Unless otherwise stated, all vector spaces in this worksheet are finite dimensional and the scalar field $F$ has characteristic zero.

The following are facts (in no particular order) that you may assume for the purposes of this worksheet (and, as far as I can tell, for the basic exam).

**Theorem 1** (Fund. Thm. of Algebra). Every complex polynomial of degree $\geq 1$ has a root.

**Definition 2.** The characteristic polynomial of an $n \times n$ matrix $A$ is the monic polynomial $\chi_A(t) \in F[t]$ we obtain by row-reducing $tI - A$ until it is in upper triangular form with monic polynomials $p_1(t), \ldots, p_n(t)$ on the diagonal, and then multiplying the diagonal entries, i.e. $\chi_A(t) = p_1(t) \cdots p_n(t)$. All of the eigenvalues of $A$ are roots of $\chi_A$.

**Lemma 3.** If $A$ has the block triangular form

$$A = \begin{bmatrix} R & * \\ 0 & S \end{bmatrix},$$

where $R$ and $S$ are square matrices, then $\chi_A(t) = \chi_R(t)\chi_S(t)$.

**Fact 4.** The determinant $\det A$ is the product of the eigenvalues of $A$ (with multiplicity). Equivalently, it is $(-1)^n$ times the constant coefficient of $\chi_A$. The trace $\operatorname{tr} A$ is the sum of the eigenvalues.

**Definition 5.** If $\chi_L(t) = (t-\lambda)^mq(t)$ with $q(\lambda) \neq 0$ then we say that the algebraic multiplicity of $\lambda$ is $m$. The geometric multiplicity of $\lambda$ is $\dim \ker(L - \lambda I)$; it is always $\leq m$.

**Definition 6.** Let $k$ be the smallest positive integer such that the set $\{1, L, L^2, \ldots, L^k\} \subset \operatorname{Hom}(V, V)$ is linearly dependent. Then there exist $\alpha_0, \ldots, \alpha_{k-1} \in F$ such that

$$L^k + \alpha_{k-1}L^{k-1} + \ldots + \alpha_1L + \alpha_0 = 0.$$

The minimal polynomial of $L$ is defined as

$$\mu_L(t) = t^k + \alpha_{k-1}t^{k-1} + \ldots + \alpha_1t + \alpha_0.$$

**Lemma 7.** Let $L : V \to V$ be a linear operator.

1. If $p(L) = 0$ for some $p \in F[t]$ then $\mu_L$ divides $p$, i.e. $p(t) = \mu_L(t)q(t)$ for some $q(t) \in F[t]$.
2. $\lambda \in F$ is an eigenvalue of $L$ if and only if $\mu_L(\lambda) = 0$.

**Lemma 8.** A polynomial $p(t) \in F[t]$ has $\lambda \in F$ as a multiple root if and only if $\lambda$ is a root of both $p$ and its derivative $p'$. 

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LINEAR ALGEBRA BOOT CAMP
WEEK 2: LINEAR OPERATORS
Warmup. This problem should have been on the first worksheet.

1. Let $A$ and $B$ be two real $5 \times 5$ matrices such that $A^2 = A$, $B^2 = B$, and $1 - (A + B)$ is invertible. Prove that $\text{rank}(A) = \text{rank}(B)$.

S18-2

Eigen–basics. These only require the definition of eigenvector and eigenvalue.

2. Let $V$ be a complex vector space at $T : V \to V$ a linear transformation. Let $v_1, \ldots, v_n$ be nonzero vectors in $V$, each an eigenvector of a different eigenvalue. Prove that $\{v_1, \ldots, v_n\}$ is linearly independent.

F01-10, W02-9, F08-7

3. Must the eigenvectors of a linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ span $\mathbb{C}^n$? Prove it.

F08-8

4. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation and $P(x)$ a polynomial such that $P(T) = 0$. Prove that every eigenvalue of $T$ is a root of $P(x)$.

F04-9

The Cayley-Hamilton Theorem.

Theorem 9 (Cayley-Hamilton). Let $L : V \to V$ be a linear operator on a finite-dimensional vector space. Then $L$ is a root of its own characteristic polynomial; i.e. $\chi_L(L) = 0$. In particular, the minimal polynomial divides the characteristic polynomial.

