MATH 132 HOMEWORK 3

- (3.3) 12, 13, 14
- (3.5) 15, 16
- (4.1) 9, 11
- (4.2) 5, 8, 10, 14
- (4.3) 2, 3, 4

- The Riemann zeta function is defined via the series
  \[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \]
  where \( s = \sigma + it \) is a complex variable (it is customary to use the \( s = \sigma + it \) notation instead of the usual \( z = x + iy \) because this is what Riemann used originally).

  a) Determine the region of absolute convergence of \( \zeta(s) \); that is, give conditions on \( \sigma, t \) which guarantee that the series
  \[ \sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| \]
  converges. (After taking absolute values, you are left with a series of real, positive terms, and you can use what you already know about series to finish the problem.) We will prove later that \( \zeta(s) \) is analytic in this region.

  b) Assuming for the moment that differentiation of series works term-by-term, compute \( \zeta'(s) \) as an infinite series of the form \( \sum_{n=1}^{\infty} \frac{a_n}{n^s} \).

  c) Polynomials are uniquely determined by their roots, up to a constant multiple. It turns out that something similar is also true for entire functions; in particular we have the following formula for \( \sin(\pi z) \):
  \[ \sin \pi z = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) = \pi z \left( 1 - \frac{z^2}{1^2} \right) \left( 1 - \frac{z^2}{2^2} \right) \left( 1 - \frac{z^2}{3^2} \right) \cdots \]
  (Hopefully we’ll have time to prove this later.) Using the product formula for \( \sin \pi z \), prove that \( \zeta(2) = \frac{\pi^2}{6} \). Hint: If you multiply out the factors on the right-hand side of the product formula above, what is the coefficient of \( z^3 \)? What is the coefficient of \( z^3 \) in the Maclaurin series for \( \sin \pi z \)?

Extra Practice (do not turn in):
(3.3) 1, 3, 15  (3.5) 1, 3, 4, 5  (4.1) 1, 5, 10  (4.2) 3, 6, 7, 9, 11, 12, 13  (4.3) 1(a–h), 5