MATH 205A: ANALYTIC NUMBER THEORY
HOMEWORK 2

This assignment is worth 100 points. You do not need to attempt every problem to earn all of the points. Submissions written in \LaTeX will receive 5 bonus points.

(1) (15 points)
(a) Use Stirling’s formula to prove that for bounded $\sigma$ and $|t| \gg 1$,

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{|\sigma-1/2|}e^{-\pi|t|/2}.$$

(b) Prove that the entire function

$$f(s) = \frac{4^s\Gamma(s)\Gamma(s+1/2)}{\Gamma(2s)}$$

is constant, and that it equals $2\sqrt{\pi}$. This proves the duplication formula.

(2) (15 points)
(a) Prove that

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right).$$

(b) (Dirichlet’s hyperbola method)
Let $d(n) = \sum_{d|n} 1$ denote the number of divisors of $n$. Then

$$\sum_{n \leq X} d(n) = \sum_{n \leq X} \sum_{d|n} 1 = \sum_{n \leq X} \sum_{ab=n} 1 = \sum_{ab \leq X} 1.$$

The latter sum counts the number of integer lattice points underneath the hyperbola $xy = X$. The hyperbola is symmetric about $y = x$ (i.e. across the point $(\sqrt{X}, \sqrt{X})$) so it makes sense to count only those $a$ which are $\leq \sqrt{X}$, then multiply the result by 2. But we’ve overcounted those points for which both $a \leq \sqrt{X}$ and $b \leq \sqrt{X}$. So we get

$$\sum_{n \leq X} d(n) = 2 \sum_{ab \leq X \ a \leq \sqrt{X}} 1 - \sum_{a, b \leq \sqrt{X}} 1.$$

Use this to prove that

$$\sum_{n \leq X} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$
(3) (20 points) In this problem, \( p \) and \( q \) denote primes. Prove the following:

(a) \( \sum_{pq \leq x} \frac{1}{pq} = (\log \log x)^2 + c \log \log x + O(1) \) for some constant \( c > 0 \).

\textit{Hint:} try the hyperbola method.

(b) \( \sum_{p \leq x} \frac{(\log p)^2}{p} = \frac{(\log x)^2}{2} + O(\log x) \).

(c) \( \sum_{pq \leq x} \frac{\log p \log q}{pq} = \frac{(\log x)^2}{2} + O(\log x) \).

(4) (15 points) Suppose that \( \{a(n)\} \) is a nonincreasing sequence of nonnegative real numbers.

(a) Prove that

\[
\sum_{n=1}^{\infty} a(n) \leq \sum_{m=0}^{\infty} 2^m a(2^m) \leq 2 \sum_{n=1}^{\infty} a(n),
\]

where all three are infinite if \( \sum a(n) \) diverges.

(b) Prove that

\[
\sum_{p} a(p) \text{ converges} \iff \sum_{n=1}^{\infty} \frac{a(n)}{\log n} \text{ converges}.
\]

(5) (20 points) Let \( \chi \neq \chi_0 \) be a Dirichlet character mod \( q \), and let \( f(n) = \sum_{d|n} \chi(d) \).

(a) Show that \( \sum_{n \leq x} \frac{f(n)}{\sqrt{n}} = 2L(1, \chi) \sqrt{x} + O(1) \).

\textit{Hint:} try the hyperbola method.

(b) Fact: \( f(n) \) is multiplicative, so \( f(n) = \prod_{p^k || n} f(p^k) \).

Suppose that \( \chi \) is a real character. Prove that

\[
f(n) \geq 0 \quad \text{for all } n, \quad f(n) \geq 1 \quad \text{for } n = m^2.
\]

(c) Prove, under the same assumption as in (b), that \( L(1, \chi) \neq 0 \).
(6) (15 points) The Kloosterman sum is defined as
\[ K(m, n; q) := \sum_{d \mod q, (d, q) = 1} e\left(\frac{md + nd}{q}\right), \]
where \( \bar{d} \) satisfies \( d\bar{d} \equiv 1 \pmod{q} \). Prove the following:
(a) \( K(m, n; q) = K(n, m; q) \).
(b) If \( (n, q) = 1 \) then \( K(m, n; q) = K(mn, 1; q) \).
(c) Given integers \( n, q_1, q_2 \) such that \( (q_1, q_2) = 1 \), show that there exist integers \( n_1 \) and \( n_2 \) such that
\[ n \equiv n_1q_2 + n_2q_1^2 \pmod{q_1q_2}, \]
and that for these integers we have
\[ K(m, n; q_1q_2) = K(m, n_1; q_1)K(m, n_2; q_2). \]

(7) (15 points)
(a) Suppose that \( f(n) \) is a multiplicative function (i.e. \( f(mn) = f(m)f(n) \) whenever \( \gcd(m, n) = 1 \)) such that
\[ \lim_{p^n \to \infty} f(p^n) = 0. \]
That is, for every \( \varepsilon > 0 \) there is an \( N(\varepsilon) \) such that \( |f(p^n)| < \varepsilon \) whenever \( p^n > N(\varepsilon) \). Prove that \( f(n) \to 0 \) as \( n \to \infty \).
(b) Fact: \( d(n) \) (see (2)) is multiplicative. Prove that \( d(n) \ll \varepsilon n^\varepsilon \) for any \( \varepsilon > 0 \).