The next few exercises outline a proof of this theorem. This is not the shortest proof, nor the most elegant; however, several of the ideas in this outline appear on basic exam questions (probably because this is the proof given in Petersen which is used as the main text for the basic exam).

5. The cyclic subspace generated by $x \in V$ is

$$C_x = \text{span}\{x, L(x), L^2(x), \ldots\}.$$ 

Clearly $C_x$ is $L$-invariant.

(a) Suppose $x \neq 0$. Prove that there is a $k \leq \dim(V)$ such that $x, L(x), \ldots, L^{k-1}(x)$ form a basis for $C_x$.

(b) What is the matrix representation $C_p$ for $L|_{C_x}$ with respect to the basis from (a)?

(c) What are the characteristic and minimal polynomials of $C_p$?
(6) **Proof of the Cayley-Hamilton Theorem.** Let \( 0 \neq x \in V \) and let \( M \subset V \) be a subspace such that \( V = C_x \oplus M \). We have \( L(C_x) \subset C_x \) and \( L(M) \subset V \), so there is a basis for \( V \) such that the matrix representation for \( L \) looks like

\[
[L] = \begin{bmatrix} C_p & B \\ 0 & D \end{bmatrix}.
\]

Use this to prove that \( \chi_L(L)(x) = 0 \). Since \( x \) was arbitrary, it follows that \( \chi_L(L) = 0 \). \( \Box \)

(7) (a) Prove that every linear transformation \( T : \mathbb{C}^n \to \mathbb{C}^n \) has an eigenvector.

(b) Is (a) true if we replace \( \mathbb{C} \) by \( \mathbb{R} \)?

(8) Suppose a 4 \( \times \) 4 integer matrix has four distinct real eigenvalues \( \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 \). Prove that \( \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \in \mathbb{Z} \).

(9) (a) Suppose that \( V \) is a real vector space of odd dimension. Prove that a linear transformation \( T : V \to V \) has a nonzero eigenvector.

(b) Show that for every even positive integer \( n \), there is a vector space \( V \) over \( \mathbb{R} \) of dimension \( n \) and an \( \mathbb{R} \)-linear transformation \( T : V \to V \) with no nonzero eigenvectors.

(10) Let \( V \cong \mathbb{R}^n \) be an \( n \)-dimensional vector space over \( \mathbb{R} \), and let \( \text{End}(V) = \text{Hom}(V,V) \). Note that \( \dim \text{End}(V) = n^2 \). Let \( T \in \text{End}(V) \) and show that the subspace \( W \subset \text{End}(V) \) spanned by powers of \( T \) satisfies \( \dim(W) \leq \dim(V) = n \).

(11) Let \( T \) be an invertible linear operator on a finite dimensional vector space \( V \) over a field \( \mathbb{F} \). Prove that there exists a polynomial \( f \) over \( \mathbb{F} \) such that \( T^{-1} = f(T) \).

(12) (a) For each \( n \geq 2 \), is there an \( n \times n \) matrix \( A \) with \( A^{n-1} \neq 0 \) but \( A^n = 0 \)? (Give an example or prove that no such matrix exists).
(b) Is there an \( n \times n \) upper triangular matrix \( A \) with \( A^n \neq 0 \) but \( A^{n+1} = 0 \)? (Give an example or prove that no such matrix exists). 

F05-10

(13) (a) Find a polynomial \( P(x) \) of degree 2 such that \( P(A) = 0 \) for

\[
\begin{pmatrix}
1 & 3 \\
4 & 2
\end{pmatrix}
\]

(b) Prove that \( P \) is unique up to multiplication by a constant.

S12-11

(14) This problem appeared on a Basic Exam in 2014, but it has a mistake. Identify the mistake and provide a counterexample.

(a) Find a real matrix \( A \) whose minimal polynomial is equal to \( t^4 + 1 \).

(b) Show that the usual real linear map \( v \mapsto Av \) has no nontrivial invariant subspace.

S14-1

(15) Suppose that \( V \) is a finite dimensional vector space over \( \mathbb{C} \) and that \( T : V \to V \) is a linear transformation. Suppose that \( F(x) \in \mathbb{C}[x] \) is a polynomial. Show that the linear transformation \( F(T) \) is invertible if and only if \( F(x) \) and the minimal polynomial of \( T \) have no common factors.

