_1$ (every singleton is closed) iff every prime ideal of $A$ is maximal iff every $A/N(A)$ module is flat over $A/N(A)$. Here $N(A)$ is the nilradical of $A$

3. Let $k$ be a field. Show that for each integer $1 \leq i \leq n$ and for each integer $m \geq 1$, $P_i = (x_1, x_2, \ldots, x_i)$ is prime in $k[x_1, x_2, \ldots, x_n]$ and $P_i^m$ is primary.