CHAPTER 6 PRACTICE QUIZ SOLUTIONS

(1) Compute the integral
\[ \int_{|z|=\pi} \frac{z}{\cos z - 1} \, dz. \]

The denominator has a zero of order 2 at 0 since \( \cos z - 1 = -z^2/2 + \ldots \). So the integrand has a pole of order one at \( z = 0 \). Since the next largest zeros of \( \cos z - 1 \) are at \( z = \pm 2\pi \), which are both outside the contour, there is only one pole to consider. Thus we have
\[ \int_{|z|=\pi} \frac{z}{\cos z - 1} \, dz = 2\pi i \text{ Res}(0). \]

To compute the residue, we find that
\[ \text{Res}(0) = \lim_{z \to 0} z \cdot \frac{z}{\cos z - 1} = \lim_{z \to 0} \frac{z^2}{-z^2/2 + z^4/4! - \ldots} = \lim_{z \to 0} \frac{1}{-1/2 + z^2/4! - \ldots} = -2. \]

It follows that the integral equals \(-4\pi i\).

(2) With the aid of residues, verify the integral formula
\[ \int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4} \, dx = \frac{\pi}{4}. \]

Proof. Since the integrand is even, we will instead show that
\[ \int_{-\infty}^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4} \, dx = \frac{\pi}{2}. \]

The zeros of the denominator are at the points where \( x^4 + 5x^2 + 4 = 0 \), i.e.
\[ 0 = (x^2 + 4)(x^2 + 1) = (x - 2i)(x + 2i)(x - i)(x + i). \]

In the upper half-plane, the relevant zeros are \( i \) and \( 2i \). Let \( \gamma_1 \) be the interval \([-R, R]\) on the real line and let \( \gamma_2 \) denote the semicircle in the upper half-plane of radius \( R \) centered at 0, positively oriented. Then
\[ \int_{\gamma_1 + \gamma_2} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} \, dz = 2\pi i(\text{Res}(i) + \text{Res}(2i)) \tag{0.1} \]

Using the lemma about computing residues of functions of the form \( P/Q \) where \( P(z_0) \neq 0 \) and \( Q \) has a simple zero at \( z_0 \), we find that
\[ \text{Res}(i) = \frac{2i^2 - 1}{4i^3 + 10i} = \frac{-3}{6i} = \frac{-1}{2i} \]
and
\[ \text{Res}(2i) = \frac{8i^2 - 1}{32i^3 + 20i} = \frac{-9}{-12i} = \frac{3}{4i}. \]

So the right-hand side of (0.1) is \( 2\pi i(1/(4i)) = \pi/2 \).

By a lemma from the book, we have \( \lim_{R \to \infty} \int_{\gamma_2} = 0 \) since the degree of the denominator is 4 and the degree of the numerator is 2. The result follows. \( \square \)
(3) Use Rouché’s Theorem to prove that the polynomial $P(z) = z^5 + 3z^2 + 1$ has exactly three zeros in the annulus $1 < |z| < 2$. Hint: prove that it has five zeros in $|z| < 2$ and two zeros in $|z| < 1$.

Proof. Apply Rouché’s Theorem to the functions $f(z) = z^5$ and $g(z) = z^5 + 3z^2 + 1$ on $|z| = 2$. We have

$$|f(z) - g(z)| = |3z^2 + 1| \leq 3|z|^2 + 1 = 3 \cdot 2^2 + 1 = 13 < 32 = |z|^5 = |f(z)|.$$ 

So $g$ has 5 zeros inside $|z| < 2$.

Now apply Rouché’s Theorem to the functions $f(z) = 3z^2$ and $g(z) = z^5 + 1$ on $|z| = 1$. We have

$$|f(z) - g(z)| = |z^5 + 1| \leq |z|^5 + 1 = 2 < 3 = 3|z|^2 = |f(z)|.$$ 

So $g$ has 2 zeros inside $|z| < 1$.

What happens on $|z| = 1$? We have $|P(z)| = |z^5 + 3z^2 + 1| \geq 3|z|^2 - |z|^5 - 1 = 1 > 0$ so $P$ has no zeros on $|z| = 1$.

We conclude that $g$ has 3 zeros in the annulus. □