S17-5

**Diagonalizability.** A linear operator \( T : V \to V \) is diagonalizable if there is a basis of \( V \) consisting of eigenvectors for \( T \). A matrix \( A \) is diagonalizable if there exists an invertible matrix \( B \) such that \( BAB^{-1} \) is diagonal.

(16) Prove that \( T \) is diagonalizable if and only if for each eigenvalue \( \lambda \) the geometric multiplicity of \( \lambda \) equals the algebraic multiplicity of \( \lambda \).

An immediate corollary is that if the characteristic polynomial has \( n \) distinct roots then \( T \) is diagonalizable.
(17) Prove the Minimal Polynomial Characterization of Diagonalizability.

**Theorem 10.** Let \( L : V \to V \) be a linear operator on an \( n \)-dimensional vector space over \( \mathbb{F} \). Then \( L \) is diagonalizable if and only if the minimal polynomial factors as

\[
\mu_L(t) = (t - \lambda_1) \cdots (t - \lambda_k),
\]

where \( \lambda_1, \ldots, \lambda_k \) are distinct.

(18) Assume \( A \) is an \( n \times n \) complex matrix such that for some positive integer \( m \) we have \( A^m = I_n \), where \( I_n \) is the \( n \times n \) identity matrix. Prove that \( A \) is diagonalizable.

(19) Let \( t \in \mathbb{R} \) such that \( t \) is not an integer multiple of \( \pi \). Prove that the matrix

\[
A = \begin{pmatrix}
\cos(t) & \sin(t) \\
-\sin(t) & \cos(t)
\end{pmatrix}
\]

is not diagonalizable. Now do the same for the matrix

\[
B = \begin{pmatrix}
1 & \lambda \\
0 & 1
\end{pmatrix}
\]

(20) Suppose \( A \) is an \( n \times n \) complex matrix with \( n \) distinct eigenvalues. Prove that if \( B \) is an \( n \times n \) complex matrix such that \( AB = BA \) then \( B \) is diagonalizable.

(21) Suppose that a complex matrix \( A \) satisfies

\[
\ker((A - \lambda I)) = \ker((A - \lambda I)^2) \quad \text{for all } \lambda \in \mathbb{C}.
\]

Show from first principles (i.e. without using the theory of canonical forms) that \( A \) must be diagonalizable.

(22) Let \( V \) be an \( n \)-dimensional vector space \( (n \geq 2) \) over \( \mathbb{C} \) with basis \( e_1, \ldots, e_n \). Let \( T \) be a linear transformation of \( V \) satisfying \( T(e_1) = e_2, \ldots, T(e_{n-1}) = e_n, T(e_n) = e_1 \).

(a) Show that \( T \) has 1 as an eigenvalue and write down an eigenvector with eigenvalue 1.

(b) Is \( T \) diagonalizable? *Hint:* calculate the characteristic polynomial.
(23) Prove that a positive power of an invertible matrix with complex entries is diagonalizable if and only if the matrix itself is diagonalizable.

(24) Let $a_1 = 1$, $a_2 = 4$, $a_{n+2} = 4a_{n+1} - 3a_n$ for all $n \geq 1$. Find a $2 \times 2$ matrix $A$ such that

$$A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}.$$ 

Compute the eigenvalues of $A$ and use them to determine the limit $\lim_{n \to \infty} (a_n)^{1/n}$.

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**Jordan Canonical Form.** I don’t think you’ll need to know how to prove this for the basic exam, so we won’t prove it here.

**Theorem 11** (The Jordan Canonical Form). Let $L : V \to V$ be a linear operator on a finite dimensional complex vector space. Then we can find $L$-invariant subspaces $M_1, \ldots, M_s$ such that

$$V = M_1 \oplus \cdots \oplus M_s$$

and each $L|_{M_i}$ has a matrix representation of the form

$$\begin{bmatrix}
\lambda_i & 1 & 0 & \cdots & 0 \\
0 & \lambda_i & 1 & \cdots & 0 \\
0 & 0 & \lambda_i & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 1 \\
0 & 0 & 0 & \cdots & \lambda_i
\end{bmatrix},$$

where $\lambda_i$ is an eigenvalue for $L$.

