MATH 131A: REAL ANALYSIS

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The textbook for the course is Ross, Elementary Analysis [2], but in these notes I have also borrowed from Tao, Analysis I [3], and Abbott, Understanding Analysis [1].

1. Introduction

Real analysis is a rigorous study of the (hopefully familiar) objects from calculus: the real numbers, functions, limits, continuity, derivatives, sequences, series, and integrals. While you are likely quite familiar with computing with these objects, this course will focus on developing the theoretical foundation for the definitions, theorems, formulas, and algorithms that you are used to using. We will start by building up the real numbers from scratch, i.e., from just a few basic axioms, then we will focus our attention on proving many of the things you already believe about functions, sequences, and series. Along the way we will encounter several “pathological” objects which will hopefully convince you that our careful approach is necessary and worthwhile.

To get an idea of how subtle some questions in analysis can be, ask yourself: what is a real number? We will answer this question in due time, but for now let’s focus on one specific real number that got a lot of attention from the ancient Greeks: $\sqrt{2}$. Prior to the discovery that $\sqrt{2}$ is an irrational number, it was assumed that: given any two line segments $AB$ and $CD$, there is a rational number $r$ so that the length of $CD$ equals $r$ times the length of $AB$. However, the length of the diagonal of a square of side length 1 (using the Pythagorean Theorem) equals $\sqrt{2}$, so by the previous assumption, we must have that $\sqrt{2}$ is a rational number. The proof of the following theorem is one of the most classic proofs in mathematics.

**Theorem 1.1.** There is no rational number whose square is 2.

**Proof.** Suppose that $x^2 = 2$ and that $x$ is a rational number. Recall that a rational number is one which can be expressed as $p/q$ for integers $p$ and $q$. To prove that there are no integers $p, q$ for which $x = p/q$ we will employ an important proof technique called proof by contradiction. That is, we will assume that $x = p/q$ for some integers $p, q$ and we will carefully follow logical steps until we end up with something absurd. Thus, our original assumption must have been faulty. Here we go.

Suppose that there are integers $p, q$ for which

$$\left(\frac{p}{q}\right)^2 = 2. \tag{1.1}$$

We may assume that $p$ and $q$ have no common factors, since we could just cancel the common factors and write $p/q$ in lowest terms. Equation (1.1) implies that

$$p^2 = 2q^2. \tag{1.2}$$

It follows that $p^2$ is an even number (it’s 2 times an integer), and therefore $p$ is an even number (you can’t square an odd number and get an even number). So we can write $p = 2r$...
for some integer $r$. Then equation (1.2) becomes (after cancelling 2s)

$$2r^2 = q^2.$$ 

By the previous discussion, this implies that $q$ is even. But that’s ridiculous! We assumed that $p$ and $q$ had no factors in common, but we just showed that $p$ and $q$ were both even. Since we have reached a contradiction, it must be that our initial assumption (1.1) was false. Thus 2 is not the square of any rational number. □

The previous theorem shows that there is a “hole” at $\sqrt{2}$ in the rational numbers. The importance of this fact cannot be overstated. Later it will lead us to the Axiom of Completeness which is an essential property that the real numbers enjoy which basically states that there are no holes in the set of real numbers. This will lead us to limits, derivatives, continuity, and eventually integrals. But first we need to take a few steps back and start at the beginning.

2. THE NATURAL NUMBERS

If our aim is to construct the real numbers and do calculus on the set of real numbers, then we must start with the simplest numbers first and build our way up. Thus our story begins with the natural numbers (a.k.a whole numbers or counting numbers) which we can informally define as the elements of the set

$$\mathbb{N} := \{1, 2, 3, 4, 5, \ldots\}.$$ 

We are no longer in the business of informal definitions, so we’ll need to build $\mathbb{N}$ from scratch. In excruciating detail.

Let’s think about what we want in a set of numbers. In mathematics, it is often desirable not to think too carefully about the actual elements in a set, but more about what you want those elements to do (i.e. what operations or functions do you want to apply to those elements?). A few moments of thought might lead you to say that the most important thing we do with the natural numbers is counting (you might have said addition or multiplication, but addition is just repeated counting, and multiplication is just repeated addition). So it stands to reason that we should construct the natural numbers so that we can count with them.

We will begin with two concepts: the number 1, and the successor $n + 1$. Note that we haven’t defined addition yet (we don’t even know what the numbers are!) so $n + 1$ doesn’t mean $n$ “plus” 1. Yet. It’s just an expression that we use to denote the successor of $n$. Informally (but less informally than before), we will define the natural numbers as the set containing 1, the successor 1 + 1, the successor $(1 + 1) + 1$, the successor $((1 + 1) + 1) + 1$, etc. This leads to our first two axioms.

Axiom 1: 1 is a natural number.

Axiom 2: If $n$ is a natural number, then $n + 1$ is a natural number.

By Axioms 1 and 2, we see that

$$((((((1 + 1) + 1) + 1) + 1) + 1) + 1) + 1$$

is a natural number. Don’t worry, we won’t write numbers like this; instead we’ll use the notation we’re all familiar with. So the number above is called 8. But for now, the symbol 8 means nothing other than a shorthand notation for the successor of the successor of the successor of the successor of the successor of the successor of the successor of 1.
It may seem like this is enough to define the natural numbers, but consider the set consisting of all natural numbers from 1 to 12, where the successor of 12 is 1 (this is not some crazy thing, it’s how clocks work!). Even though this number system obeys Axioms 1 and 2, it doesn’t even allow us to count how many fingers and toes we have, so it must not be right. Let’s add another axiom.

**Axiom 3:** 1 is not the successor of any natural number; i.e. \( n + 1 \neq 1 \) for all \( n \).

Now we can prove statements like the following.

**Lemma 2.1.** \( 4 \neq 1 \).

*Proof.* By definition \( 4 = 3 + 1 \). By Axioms 1 and 2, 3 is a natural number (since \( 3 = (1 + 1) + 1 \)). Thus by Axiom 3, \( 3 + 1 \neq 1 \). Therefore \( 4 \neq 1 \). \( \Box \)

At this rate we’ll never get to derivatives! (Don’t worry, we’re going to go through the construction of the natural numbers in painful detail so that you can see what goes into a rigorous mathematical foundation of analysis. Then we will move a bit faster so that we can cover other things.)

Have we constructed \( \mathbb{N} \) yet? Unfortunately, there are still weird pathological number systems which satisfy the first three axioms, but which are not the natural numbers (as we would like them to be). Consider the number system

\[ 1, \ 1+1 = 2, \ 2+1 = 3, \ 3+1 = 4, \ 4+1 = 4, \ 4+1 = 4, \ldots. \]

You can check that this doesn’t break our first three axioms, but it’s still definitely not right. Let’s add another axiom.

**Axiom 4:** If \( n \) and \( m \) are natural numbers and \( n \neq m \) then \( n+1 \neq m+1 \).

Equivalently, if \( n+1 = m+1 \) then \( n = m \).

