MATH 132 QUIZ 4 (PRACTICE)

(1) Compute the Taylor series for \( g(z) = \frac{1}{(1 - z)^3} \) around \( z = 0 \) and find the radius of convergence \( R \).

True or false: this series converges at every point on the boundary \( |z| = R \).

First we notice that \( \frac{1}{(1 - z)^3} = \frac{1}{2} f''(z) \), where \( f(z) = \frac{1}{1 - z} \). So

\[
\frac{1}{(1 - z)^3} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{d^2}{dx^2} \sum_{n=0}^{\infty} z^n = \frac{1}{2} \sum_{n=2}^{\infty} n(n - 1)z^{n-2} = \frac{1}{2} \sum_{n=2}^{\infty} (n-1)z^{n-2} = 1 + 3z + 6z^2 + 10z^3 + 15z^4 + \ldots
\]

Since the radius of convergence of the Taylor series for \( f \) around \( z = 0 \) is \( R = 1 \), the radius of convergence of the Taylor series for \( g \) around \( z = 0 \) is also \( R = 1 \). The series doesn’t converge for any \( z \) on the boundary \( |z| = 1 \) (but you only need to show this for a single value of \( z \) to conclude that the statement above is false). For example, at \( z = 1 \) we have the series \( 1 + 3 + 6 + 10 + 15 + \ldots \) which is obviously divergent.

(2) Compute the Laurent series of \( f(z) = \frac{1}{z^3(1 + z)} \) in the regions

(a) \( 0 < |z| < 1 \)
(b) \( |z + 1| > 1 \)

Hint: for part (b) you’ll want to use your answer from #1 above.

(a) In \( 0 < |z| < 1 \) we have

\[
f(z) = \frac{1}{z^3} \cdot \frac{1}{1 - (-z)} = \frac{1}{z^3} \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^{n-3} = z^{-3} - z^{-2} + z^{-1} - 1 + z - \ldots
\]

(b) In \( |z + 1| > 1 \) we have

\[
f(z) = \frac{1}{1 + z} \cdot \frac{1}{((1 + z) - 1)^3} = \frac{1}{(1 + z)^4} \cdot \frac{1}{(1 - (1 + z)^{-1})^3}
\]

\[
= \frac{1}{(1 + z)^4} \sum_{n=2}^{\infty} n(n - 1)(1 + z)^{-n+2} = (1 + z)^{-4} + 3(1 + z)^{-5} + 6(1 + z)^{-6} + 10(1 + z)^{-7} + \ldots
\]

(3) Classify the zero or singularity of the function

\[
h(z) = \frac{\log^2(z)}{(z - 1)e^z}
\]

at the point \( z = 1 \). If the singularity is a zero or pole, state its order.
The function $e^z$ is analytic and nonzero at $z = 1$, so it will not contribute to a zero or singularity of $h(z)$ there. (It will affect the coefficients of the Laurent series, just not the lowest exponent). Thus it is safe (for this problem) to ignore the $e^z$ factor.

The factor $(z - 1)$ is already in the right form for a Laurent series centered at $z = 1$, so let’s focus first on the log term. Near $z = 1$ we have

\[ \log(z) = \log(1 - (1 - z)) = -\sum_{n=1}^{\infty} \frac{(1 - z)^n}{n} = -\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z - 1)^n}{n} = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \ldots. \]

It follows that

\[ \log^2(z) = \left( (z - 1) - \frac{1}{2}(z - 1)^2 + \frac{1}{3}(z - 1)^3 - \ldots \right)^2 = (z - 1)^2 - (z - 1)^3 + \frac{11}{12}(z - 1)^4 - \ldots \]

and therefore

\[ \frac{\log^2(z)}{z - 1} = (z - 1) - (z - 1)^2 + \ldots \]

which has a simple zero (a zero of order one) at $z = 1$. 