CHEAT SHEET

Definition 1. If \( z = x + iy \) then \( e^z \) is defined to be the complex number \( e^x(\cos y + i \sin y) \).

Proposition 2 (De Moivre’s Formula). If \( n = 1, 2, 3, \ldots \) then
\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.\]

Definition 3. A domain is an open connected set.

Definition 4. A complex-valued function \( f \) is said to be analytic on an open set \( D \) if it has a derivative at every point of \( D \).

Theorem 5. If \( f(z) = u(x, y) + iv(x, y) \) is analytic in \( D \) then the Cauchy-Riemann equations
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]
must hold at every point in \( D \).

Theorem 6 (The Fundamental Theorem of Algebra). Every nonconstant polynomial with complex coefficients has at least one zero in \( \mathbb{C} \).

Theorem 7. Given any complex number \( z \), we define
\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.
\]

Definition 8. The principal value of the logarithm is defined as
\[
\text{Log } z := \text{Log } |z| + i \text{Arg } z.
\]

Theorem 9 (The ML-inequality). If \( f \) is continuous on the contour \( \Gamma \) and if \( |f(z)| \leq M \) for all \( z \) on \( \Gamma \) then
\[
\left| \int_{\Gamma} f(z) \, dz \right| \leq M \cdot \text{length}(\Gamma).
\]

Theorem 10 (Independence of Path). Suppose that the function \( f(z) \) is continuous in a domain \( D \) and has an antiderivative \( F \) throughout \( D \). Then for any contour \( \Gamma \) in \( D \) with initial point \( z_I \) and terminal point \( z_T \) we have
\[
\int_{\Gamma} f(z) \, dz = F(z_T) - F(z_I).
\]

Definition 11. A domain \( D \) is simply connected if one of the following (equivalent) conditions holds:
- every loop in \( D \) can be continuously deformed in \( D \) to a point.
- if \( \Gamma \) is any simple closed contour in \( D \) then \( \text{int}(\Gamma) \subseteq D \).

Theorem 12 (Cauchy’s Integral Theorem). If \( f \) is analytic in a simply connected domain \( D \) then
\[
\int_{\Gamma} f = 0 \quad \text{for all loops } \Gamma \subseteq D.
\]

Theorem 13 (Morera’s Theorem). If \( f \) is continuous in a domain \( D \) and if
\[
\int_{\Gamma} f = 0 \quad \text{for all loops } \Gamma \subseteq D
\]
then \( f \) is analytic in \( D \).
Theorem 14 (Cauchy’s (generalized) Integral Formula). If \( f \) is analytic inside and on the simple positively oriented loop \( \Gamma \) and if \( z \) is any point inside \( \Gamma \) then

\[
f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta
\]

for \( n = 1, 2, 3, \ldots \).

Theorem 15 (Liouville’s Theorem). The only bounded entire functions are the constant functions.

Theorem 16 (The Maximum Modulus Principle). A function analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary.

Theorem 17 (The Taylor Series Theorem). If \( f \) is analytic in the disk \( |z - z_0| < R \) then the Taylor series

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n
\]

converges to \( f(z) \) for all \( z \) in the disk. Furthermore, the convergence is uniform in any closed subdisk \( |z - z_0| \leq R' < R \).

Definition 18. A point \( z_0 \) is called a zero of order \( m \) for \( f \) if \( f \) is analytic at \( z_0 \) and if \( f^{(n)}(z_0) = 0 \) for \( 0 \leq n \leq m-1 \) but \( f^{(m)}(z_0) \neq 0 \).

Definition 19. If \( f \) has an isolated singularity at \( z_0 \) and if \( f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \) is its Laurent expansion around \( z_0 \) then

1. If \( a_n = 0 \) for all \( n < 0 \) then \( f \) has a removable singularity at \( z_0 \).
2. If \( a_{-m} \neq 0 \) for some \( m > 0 \) but \( a_n = 0 \) for all \( n < -m \) then \( f \) has a pole of order \( m \) at \( z_0 \).
3. If \( a_n \neq 0 \) for infinitely many negative \( n \) then \( f \) has an essential singularity at \( z_0 \).

Theorem 20 (Picard’s Theorem). A function with an essential singularity assumes every complex number, with possibly one exception, as a value in any neighborhood of this singularity.

Definition 21. If \( f \) has an isolated singularity at \( z_0 \) then the coefficient \( a_{-1} \) of \( 1/(z - z_0) \) in the Laurent expansion of \( f \) around \( z_0 \) is called the residue of \( f \) at \( z_0 \).

Theorem 22 (The Residue Theorem). If \( \Gamma \) is a simple positively oriented loop and \( f \) is analytic inside and on \( \Gamma \) except at the points \( z_1, \ldots, z_n \) inside \( \Gamma \) then

\[
\frac{1}{2\pi i} \int_{\Gamma} f(z) \, dz = \sum_{k=1}^{n} \text{Res}(f; z_k).
\]

Theorem 23 (The Argument Principle). If \( f \) is analytic and nonzero at each point of a simple positively oriented loop \( \Gamma \) and is meromorphic inside \( \Gamma \) then

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz = Z - P,
\]

where \( Z \) is the number of zeros of \( f \) inside \( \Gamma \) and \( P \) is the number of poles inside \( \Gamma \), counted with multiplicity.

Theorem 24 (Rouché’s Theorem). If \( f \) and \( g \) are analytic inside and on a simple loop \( \Gamma \) and if

\[
|f(z) - g(z)| < |f(z)| \quad \text{for all } z \text{ on } \Gamma
\]

then \( f \) and \( g \) have the same number of zeros (counting multiplicities) inside \( \Gamma \).
Final Exam Information

The final exam will be worth 40 total points, broken up in the following way:

15 points: Chapter 6 Quiz (3 questions)
15 points: Cumulative Quiz (3 questions very similar to exercises from HW 1–4)
10 points: Definitions / Theorems (See Cheat Sheet above)

To help you practice for the Chapter 6 Quiz portion, here are some exercises to do from Sec. 6.7:

(6.7) 1, 2, 3, 6, 7, 8

Practice Chapter 6 Quiz:

1. Compute the integral
   \[ \int_{|z|=\pi} \frac{z}{\cos z - 1} \, dz. \]

2. With the aid of residues, verify the integral formula
   \[ \int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4} \, dx = \frac{\pi}{4}. \]

3. Use Rouché’s Theorem to prove that the polynomial \( P(z) = z^5 + 3z^2 + 1 \) has exactly three zeros in the annulus \( 1 < |z| < 2 \). **Hint:** prove that it has five zeros in \( |z| < 2 \) and two zeros in \( |z| < 1 \).