Now we can’t have the above pathology.

**Lemma 2.2.** \( 4 \neq 2 \).

*Proof.* Suppose, by way of contradiction, that \( 4 = 2 \). Then \( 3+1 = 1+1 \), so by Axiom 4 we have \( 3 = 1 \). But that contradicts Axiom 3, so our original assumption must have been wrong. Thus \( 4 \neq 2 \). \( \Box \)

We’re not out of the woods yet. We have constructed a set of axioms which confirms that all of the numbers that we think should be natural numbers (i.e. 1, 2, 3, \ldots) are elements of \( \mathbb{N} \). But we can’t rule out the existence of other numbers masquerading as natural numbers. For example, the set

\[ \{5, \ 1, \ 1.5, \ 2, \ 2.5, \ 3, \ 3.5, \ 4, \ldots\} \]

satisfies all of the axioms so far. So we’ll need one final axiom. This one is so important that it gets its own name. (You’ll want to chew on this one for a bit.)

**Axiom 5 (The principle of mathematical induction):**

Let \( P_n \) be any statement or proposition that may or may not be true.

Suppose that \( P_1 \) is true, and that whenever \( P_n \) is true, \( P_{n+1} \) is also true.

Then \( P_n \) is true for every natural number \( n \).

The principle of mathematical induction allows us to prove that a statement is true by simply checking two things: first we check that the statement is true for \( n = 1 \), then,
assuming it is true for \( n \), we check that it is true for \( n + 1 \). Here is an example (note: this example really belongs later in the course, after we’ve defined addition and multiplication and division, but I’m happy to time travel for a few seconds if you are).

**Proposition 2.3.** For all natural numbers \( n \) we have

\[
1 + 2 + \ldots + n = \frac{n(n + 1)}{2}.
\]

**Proof.** For each \( n \), the statement we want to prove is

\[ P_n : "1 + 2 + \ldots + n = \frac{n(n + 1)}{2}."
\]

We proceed by induction. We begin with \( P_1 \), which states that \( 1 = \frac{1(1+1)}{2} \). This is certainly true. Suppose that \( P_n \) is true, i.e. suppose that

\[
1 + 2 + \ldots + n = \frac{n(n + 1)}{2}
\]

is a true statement. We wish to show that \( P_{n+1} \) is true. Add \( n + 1 \) to both sides to obtain

\[
1 + 2 + \ldots + n + n + 1 = \frac{n(n + 1)}{2} + n + 1
\]

\[
= \frac{(n + 1)(n + 1 + 1)}{2}.
\]

Thus \( P_{n+1} \) holds if \( P_n \) holds. By the principle of mathematical induction, \( P_n \) holds for all natural numbers \( n \). \( \Box \)

Note that we didn’t prove \( P_n \) directly for any \( n \) except for \( n = 1 \). We just proved \( P_1 \), and we proved that if \( P_1 \) is true, so is \( P_2 \) (thus \( P_2 \) is true), and we proved that if \( P_2 \) is true, so is \( P_3 \) (thus \( P_3 \) is true), and we proved that if \( P_3 \) is true, so is \( P_4 \) (thus \( P_4 \) is true), etc. It’s like stacking up an infinite line of dominoes and knocking over the first one.

Since we will use induction regularly to prove things in lecture, homework, and exams, it might be useful to have a template for such proofs.

**Proposition 2.4.** A property \( P_n \) is true for all natural numbers \( n \).

**Proof.** We proceed by induction on \( n \) (it’s good to specify the variable if there are several variables in the statement you want to prove). We first verify the base case \( n = 1 \), i.e. we prove that \( P_1 \) is true. [Insert proof of \( P_1 \) here]. Now suppose that \( P_n \) has already been proven. We show that \( P_{n+1} \) is true. [Insert proof of \( P_{n+1} \), assuming \( P_n \)]. It follows that \( P_n \) is true for all natural numbers \( n \). \( \Box \)

3. **The integers and the rationals**

At this point, if we had all the time in the world, we would maintain the glacial pace of the last section and carefully develop the theory of addition and multiplication on the natural numbers. But for the sake of time (and to maintain our sanity) we will take some things for granted as we build up the integers and the rational numbers. This means that the discussion in the section will be quite informal. I hope our careful approach to the natural numbers was enough to convince you that there is a careful way to construct the integers and the rational numbers from the natural numbers. If you would like to learn about this in more detail, see [3].
Informally, the set of integers is made up of the positive integers, the negative integers and 0. We know the positive integers well; these are just the natural numbers. We can add \((2 + 3 = 5)\) and multiply \((2 \cdot 3 = 6)\) natural numbers as usual. (Formally, \(m + n\) is defined as applying the successor to \(m\), \(n\) times; multiplication is then defined as repeated addition.) These operations satisfy the rules

\[
\begin{align*}
    a + b &= b + a, & \text{(commutative law for addition)} \\
    (a + b) + c &= a + (b + c), & \text{(associative law for addition)} \\
    a \cdot b &= b \cdot a, & \text{(commutative law for multiplication)} \\
    (a \cdot b) \cdot c &= a \cdot (b \cdot c), & \text{(associative law for multiplication)} \\
    a \cdot 1 &= a, & \text{(multiplicative identity)} \\
    (a + b) \cdot c &= a \cdot c + b \cdot c. & \text{(distributive law)}
\end{align*}
\]

In order to define subtraction, we introduce the additive inverse of a number and the additive identity element. The additive identity element, 0, is defined by its behavior under addition via

\[
n + 0 = 0 + n = n.
\]

(additive identity)

(Note: we could have defined the natural numbers to include 0, and some authors do this. But Ross [2] doesn’t, so we won’t.) For each natural number \(n\), the additive inverse of \(n\), which we will call \(-n\), is defined by the property

\[
n + (-n) = (-n) + n = 0.
\]

(additive inverse)

Subtraction is then defined via

\[
a - b := a + (-b).
\]

(Note: here I’m being quite informal and sweeping some subtleties under the rug. For example, we would like to say that the integers 1 – 5 and 2 – 6 and 3 – 7, etc., are all the same, but the definition I’ve given you above doesn’t account for that. But to do this properly, we would have to introduce the notion of equivalence classes, which we won’t do here. See Tao’s book [3] if you are interested in learning more.)

We define the integers as the elements of the set

\[
\mathbb{Z} := \mathbb{N} \cup \{0\} \cup (-\mathbb{N}).
\]

Here \(-\mathbb{N}\) is the set consisting of the additive inverses of all the natural numbers, i.e., the negative integers. The notation \(\cup\) means union. For any two sets \(A\) and \(B\), the set \(A \cup B\) consists of all the elements that are in \(A\) and/or \(B\). As you are already aware, the laws listed above extend to the entire set \(\mathbb{Z}\) of integers. Using these laws, we can prove some familiar properties of integers.