(25) Show that if $L : V \to V$ has minimal polynomial $\mu_L(t) = (t - \lambda_1)^{m_1} \cdots (t - \lambda_k)^{m_k}$, then $m_i$ is the size of the largest Jordan block that has $\lambda_i$ on the diagonal.

(26) (a) Explain the following (overly informal) statement:

Every matrix can be brought to Jordan normal form; moreover the normal form is essentially unique.

No proofs are required; however, all statements must be clear and precise. All required hypotheses must be included. The meaning of the phrases ‘brought to’, ‘Jordan normal form’, and ‘essentially unique’ must be defined explicitly.

(b) Define the *minimal polynomial* of a matrix. How may it be determined for a matrix in Jordan normal form?
(27) Let $A, B$ be two $n \times n$ complex matrices which have the same minimal polynomial $M(t)$ and the same characteristic polynomial $P(t) = (t - \lambda_1)^{a_1} \cdots (t - \lambda_k)^{a_k}$, with distinct $\lambda_1, \ldots, \lambda_k$. Prove that if $P(t)/M(t) = (t - \lambda_1) \cdots (t - \lambda_k)$ then these matrices are similar.

(28) Let 
\[ A = \begin{pmatrix} 4 & -4 \\ 1 & 0 \end{pmatrix}. \]

(a) Find the Jordan form $J$ of $A$ and a matrix $P$ for which $P^{-1}AP = J$.

(b) Compute $A^{100}$ and $J^{100}$.

(c) Find a formula for $a_n$, when $a_{n+1} = 4a_n - 4a_{n-1}$ and $a_0 = a, a_1 = b$.

(29) Let $A$ be a $3 \times 3$ matrix with complex entries. Consider the set of such $A$ that satisfy $\text{tr}(A) = 4$, $\text{tr}(A^2) = 6$, and $\text{tr}(A^3) = 10$. For each similarity (i.e. conjugacy) class of such matrices, give one member in Jordan normal form. The following identity may be helpful:

\[
\begin{align*}
    b_1 &= a_1 + a_2 + a_3, \\
    b_2 &= a_1^2 + a_2^2 + a_3^2, \\
    b_3 &= a_1^3 + a_2^3 + a_3^3,
\end{align*}
\]

if \( b_1 = b_2 = b_3 \), then \( 6a_1a_2a_3 = b_1^3 + 2b_3 - 3b_1b_2 \).

(30) Let $A$ be a linear operator on a four dimensional complex vector space that satisfies the polynomial equation
\[ P(A) = A^4 + 2A^3 - 2A - I = 0. \]

Let $B = A + I$ and suppose that the rank($B$) = 2. Finally, suppose that $|\text{tr}(A)| = 2$. Give a Jordan canonical form of $A$.

(31) Suppose $A$ is an endomorphism of a complex vector space with characteristic polynomial
\[ C_A(x) = x^4 - 6x^3 + 13x^2 - 12x + 4. \]

How many similarity (i.e. conjugacy) classes of elements can have this characteristic polynomial? Suppose also that the minimal polynomial $M_A(x)$ is equal to $C_A(x)$. How many classes satisfy this additional condition? Prove your answers, quoting any general theorems you need.
(32) Let \( a, b, c, d \in \mathbb{R} \) and let
\[
M = \begin{pmatrix}
1 & 0 & a & b \\
0 & 1 & 0 & 0 \\
0 & c & 3 & -2 \\
0 & d & 2 & -1
\end{pmatrix}.
\]
(a) Determine the conditions on \( a, b, c, d \) so that there is only one Jordan block for each eigenvalue of \( M \) in the Jordan form of \( M \).

(b) Find the Jordan form of \( M \) when \( a = c = d = 2 \) and \( b = -2 \).

(33) Consider the matrices
\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad
B = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
C = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
D = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}, \quad
E = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
F = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]
Which pairs of these matrices are similar over \( \mathbb{R} \)? You must fully justify your answer.

