1. (5 points)

(a) What is the inverse image of the positive real line

\[ \mathbb{R}_{>0} = \{ z \in \mathbb{C} : \text{Im} z = 0, \text{Re} z > 0 \} \]

under the map \( z \mapsto \frac{1}{z+i} \)? That is, what subset of \( \mathbb{C} \) maps to \( \mathbb{R}_{>0} \) under \( z \mapsto \frac{1}{z+i} \)?

I would prefer to draw this out graphically, but that’s hard to do by typing so I’ll do it another way. Suppose \( \frac{1}{z+i} = x \) where \( x \) is a positive real number. Then \( z = \frac{1}{x} - i \).

The set of all such numbers is the set \( \{ z \in \mathbb{C} : \text{Im} z = -1, \text{Re} z > 0 \} \).

(b) Find a branch of the multi-valued function \( \log \left( \frac{1}{z+i} \right) \) that is analytic in the upper half-plane \( \{ \text{Im}(z) > 0 \} \).

The branch cut of the function \( \log_0(z) \) is \( \mathbb{R}_{\geq 0} \), so by (a), \( \log_0 \left( \frac{1}{z+i} \right) \) is analytic in \( \mathbb{C} \setminus \{ z \in \mathbb{C} : \text{Im} z = -1, \text{Re} z \geq 0 \} \). Since this set contains the upper half-plane \( \{ \text{Im} z > 0 \} \) we are done.

(c) Compute the value of your function in (b) at the point \( z = i \).

\[ \log_0 \left( \frac{1}{2i} \right) = \log \left( \frac{1}{2} \right) + i \arctan \left( \frac{1}{2} \right) = -\log 2 + \pi i/2. \]

2. (5 points) Show that if the rational function \( R(z) \) has a pole of order \( m \) at \( z = 0 \) then its derivative has a pole of order \( m+1 \) at \( z = 0 \).

Write \( R(z) = \frac{p(z)}{z^m q(z)} \) where \( p, q \) are polynomials and \( p(0) \neq 0 \) and \( q(0) \neq 0 \). Then

\[ R'(z) = -m z^{-m-1} \frac{p(z)}{q(z)} + z^{-m} \left( \frac{p(z)}{q(z)} \right)' = \frac{-mp(z)/q(z) + z(p(z)/q(z))'}{z^{m+1}}. \]

This looks like it has a pole of order \( m+1 \) at \( z = 0 \), but we’re not done yet! We need to make sure that the numerator isn’t zero at \( z = 0 \) (if it is, then we’ll have cancellation and the pole order will be smaller). At \( z = 0 \) the numerator equals \( -mp(0)/q(0) \neq 0 \) by assumption. That finishes the proof.

3. (5 points) Consider the multi-valued function \( (z^4-1)^{1/4} \). Verify that the function

\[ g(z) = e^{\pi i/4} e^{\frac{1}{4} \log(1-z^4)} \]

is a branch of \( (z^4-1)^{1/4} \) that is analytic in the interior of the unit disk \( |z| < 1 \).

Note: you must show two things: that \( |g(z)|^4 = z^4 - 1 \), and that \( g \) is analytic in \( |z| < 1 \).
Let’s first check that \([g(z)]^4 = z^4 - 1\). We have

\[
g(z)^4 = e^{\pi i} e^{\text{Log}(1 - z^4)} = e^{\pi i} (1 - z^4) = z^4 - 1.
\]

Now we need to show that \(g(z)\) is analytic in \(|z| < 1\). Again, normally I’d do this graphically but I’ll do it a different way here so I can type it easier.

Suppose that \(1 - z^4\) is on the branch cut of \(\text{Log}\), i.e. that \(1 - z^4 = x\) where \(x < 0\). Then \(z^4 = 1 - x > 1\). Therefore \(|z| > 1\). So in the open unit disk \(|z| < 1\) the function \(\text{Log}(1 - z^4)\) is analytic since \(1 - z^4\) avoids the negative real line when \(|z| < 1\). Thus \(g\) is analytic in \(|z| < 1\).