**Proposition 3.1.** Suppose that \(a, b, c \in \mathbb{Z}\). Then

(1) \(a + c = b + c\) implies that \(a = b\),

(2) \(a \cdot 0 = 0\),

(3) \((-a) \cdot b = -(a \cdot b)\),

(4) \((-a) \cdot (-b) = a \cdot b\),

(5) \(a \cdot b = 0\) implies that \(a = 0\) or \(b = 0\) (or both),

(6) \(a \cdot c = b \cdot c\) and \(c \neq 0\) together imply that \(a = b\).
Proof. (1) If \( a + c = b + c \) then \((a + c) + (−c) = (b + c) + (−c)\). So by associativity of addition we have \( a + (c + (−c)) = b + (c + (−c)) \). Then by the additive inverse property, we have \( a + 0 = b + 0 \), and by the additive identity property, we conclude that \( a = b \).

(2) By the additive identity property and the distributive law we have

\[
0 + a \cdot 0 = a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0.
\]

Using (1) we find that \( 0 = a \cdot 0 \).

(3) Starting with \( a + (−a) = 0 \), we multiply both sides by \( b \) and use the distributive law to see that

\[
a \cdot b + (−a) \cdot b = (a + (−a)) \cdot b = 0 \cdot b = 0,
\]

where we have used (2) in the last equality. Thus we see that \( (−a) \cdot b = −(a \cdot b) \).

(4) Exercise.

(5) We will prove that \( a, b \neq 0 \) implies that \( a \cdot b \neq 0 \) (convince yourself that this is enough! This is an example of proof using contrapositive).

First assume that \( a, b \in \mathbb{N} \). Then since multiplication is repeated addition, which is the same as taking the successor repeatedly, by Axiom 3 we cannot have \( a \cdot b = 0 \) (this is slightly informal, but since we never formally defined addition or multiplication, I’m going to let it slide).

Now suppose that \( a \in (−\mathbb{N}) \) and \( b \in \mathbb{N} \). Then \( a = −n \) for some \( n \in \mathbb{N} \), and by (3) we have \( a \cdot b = (−n) \cdot b = −(n \cdot b) \). Since \( n, b \in \mathbb{N} \), the previous case shows that \( n \cdot b \neq 0 \), so \( −(n \cdot b) \neq 0 \).

Now suppose that \( a, b \in (−\mathbb{N}) \). Then \( a = −m \) and \( b = −n \) for some \( m, n \in \mathbb{N} \), and by (4) we have \( a \cdot b = (−m) \cdot (−n) = m \cdot n \). Since \( m, n \in \mathbb{N} \), the first case applies again and we have \( m \cdot n \neq 0 \).

(6) Exercise.

□

Just as we defined subtraction by creating the negative integers and using addition, we can define division by creating reciprocals and using multiplication. For each nonzero integer \( n \), we define the multiplicative inverse \( n^{-1} \) by the relation

\[
n \cdot n^{-1} = n^{-1} \cdot n = 1.
\]

(multiplicative inverse)

Division is then defined by (assuming that \( b \neq 0 \))

\[
a/b := a \cdot b^{-1},
\]

and the rational numbers are all of the numbers \( a/b \) for \( a, b \in \mathbb{Z} \) with \( b \neq 0 \). We use \( \mathbb{Q} \) to denote the set of rational numbers. In set-builder notation,

\[
\mathbb{Q} := \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.
\]

(As with subtraction, we have some subtlety here: we would like to think of the rational numbers 1/2 and 2/4 and 3/6 as being equal. To do this properly requires an equivalence relation; see [3] for more details if you are interested.)

As you know already, the rational numbers \( \mathbb{Q} \) inherit the same properties as the integers, along with the properties listed in Proposition 3.1. For completeness, we list them all together...
here. If \( a, b, c \in \mathbb{Q} \) then

\[
\begin{align*}
  a + b &= b + a, & \text{(commutative law for addition)} \\
  (a + b) + c &= a + (b + c), & \text{(associative law for addition)} \\
  n + 0 &= n, & \text{(additive identity)} \\
  n + (-n) &= 0, & \text{(additive inverse)} \\
  a \cdot b &= b \cdot a, & \text{(commutative law for multiplication)} \\
  (a \cdot b) \cdot c &= a \cdot (b \cdot c), & \text{(associative law for multiplication)} \\
  a \cdot 1 &= a, & \text{(multiplicative identity)} \\
  n \cdot n^{-1} &= 1, & \text{(multiplicative inverse)} \\
  (a + b) \cdot c &= a \cdot c + b \cdot c. & \text{(distributive law)}
\end{align*}
\]

A number system which satisfies all of these properties (and for which \( 0 \neq 1 \); that would be weird) is called a field.

The set field \( \mathbb{Q} \) is actually an ordered field, which means that it has an order structure \( \leq \) which obeys the following rules (which you are already familiar with):

\begin{enumerate}
  \item[(O1)] For all \( a, b \in \mathbb{Q} \), either \( a \leq b \) or \( b \leq a \).
  \item[(O2)] If \( a \leq b \) and \( b \leq a \) then \( a = b \).
  \item[(O3)] If \( a \leq b \) and \( b \leq c \) then \( a \leq c \).
  \item[(O4)] If \( a \leq b \) then \( a + c \leq b + c \).
  \item[(O5)] If \( a \leq b \) and \( c \geq 0 \) then \( ac \leq bc \).
\end{enumerate}

Using these rules, we can prove the following properties. Note that \( a > b \) means that \( a \geq b \) and \( a \neq b \).

**Proposition 3.2.** If \( a, b, c \in \mathbb{Q} \) then the following properties hold.

\begin{enumerate}
  \item[(1)] If \( a \leq b \) then \(-a \geq -b\).
  \item[(2)] If \( a \leq b \) and \( c \leq 0 \) then \( ac \geq bc \).
  \item[(3)] If \( a \geq 0 \) and \( b \geq 0 \) then \( ab \geq 0 \).
  \item[(4)] \( a^2 \geq 0 \) for all \( a \).
  \item[(5)] \( 0 < 1 \).
  \item[(6)] If \( a > 0 \) then \( a^{-1} > 0 \).
  \item[(7)] If \( 0 < a < b \) then \( 0 < b^{-1} < a^{-1} \).
\end{enumerate}

**Proof.**

1. We apply (O4) with \( c = (-a) + (-b) \). If \( a \leq b \) then

\[
-b = -b + a + (-a) = a + (-a) + (-b) \leq b + (-a) + (-b) = (-b) + b + (-b) = -a,
\]

as desired.

2. Suppose that \( c \leq 0 \). Then by (1) we have \(-c \geq 0 \). If \( a \leq b \) then by (O5) we have

\( (-c)a \leq (-c)b \) which implies that \(-ac \leq -(bc)\). Applying (1) again we find that \( ac \geq bc \).