(34) Let \( T : V \to V \) be a linear operator such that \( T^6 = 0 \) and \( T^5 \neq 0 \). Suppose that \( V \cong \mathbb{R}^6 \).
Prove that there is no linear operator \( S : V \to V \) such that \( S^2 = T \). Does the answer change if \( V \cong \mathbb{R}^{12} \)?

Extra Problems.
(35) Suppose that \( p \in \mathbb{F}[t] \) is a polynomial. Show that
\[
\text{deg} \mu_{p(L)}(t) \leq \text{deg} \mu_L(t).
\]

(36) Show that if \( E : V \to V \) is a projection then \( \text{tr}(E) = \dim(\text{im}(E)) \).

(37) Suppose that \( L \) is an involution, i.e. \( L^2 = 1 \).
(a) Show that \( x \pm L(x) \) is an eigenvector for \( L \) with eigenvalue \( \pm 1 \).
(b) Show that \( V = \ker(L + I) \oplus \ker(L - I) \).
(c) Conclude that \( L \) is diagonalizable.
(38) Let $L : V \to V$ be a linear operator on a finite dimensional vector space and suppose that $M \subset V$ is an $L$-invariant subspace. Prove the following.

(a) If $x + y \in M$, where $Lx = \lambda x$, $Ly = \mu y$, and $\lambda \neq \mu$, then $x, y \in M$.
(b) If $x_1 + \ldots + x_k \in M$ and $L(x_i) = \lambda_i x_i$, with the $x_i$ distinct, then $x_1, \ldots, x_k \in M$.
(c) If $L : V \to V$ is diagonalizable then $L : M \to M$ is diagonalizable.
(d) Prove (c) again using the minimal polynomial characterization of diagonalizability.

(39) Suppose that $L, K : V \to V$ are both diagonalizable and that $KL = LK$. Show that we can find a basis for $V$ that diagonalizes both $L$ and $K$. One strategy is:

(a) Show that $\ker K$ is an $L$-invariant subspace.
(b) Use the previous exercise with $M$ as an eigenspace for $K$.

Proof of Jordan Canonical Form. These exercises will outline a proof of the Jordan Canonical Form theorem.

(40) Prove the Jordan-Chevalley Decomposition:

**Theorem 12** (The Jordan-Chevalley Decomposition). Let $L : V \to V$ be a linear operator on an $n$-dimensional complex vector space. Then $L = S + N$, where $S$ is diagonalizable, $N^n = 0$, and $SN = NS$.

One strategy is:

(a) Prove that

$$V = \ker(L - \lambda_1)^{m_1} \oplus \cdots \oplus \ker(L - \lambda_k)^{m_k},$$

where $\mu_L(t) = (t - \lambda_1)^{m_1} \cdots (t - \lambda_k)^{m_k}$.

(b) Let $M_i = \ker(L - \lambda_i)^{m_i}$. Decide on a reasonable definition for $S|_{M_i}$ and $N|_{M_i}$ so that $L = S + N$, $S$ is diagonalizable, and $SN = NS$. Prove that $N^n = 0$.

(41) Proof of the Jordan Canonical Form theorem:

(a) Use the Jordan-Chevalley Decomposition to reduce to the case where $L$ is an operator of the form $L = N$, where $N^n = 0$.

(b) Prove that there is a basis for $V$ in which $[N]$ is a Jordan matrix with zeros on the diagonal.

(i) First do this in the case $N^{n-1} \neq 0$.

(ii) Suppose that $N^m = 0$, $N^{m-1} \neq 0$, and $m < n$. Show that there is an $x \in V$ such that the cyclic subspace $C_x$ has dimension $m$. Find an $N$-invariant subspace $M$ such that $V = C_x \oplus M$, and finish the proof.

(42) With the same notation as in the Jordan-Chevalley Decomposition:

(a) Show that $S$ and $N$ commute with $L$.

(b) Show that $L$ and $S$ have the same eigenvalues.

(c) Show that if $\lambda$ is an eigenvalue for $L$, then

$$\ker((L - \lambda I)^n) = \ker(S - \lambda I).$$

(d) Show that the Jordan-Chevalley decomposition is unique.

(e) Find polynomials $p, q$ such that $S = p(L)$ and $N = q(L)$. 