MATH 215A Fall 2017

Homework 1

Due October 11

Please turn in the solutions ONLY to the problems marked mandatory (4×10 pts).

Practice problems

1. Let k be a field and let A be a k-algebra (A is a ring with $k \subset A$ say). Assume A is a domain and of finite dimension as a k-vector space. Show that A is also a field.

2. Let A be a ring and N, its nil radical. Show that A has exactly one prime ideal iff every element of A is a unit or nilpotent element iff A/N is a field. Can you give an example of such a ring ?

3. Let $f : M \to N$ be a homomorphism of A-modules and let $\bar{f} : M/JM \to N/JN$ be the induced morphism of A/J modules where J is the jacobson radical of A. Show that if N is finitely generated, then f is onto if and only if $\bar{f}$ is onto.

4. Give an example where the Nakayamma Lemma fails when the module under consideration is not finitely generated.

5. Show that $A[x]$ is a flat A-algebra.

Try and try again. And then if needed, look up

1. Show that prime ideals of $k[x,y]$ where k is a field are as follows : (0), (f) where $f \in k[x,y]$ is irreducible or m where m is maximal. Show further that $m = (p,g)$ for some p, g where $p(x) \in k[x]$ is irreducible and $g \in k[x,y]$ such that mod p, $\bar{g}$ is irreducible in $(k[x]/(p))[y]$.

2. Let A be a ring. Show that the Jacobson radical of $A[x]$ is equal to its nilradical.

3. Show using Nakayamma's Lemma that every finitely generated projective module over a local ring is free.

Mandatory problems

1. Find the radical of (x^2+y^2-1) in $\mathbb{R}[x,y]$. Does the answer change over $\mathbb{C}[x,y]$? Why or why not. Also find the radical of $(x-1, x^2+y^2-1)$ in $\mathbb{R}[x,y]$.

2. Show that $Spec(A)$ is disconnected if and only if A contains an idempotent $(x = x^2)$ which is not 0 or 1. Hence show that $Spec(A)$ is connected if A is a local ring.

3. Let M be a finitely generated A module and $f : M \to M$ be a surjective A module homomorphism. Show that f is an isomorphism.
   (Hint : Make M into an $A[x]$ module via action of f and use the determinant trick to construct an inverse for f.)

4. Let k be a field. Show that $k[t,y]/(ty-t)$ is NOT flat over $k[t]$.
   (Hint : Look at $k[t] \xrightarrow{\times t} k[t]$)