3. Follows immediately from (O5) with \( a = 0 \).

4. By (O1), either \( a \geq 0 \) or \( a \leq 0 \). If \( a \geq 0 \) then we apply (3) with \( b = a \). If \( a \leq 0 \) then by (1) we have \(-a \geq 0 \). By (3) again we have \((-a)(-a) \geq 0 \), i.e. \( a^2 \geq 0 \).

5. Exercise.
Suppose, by way of contradiction, that $a > 0$ but $a^{-1} \leq 0$. Applying (2) to $0 \leq a$ we find that

$$0 = 0 \cdot a^{-1} \geq aa^{-1} = 1,$$

which contradicts (5).

(7) Exercise. \hfill \square

4. Absolute value

One of the most important basic concepts in analysis is that of absolute value. The purpose of absolute value is to measure distance between two numbers; for example (we will see this over and over again in this course) if we want to say that two numbers $x$ and $y$ are very close, we would write

$$|x - y| < \varepsilon,$$

where $\varepsilon > 0$ is some very small positive number. The absolute value is defined by

$$|x| = \begin{cases} 
  x & \text{if } x > 0, \\
  0 & \text{if } x = 0, \\
  -x & \text{if } x < 0.
\end{cases}$$

From this definition, we immediately see that

$$|x| \geq 0 \quad \text{for all } x.$$

The absolute value plays nicely with multiplication:

$$|xy| = |x| \cdot |y|,$$

which you can prove easily by splitting into the four cases $x, y \geq 0$, $x, -y \geq 0$, $-x, y \geq 0$, and $x, y \leq 0$.

The following theorem is probably the most important and most used inequality in analysis. It’s so important that it has its own name.

**Theorem 4.1** (The Triangle Inequality). For all $x, y \in \mathbb{Q}$ we have

$$|x + y| \leq |x| + |y|.$$

To prove the triangle inequality, we require the following lemmas which you will prove as exercises.

**Lemma 4.2.** For all $x \in \mathbb{Q}$ we have $-|x| \leq x \leq |x|$.

**Lemma 4.3.** For all $x, y \in \mathbb{Q}$ we have $|x| \leq |y|$ if and only if $-|y| \leq x \leq |y|$.

**Proof of Theorem 4.1.** Using Lemma 4.3, it is enough to prove that

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

But this follows by adding together the inequalities

$$-|x| \leq x \leq |x|$$

$$-|y| \leq y \leq |y|$$

from Lemma 4.2. \hfill \square
5. The real numbers

We are now ready to construct the real numbers $\mathbb{R}$. Recall that the property which sets the real numbers apart from the rational numbers is that, in the reals, all of the "holes" are filled (i.e. $\sqrt{2}$ is not a rational number, but it is a real number). The purpose of this section is to make this property precise and then to construct the real numbers to satisfy it. We will freely use all the properties of the rational numbers, ordering, and the absolute value without explicitly stating them.

Let’s start with $\sqrt{2}$. It may help to time travel for a bit; in the following discussion it will be helpful to know that $\sqrt{2} = 1.414213562373095048\ldots$.

Although we haven’t defined the decimal expansion of a number, you already know that $1.414213 = \frac{1414213}{1000000}$, etc. While we can’t find a rational number whose square is 2, we can find rational numbers whose square is arbitrarily close to 2, as follows:

\[
\begin{align*}
1^2 &= 1, \\
1.4^2 &= 1.96, \\
1.41^2 &= 1.9881, \\
1.414^2 &= 1.999396, \\
1.4142^2 &= 1.9996164, \\
1.41421^2 &= 1.9999899241.
\end{align*}
\]

Assuming we can continue this process indefinitely (and we can; there will be a homework problem about it later) this defines a sequence of rational numbers that converges to $\sqrt{2}$ (if we can make sense of what those words mean). Using this example as a template, we will “fill in the gaps” in the rational numbers by asserting that every sequence that should converge does converge. Let’s try to make this precise.

First, a sequence is just a list of numbers

$a_1, a_2, a_3, a_4, \ldots$

indexed by the natural numbers $1, 2, 3, \ldots$. For now, all of the $a_i \in \mathbb{Q}$ (because we haven’t defined $\mathbb{R}$) but later they can be in $\mathbb{R}$. We will usually start the index of our lists at 1, but sometimes it is convenient to start them at some other integer. We usually think of sequences as being infinitely long lists, but often it is useful consider finite sequences; sometimes we will think of a finite sequence as a finite list, and sometimes we will just implicitly repeat the last element of the list indefinitely, i.e. $a_1, a_2, a_3, \ldots a_n, a_n, a_n, \ldots$. Usually this distinction will not make too big a difference.

We would like our sequences to be convergent; roughly speaking, that means that $a_n$ and $a_{n+1}$ should be getting closer together as $n$ gets larger. Actually, this is not quite enough; instead we will use the following definition.

**Definition 5.1.** A Cauchy sequence (of rational numbers) is a sequence $(a_1, a_2, a_3, a_4, \ldots)$ such that for every rational $\varepsilon > 0$ there exists a positive integer $N$ (which is allowed to depend on $\varepsilon$) such that

$|a_m - a_n| < \varepsilon$ whenever $m, n \geq N$. 
Note: the restriction that \( \varepsilon \) needs to be a rational number is there purely because we don’t know what a real number is yet. Later we will consider Cauchy sequences of real numbers and we will think of \( \varepsilon \) as being any positive real number. You should not think about this distinction too much, as it will not be important in the long run.

We should unpack Definition 5.1; it’s quite important to understand this well. It’s saying that a Cauchy sequence is a sequence whose terms eventually (i.e. there exists a positive integer \( N \)) get really close together (i.e. \( |a_m - a_n| < \varepsilon \)). Note that this needs to hold for all \( m, n \geq N \), not just \( n \) and \( n + 1 \), say. You should think of \( \varepsilon \) as being a microscopically tiny number; so for \( \varepsilon = \frac{1}{10} \) or \( \frac{1}{100} \) or \( \frac{1}{10^{10}} \) or \( \frac{1}{10^{100}} \), etc. the terms of our sequence are eventually at most a distance of \( \varepsilon \) apart.

**Proposition 5.2.** The sequence \((1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots)\) is a Cauchy sequence.

**Proof.** We would like to come up with a natural number \( N \) so that
\[
|\frac{1}{m} - \frac{1}{n}| < \varepsilon \quad \text{whenever} \quad m, n \geq N.
\]
Using the fact that \( m, n \geq N \), together with the triangle inequality, we find that
\[
|\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{m} + \frac{1}{n} \leq \frac{2}{N}.
\]
So it is enough to come up with a number \( N \) for which \( \frac{2}{N} < \varepsilon \) or, in other words, \( N > \frac{2}{\varepsilon} \). But this is possible by the Archimedean property (the following proposition). Now that we’ve done some scratch work, let’s write out a formal proof.

