MATH 132 QUIZ 3 (PRACTICE) SOLUTIONS

(1) Let $Γ$ be the contour shown below from $-1 - i$ to $-1 + i$.

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{contour}\end{array}
\]

Compute $\int_{Γ} \frac{dz}{z}$.

We need an antiderivative for $\frac{1}{z}$ which is analytic in an open neighborhood of the contour $Γ$. There are many reasonable choices, but I’ll pick $F(z) = \log(0, z)$ (this is analytic in $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$). Then the integral equals

\[
F(-1 + i) - F(-1 - i) = \log(0, -1 + i) - \log(0, -1 - i)
\]

\[
= \log \sqrt{2} + i \text{Arg}(0, 2\pi)(-1 + i) - \log \sqrt{2} - i \text{Arg}(0, 2\pi)(-1 - i) = 3\pi i/4 - 5\pi i/4 = -\pi i/2.
\]

(2) Let $C$ be the positively oriented circle $|z| = 2$. Compute $\int_{C} \frac{e^{2z}}{(1 - z)^{3}} dz$.

By Cauchy’s Integral Formula, this equals $-\frac{2\pi i}{2!}f''(1)$ (you need to factor out $-1$ from the denominator to put it in the right form), where $f(z) = e^{2z}$. Since $f''(z) = 4e^{2z}$ the integral equals $-4\pi i e^{2}$.

(3) Suppose that $P(z)$ is a polynomial with a zero of order $m$ at $z = 0$ and no other zeros inside the closed unit disk $|z| \leq 1$. Prove that

\[
\frac{1}{2\pi i} \int_{|z|=1} \frac{P'(z)}{P(z)} \, dz = m.
\]

Hint: first find a way to simplify $P'(z)/P(z)$.

Writing $P(z) = z^{m}Q(z)$ where $Q(z) \neq 0$ for all $|z| \leq 1$, we find that

\[
\frac{P'(z)}{P(z)} = \frac{mz^{m}Q(z) + z^{m}Q'(z)}{z^{m}Q(z)} = \frac{m}{z} + \frac{Q'(z)}{Q(z)}.
\]
The function $Q'/Q$ is analytic in $|z| \leq 1$ since $Q(z) \neq 0$ there. So its loop integrals are zero by Cauchy’s integral theorem. Thus
$$\frac{1}{2\pi i} \int_{|z|=1} \frac{P'(z)}{P(z)} \, dz = \frac{1}{2\pi i} \int_{|z|=1} \frac{m}{z} \, dz = m.$$ 

(4) Suppose that $f$ is an entire function with $\text{Re} f(z) \leq M$ (for some positive constant $M$) for all $z \in \mathbb{C}$. Prove that $f$ is constant.

Hint: First prove that $e^{f(z)}$ is constant. Then conclude that $f'(z) = 0$ and thus $f(z)$ is constant.

Let $g(z) = e^{f(z)}$. Then $|g(z)| = e^{\text{Re} f(z)} \leq e^M$. Since $g$ is entire and bounded, it must be a constant. Therefore
$$0 = g'(z) = f'(z)e^{f(z)}.$$ 

It follows that $f'(z) = 0$ and thus $f$ is a constant.