Given \( \varepsilon > 0 \), choose a natural number \( N > \frac{2}{\varepsilon} \). Then if \( m, n \geq N \) we have
\[
|a_m - a_n| = \left| \frac{1}{m} - \frac{1}{n} \right| \leq \frac{1}{m} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Therefore \((a_i)\) is a Cauchy sequence. \( \square \)

To complete the proof we need the Archimedean property.

**Proposition 5.3** (The Archimedean property). For each \( x \in \mathbb{Q} \) there exists an \( n \in \mathbb{N} \) such that \( n > x \).

**Proof.** Since \( x \in \mathbb{Q} \) there exist integers \( p, q \) with \( q \geq 1 \) such that \( x = p/q \). If \( p \leq 0 \), we can just take \( n = 1 \). So we may assume that \( p \geq 1 \). Then we can take \( n = p \) since \( q \geq 1 \) implies that \( p/q \leq p < p + 1 \).

Here’s a non-example.

**Proposition 5.4.** For each \( n \geq 1 \) let \( H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \). Then the sequence \((H_1, H_2, H_3, \ldots)\) is not a Cauchy sequence.

**Proof.** In order to show that a sequence is not Cauchy, it suffices to find a specific \( \varepsilon > 0 \) such that for all \( N \in \mathbb{N} \) we have
\[
|a_m - a_n| \geq \varepsilon \quad \text{for some} \quad m, n \geq N.
\]
That is, no matter how big you take \( N \), we will always be able to find a pair \( m, n \) for which \( a_m \) and \( a_n \) are greater than \( \varepsilon \) apart.

For this particular sequence we can take \( \varepsilon = \frac{1}{2} \). For each \( n \in \mathbb{N} \), consider the difference
\[
|H_{2n} - H_n| = \frac{1}{n + 1} + \frac{1}{n + 2} + \frac{1}{n + 3} + \cdots + \frac{1}{2n}.
\]
Notice that there are exactly $n$ terms in the sum, and each term is larger than (or equal to) the smallest one $\frac{1}{2n}$. Thus
\[|H_{2n} - H_n| \geq \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.\]
Therefore, no matter how big you choose $N$, we will always be able to find a pair $n, m = 2n$ such that $|a_m - a_n| \geq \frac{1}{2}$, which shows that the sequence $(H_n)$ is not Cauchy. \hfill \square

Right now, proving that a sequence is Cauchy requires a bit of work, even for a sequence as simple as $(\frac{1}{n})$. Later we will prove some limit laws that allow us to determine when a sequence is Cauchy more easily.

We would like to do arithmetic on Cauchy sequences like addition and multiplication, but first we need to develop the notion of boundedness.

**Definition 5.5.** A sequence $(a_i)$ is **bounded** if there exists a (rational) number $M$ such that $|a_i| \leq M$ for all $i \geq 1$. In this case, we say that $(a_i)$ is bounded by $M$.

**Lemma 5.6.** Every finite sequence $(a_1, \ldots, a_n)$ is bounded.

**Proof.** We proceed by induction on the length $n$. If $n = 1$ then the statement of the lemma is clear by taking $M = |a_1|$. We want to show that the sequence $(a_1, \ldots, a_n, a_{n+1})$ is bounded. By the induction hypothesis, every list of length $n$ is bounded; in particular $(a_1, \ldots, a_n)$ is bounded. Then there exists some $M$ such that $|a_i| \leq M$ for $1 \leq i \leq n$. Since the list $(a_1, \ldots, a_{n+1})$ is bounded by $M + |a_{n+1}|$, we are done. \hfill \square

**Proposition 5.7.** Every Cauchy sequence is bounded.

**Proof.** Suppose that $(a_i)$ is a Cauchy sequence. Taking $\varepsilon = 1$, this means that there is some $N \in \mathbb{N}$ for which $|a_m - a_n| \leq 1$ for all $m, n \geq N$. This splits our sequence into a finite piece (indices 1 through $N$) and an infinite piece (index $N$ and after). (Note that we are using $a_N$ in both pieces.) By Lemma 5.6, the finite piece is bounded, say by $M$. Using the Triangle Inequality we find that
\[|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| < 1 + M.\]
It follows that $(a_n)$ is bounded by $M + 1$. \hfill \square

The trick we used in the preceding proof of adding and subtracting $a_N$ (clever addition of 0) comes up all the time in analysis (and throughout mathematics). You should keep this trick in your back pocket for quick use at all times.

We can now add and multiply Cauchy sequences.

**Proposition 5.8.** If $(a_i)$ and $(b_i)$ are Cauchy sequences then so are $(a_i + b_i)$ and $(a_i b_i)$.

**Proof.** Given $\varepsilon > 0$ there are natural numbers $N_a$ and $N_b$ such that
\[|a_m - a_n| < \frac{\varepsilon}{2} \quad \text{and} \quad |b_m - b_n| < \frac{\varepsilon}{2}\]
if $m, n \geq N_a$ and $m, n \geq N_b$ (you’ll see in a minute why we chose $\varepsilon/2$). Let $N = \max\{N_a, N_b\}$ and suppose that $m, n \geq N$. Then
\[|(a_m + b_m) - (a_n + b_n)| = |(a_m - a_n) + (b_m - b_n)| \leq |a_m - a_n| + |b_m - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\]
Thus $(a_i + b_i)$ is a Cauchy sequence.
For the product, use Proposition 5.7 to find \( M_a \) and \( M_b \) such that \( |a_i| \leq M_a \) and \( |b_i| \leq M_b \), and set \( M = \max\{M_a, M_b\} \). Since \((a_i)\) and \((b_i)\) are Cauchy, there are natural numbers \( N_a \) and \( N_b \) (likely different than before) such that

\[
|a_m - a_n| < \frac{\varepsilon}{2M} \quad \text{and} \quad |b_m - b_n| < \frac{\varepsilon}{2M}
\]

if \( m, n \geq N_a \) and \( m, n \geq N_b \). Let \( N = \max\{N_a, N_b\} \). Then if \( m, n \geq N \) we have (cleverly adding 0 again)

\[
|a_m b_m - a_n b_n| = |a_m b_m - a_m b_n + a_m b_n - a_n b_n| \\
\leq |a_m| |b_m - b_n| + |b_n| |a_m - a_n| < M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M} < \varepsilon.
\]

Thus \((a_i b_i)\) is Cauchy.

It should be clear that since the rational numbers satisfy the commutative ring axioms (associativity and commutativity of addition and multiplication, the distributive law, etc.), so do Cauchy sequences. Is there an additive identity element? Sure, take \((0, 0, 0, \ldots)\). What about a multiplicative identity element? Yes, use \((1, 1, 1, \ldots)\).

What about division? It is tempting to define division of Cauchy sequences by \((a_i/b_i)\) but this only works if \( b_n \neq 0 \) for all \( n \). Fine, but we don’t want to divide by the zero element \((0, 0, 0, \ldots)\) anyway. However, there is a catch: the sequences \((1, 1, 1, 1, 1, \ldots)\) and \((1, 0, 1, 1, 1, 1, \ldots)\) are both Cauchy sequences and they seem to have the same “limit,” namely 1. So we should be able to divide by either of them; but we can’t divide by the second one in the usual way because of that pesky 0.

A more interesting example comes from the two Cauchy sequences

\[
1, 1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots, \\
2, 1.5, 1.42, 1.415, 1.4143, 1.41422, \ldots.
\]

Both “look like” they are converging to \( \sqrt{2} \). But none of the terms in the first sequence are equal to their counterpart in the second sequence, so the sequences are definitely not the same. Our aim is eventually to define the real numbers as limits of Cauchy sequences of rational numbers, so we should consider the two sequences above to be “the same.”

**Definition 5.9.** Two sequences \((a_i)\) and \((b_i)\) are equivalent if for each \( \varepsilon > 0 \) there exists a natural number \( N \) (which can depend on \( \varepsilon \)) such that

\[
|a_n - b_n| < \varepsilon \quad \text{whenever} \quad n \geq N.
\]

The following two propositions will allow us to define division.

**Proposition 5.10.** Suppose that \((a_i)\) is a Cauchy sequence that is not equivalent to the zero sequence \((0, 0, 0, \ldots)\). Then there exists a Cauchy sequence \((b_i)\) and a number \( m > 0 \), with \( |b_i| \geq m \) for all \( i \), such that \((a_i)\) and \((b_i)\) are equivalent.

**Proof.** Since \((a_i)\) is not equivalent to the zero sequence, there exists a fixed number \( \varepsilon_0 \) for which the definition of equivalent sequences fails. This means that for each \( N \geq 1 \) there exists a number \( n \geq N \) for which

\[
|a_n - 0| \geq \varepsilon_0.
\]

Now, \((a_i)\) is a Cauchy sequence, so there exists some (fixed) \( N_0 \) for which

\[
|a_m - a_n| < \frac{\varepsilon_0}{2} \quad \text{whenever} \quad m, n \geq N_0.
\]


And for this fixed $N_0$ there is a fixed $n_0 \geq N_0$ for which
\[ |a_{n_0}| = |a_{n_0} - 0| \geq \varepsilon_0. \tag{5.1} \]

I claim that $|a_n| \geq \varepsilon_0/2$ for every $n \geq N_0$. Why? Suppose, by way of contradiction, that $|a_{n_1}| < \varepsilon_0/2$ for some $n_1 \geq N_0$. Then, by the triangle inequality
\[ |a_{n_0}| = |a_{n_0} - a_{n_1} + a_{n_1}| \leq |a_{n_0} - a_{n_1}| + |a_{n_1}| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0. \]
But this contradicts (5.1), so we must have $|a_n| \geq \varepsilon_0/2$ for every $n \geq N_0$.

This almost does it. We have shown that the sequence $a_n$ eventually satisfies $|a_n| \geq m$ for $m = \varepsilon_0/2$. We simply define $b_i = m$ for $1 \leq i < N_0$ and $b_i = a_i$ for $i \geq N_0$. It is clear that $(a_i)$ and $(b_i)$ are equivalent because $|a_i - b_i| = 0$ for all $i \geq N_0$. This finishes the proof.

Sequences such as $(b_i)$ from the previous proposition which satisfy $|b_i| \geq m$ for some $m > 0$ are said to be bounded away from zero.

**Proposition 5.11.** Suppose that $(a_i)$ is a Cauchy sequence which is bounded away from zero. Then the sequence $(a_i^{-1})$ is also a Cauchy sequence.

**Proof.** Since $(a_i)$ is bounded away from zero, there exists some $m > 0$ for which $|a_i| \geq m$ for all $i$. Thus
\[ |a_i^{-1} - a_n^{-1}| = |\frac{a_m - a_n}{a_m a_n}| \leq \frac{|a_m - a_n|}{m^2}. \]
So it suffices to find an $N$ for which $m, n \geq N$ implies that $|a_m - a_n| < m^2 \varepsilon$. But $(a_i)$ is a Cauchy sequence, so we can certainly do that. \[ \square \]

Now we can define division of sequences via $(a_i b_i^{-1})$ as long as the sequence $(b_i)$ is bounded away from zero. Furthermore, Proposition 5.10 shows that any sequence $(b_i)$ which is not equivalent to the zero sequence is equivalent to a sequence $(c_i)$ which is bounded away from zero. So we should be able to divide by any sequence which is not equivalent to zero by just swapping $(b_i)$ out for $(c_i)$. But how do we know that this is well-defined? That is, does the result depend on which sequence $(c_i)$ we choose to swap out for $(b_i)$? There will be many choices in general. This opens up a whole new can of worms.

It’s time to finally define what we mean by a real number, and then we will be able to answer some of the questions posed above. To do this we will introduce a formal symbol “$\lim$”$a_n$ which you should think of as merely a symbol (hence the quotation marks). We are using “$\lim$” so that your brain thinks of this as the familiar limit of a sequence, but since we haven’t defined what we mean by limit yet, this is still just a symbol.

**Definition 5.12.** A real number is defined to be an object of the form “$\lim$”$a_n$ where $(a_n)$ is a Cauchy sequence of rational numbers. Two real numbers “$\lim$”$a_n$ and “$\lim$”$b_n$ are said to be equal if $(a_n)$ and $(b_n)$ are equivalent Cauchy sequences. The set of real numbers is $\mathbb{R}$.

It is a good idea to check that this definition of equal is consistent.

**Lemma 5.13.** Suppose that $x = \lim a_n$, $y = \lim b_n$, and $z = \lim c_n$ are real numbers. Then
\[ \begin{align*}
(1) & \ x = x, \\
(2) & \text{if } x = y \text{ then } y = x, \\
(3) & \text{if } x = y \text{ and } y = z \text{ then } x = z.
\end{align*} \]

**Proof.** Exercise. \[ \square \]
Are the rational numbers also real numbers? In a formal sense, the answer is “no” because real numbers are technically objects of the form \( \lim a_n \), and rational numbers are not. But there is a sense in which we can embed \( \mathbb{Q} \) into \( \mathbb{R} \): for each rational number \( x \), the sequence \((x, x, x, \ldots)\) is Cauchy (why?) and so the object \( \lim x \) is a real number. In this way we can think of the rational numbers as sitting inside the reals, i.e. \( \mathbb{Q} \subseteq \mathbb{R} \).

Since we know how to add, subtract, multiply, and divide sequences, we can do those operations immediately on real numbers. But there is one subtlety that we should take care of: if \( x, y, \) and \( z \) are real numbers and \( x = y \), then is it the case that \( x + z = y + z \)? Similarly, is it the case that \( x \cdot z = y \cdot z \)? Since two different Cauchy sequences can represent the same real number, it’s not immediately clear. The following lemma takes care of this.

**Lemma 5.14.** Addition, multiplication, and reciprocation are well defined. That is, suppose that \( x = \lim a_n, \ x' = \lim a'_n, \) and \( y = \lim b_n \) with \( x = x' \). Then we have

1. \( x + y = x' + y \),
2. \( x \cdot y = x' \cdot y \),
3. if \( x \neq 0 \), then \( x^{-1} = (x')^{-1} \).

**Proof.** (1) We need to show that the sequences \((a_n + b_n)\) and \((a'_n + b_n)\) are equivalent. Let \( \varepsilon > 0 \) be given. Since \( x = x' \), the sequences \((a_n)\) and \((a'_n)\) are equivalent, so there exists a number \( N \) for which

\[
|a_n - a'_n| < \varepsilon \quad \text{whenever} \quad n \geq N.
\]

So if \( n \geq N \) we have

\[
|(a_n + b_n) - (a'_n + b_n)| = |a_n - a'_n| < \varepsilon.
\]

Thus \((a_n + b_n)\) is equivalent to \((a'_n + b_n)\).

(2) Exercise.

(3) Suppose that \( x \neq 0 \). Then \( x' \neq 0 \) by Lemma 5.13. So \( x = \lim a_n \) and \( x' = \lim b_n \) for some Cauchy sequences \((a_n)\) and \((b_n)\) which, by Proposition 5.10 we can take to be bounded away from zero. Consider the product \( P \) defined by

\[
P := x^{-1}x(x')^{-1} = \lim a_n^{-1} \lim a_n \lim b_n^{-1} \lim b_n.
\]

By the definition of multiplication, we have \( P = \lim (a_n^{-1}a_n b_n^{-1}) = \lim b_n^{-1} = (x')^{-1} \). On the other hand, by (2) we have

\[
P = x^{-1}x(x')^{-1} = \lim a_n^{-1} \lim b_n \lim b_n^{-1} = \lim (a_n^{-1}b_n b_n^{-1}) = \lim a_n^{-1} = x^{-1}.
\]

Comparing the two expressions for \( P \), we find that \( x^{-1} = (x')^{-1} \). \( \square \)

We now define division of real numbers via \( x/y = x \cdot y^{-1} \). One can show without too much difficulty (and we have almost completely done it already) that the real numbers satisfy all of the field properties on page 7. It remains to define an ordering on the reals as well as the absolute value.

**Definition 5.15.** A nonzero real number \( x \) is positive (and we write \( x > 0 \)) if it can be represented by a Cauchy sequence \((a_i)\) with \( a_i > 0 \) for all \( i \). Similarly, \( x \) is negative (and we write \( x < 0 \)) if it can be represented by a Cauchy sequence \((b_i)\) with \( b_i < 0 \) for all \( i \).

**Lemma 5.16.** Every real number is either positive, negative, or zero.
**Proof.** Let \( x \in \mathbb{R} \) and suppose that \( x \neq 0 \). Then we need to prove that \( x > 0 \) or \( x < 0 \).

Since \( x \neq 0 \), Proposition 5.10 implies that there is a Cauchy sequence \((a_n)\) which is bounded away from zero such that \( x = \lim a_n \). This means that there is a number \( m > 0 \) such that \( |a_n| \geq m \) for all \( n \). I claim that there is a natural number \( N \) such that \( a_n \) has the same sign (i.e. \( a_n \) is always positive or always negative) for all \( n \geq N \).

Indeed, let \( \varepsilon = m \). Then there exists a number \( N \) such that \( |a_m - a_n| < m \) whenever \( m, n \geq N \). (5.2)

Suppose that \( a_m \) and \( a_n \) have different sign. Without loss of generality, say \( a_m > 0 \) and \( a_n < 0 \) (if not, we can just swap the indices). Then, since \((a_n)\) is bounded,

\[
|a_m - a_n| = |a_m + |a_n|| = a_m + |a_n| \geq m + m = 2m > m.
\]

But this contradicts (5.2). Thus \( a_m \) and \( a_n \) have the same sign (both are positive or both are negative).

So all of the terms \( a_n \) with \( n \geq N \) have the same sign. In order to ensure that every term \( a_n \) with \( n \geq 1 \) has the same sign, we can change the beginning of the sequence (since changing finitely many terms at the beginning of a sequence yields an equivalent sequence) to have this property. For instance, if \( a_N > 0 \) we can define a new sequence \((b_n)\) with \( b_n = |a_n| \). This new sequence is equivalent to \( a_n \) and has the property that \( b_n > m \) for all \( n \). \( \square \)

We can now define absolute value in the same way we did before:

\[
|x| = \begin{cases} 
  x & \text{if } x > 0, \\
  0 & \text{if } x = 0, \\
  -x & \text{if } x < 0. 
\end{cases}
\]

Note that this definition extends the original one in the sense that if we think of \( r \in \mathbb{Q} \) and \( x = (r, r, r, \ldots) \in \mathbb{R} \) we have \( |x| = (|r|, |r|, |r|, \ldots) \). In fact, this is true for every Cauchy sequence of rationals.

**Lemma 5.17.** If \( x = \lim a_n \in \mathbb{R} \) then \( |x| = \lim |a_n| \).

**Proof.** Exercise. \( \square \)

We also have all of the familiar ordering properties that we had for the rational numbers. Unfortunately we don’t have time to prove each one of these, so you should check them on your own.

**Proposition 5.18.** The ordering on the real numbers satisfies all of the properties that the ordering on the rationals satisfies.

The following is another important property which shows how \( \mathbb{Q} \) sits inside \( \mathbb{R} \).

**Proposition 5.19.** \( \mathbb{Q} \) is dense in \( \mathbb{R} \); i.e. for every pair of real numbers \( x, y \) with \( x < y \) there exists a rational number \( q \) such that

\[
x < q < y.
\]

**Proof.** Exercise. \( \square \)
6. Sequences of real numbers and limits

To sum up the previous section, \( \mathbb{R} \) is an ordered field which contains \( \mathbb{Q} \) as a dense subset. Recall that what we really want out of \( \mathbb{R} \) is a number system which has no holes (e.g. we want \( \sqrt{2} \in \mathbb{R} \)). But so far all we’ve done is constructed something that may not even be better than \( \mathbb{Q} \). The purpose of this section is to define limits of sequences of real numbers and to show that \( \mathbb{R} \) is complete; i.e.

*every Cauchy sequence of real numbers converges to a real number.*

The following definition should look familiar.

**Definition 6.1.** Let \((a_i)\) be a sequence of real numbers. We say that \((a_i)\) is a Cauchy sequence if for every \(\varepsilon > 0\) there exists a natural number \(N\) for which

\[
|a_m - a_n| < \varepsilon \quad \text{whenever} \quad m, n \geq N.
\]

The concept of convergence is quite similar.

**Definition 6.2.** Let \((a_i)\) be a sequence of real numbers. We say that \((a_i)\) converges to \(L \in \mathbb{R}\) (and we write \(\lim_{n \to \infty} a_n = L\)) if for every \(\varepsilon > 0\) there exists a natural number \(N\) for which

\[
|a_n - L| < \varepsilon \quad \text{for all} \quad n \geq N.
\]

It should come as no surprise that these notions are related.

**Proposition 6.3.** Let \((a_i)\) be a sequence of real numbers. If \((a_i)\) converges to \(L\) then \((a_i)\) is a Cauchy sequence.

**Proof.** Let \(\varepsilon > 0\). Since \((a_i)\) converges to \(L\) there is a number \(N\) such that

\[
|a_n - L| < \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq N.
\]

It follows that if \(m, n \geq N\) we have (by the triangle inequality)

\[
|a_n - a_m| = |a_n - L - (a_m - L)| \leq |a_n - L| + |a_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus \((a_i)\) is Cauchy. \(\square\)

Perhaps more interesting is the converse of the previous proposition, namely that every Cauchy sequence converges. This implies that there are no analogues of \(\sqrt{2}\) for the real numbers; i.e. there are no “holes” in \(\mathbb{R}\). To prove this, we first show that every Cauchy sequence of rationals converges to a real number (the following statement might look completely trivial, but this is likely an artifact of our choice of notation; now you can see why we are using “lim” \(a_n\) to denote a real number).

**Proposition 6.4.** If \((a_i)\) is a Cauchy sequence of rational numbers then \(\lim_{n \to \infty} a_n = \lim \ a_n\).

**Proof.** Let \(L = \lim a_k\) (we will use \(k\) here so as to avoid confusion with \(\lim a_n\)) and let \(\varepsilon > 0\) be given. Since \((a_k)\) is Cauchy, there is a number \(N \geq 1\) such that

\[
|a_k - a_\ell| < \varepsilon
\]

for all \(k, \ell \geq N\).

To prove that \(\lim_{n \to \infty} a_n = L\) we need to show that

\[
|a_n - L| < \varepsilon
\]
for sufficiently large $n$ (say $n \geq N$). Let’s think about what this means: the real number $a_n - L$ is the difference of the rational number $a_n$ and the real number $L$. To add/subtract real numbers, we need to represent each one by a Cauchy sequence. This is already done for $L$, and for $a_n$ we can use the constant sequence $(a_n, a_n, a_n, \ldots)$. Thus

$$|a_n - L| = \left| \lim (a_n, a_n, a_n, \ldots) - \lim (a_1, a_2, a_3, \ldots, a_k, \ldots) \right|$$

$$= \left| \lim (a_n - a_1, a_n - a_2, \ldots, a_n - a_k, \ldots) \right|$$

$$= \left| \lim ([a_n - a_1], [a_n - a_2], \ldots, [a_n - a_k], \ldots) \right|,$$

where we have use Lemma 5.17 in the last step. For all $k \geq N$ we have

$$|a_n - a_k| < \varepsilon,$$

so the real number $|a_n - L|$ is less than $\varepsilon$ (think carefully about why this is). Therefore

$$\lim_{n \to \infty} a_n = L.$$  

We can now prove that $\mathbb{R}$ is complete.

**Theorem 6.5.** $\mathbb{R}$ is complete. Equivalently, every Cauchy sequence of real numbers converges to a real number.

**Proof.** Suppose that $(x_n)$ is a Cauchy sequence of real numbers. Let $\varepsilon > 0$ be given. For each $n$, choose a rational number $q_n$ such that

$$|x_n - q_n| < \frac{\varepsilon}{3}.$$  

This is possible because $\mathbb{Q}$ is dense in $\mathbb{R}$. Then the sequence $(q_n)$ is Cauchy because

$$|q_m - q_n| = |q_m - x_m + x_m - x_n + x_n - q_n| \leq |q_m - x_m| + |x_m - x_n| + |x_n - q_n|$$

and for $m, n$ sufficiently large this quantity is less than $\varepsilon$ because we can make $|x_m - x_n| < \varepsilon/3$.

Let $L = \lim_{n \to \infty} q_n$. We will show that $\lim_{n \to \infty} x_n = L$. Let $\varepsilon > 0$ be given. By the previous proposition, $(q_n)$ converges to $L$, so there is some $N \geq 1$ such that

$$|q_n - L| < \frac{\varepsilon}{2}$$

for all $n \geq N$. It follows that

$$|x_n - L| = |x_n - q_n + q_n - L| \leq |x_n - q_n| + |q_n - L| < \frac{\varepsilon}{3} + \frac{\varepsilon}{2} < \varepsilon$$

for all $n \geq N$, as desired.  

We conclude this section by listing some additional properties of convergent sequences. First, since convergent sequences are Cauchy, we can extend the properties of Cauchy sequences of rational numbers to convergent sequences of real numbers (via essentially the same proofs). In particular, we have the following.

**Proposition 6.6.** Every convergent sequence of real numbers is bounded.

Is the converse true? That is, is every bounded sequence convergent? Here is a quick counterexample: consider the sequence $1, 0, 1, 0, 1, 0, 1, 0, \ldots$. This sequence is bounded because $|a_n| \leq 1$ for all $n$, but it is clearly not convergent. However, it is the case that bounded monotonic sequences are convergent. There are two versions of this: increasing sequences which are bounded above are convergent, and decreasing sequences which are bounded below are convergent. We will state and prove the first version in the next section.
We can also do arithmetic with limits. One way to prove the following properties is to replace every convergent sequence of reals with a convergent sequence of rationals that has the same limit (see the proof of the previous theorem). Then arithmetic of limits is the same as arithmetic of real numbers of the form  \( \lim a_n \), which we have already covered.

**Proposition 6.7.** Suppose that \((a_n)\) and \((b_n)\) are convergent sequences of real numbers. Then the following are true.

1. \( \lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n \)
2. \( \lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \lim_{n \to \infty} b_n \)
3. If \( c \in \mathbb{R} \) then \( \lim_{n \to \infty} ca_n = c \lim_{n \to \infty} a_n \)
4. If \( b_n \neq 0 \) for all \( n \) and if \( \lim_{n \to \infty} b_n \neq 0 \) then \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \)

The following proposition gives a list of basic limits which will come in handy later on. See Theorem 9.7 in your book for a proof.

**Proposition 6.8.** The following are true.

1. If \( p > 0 \) then \( \lim_{n \to \infty} \frac{1}{n^p} = 0 \).
2. If \( |a| < 1 \) then \( \lim_{n \to \infty} a^n = 0 \). If \( a = 1 \) then \( \lim_{n \to \infty} a^n = 1 \). If \( a = -1 \) or if \( |a| \geq 1 \) then the sequence \((a^n)\) does not converge.
3. If \( a > 0 \) then \( \lim_{n \to \infty} a^{1/n} = 1 \).

Finally, the ordering on the reals plays nicely with limits, too. We will frequently use this property without stating it.

**Lemma 6.9.** Suppose that \((a_n)\) is a convergent sequence. If \( a_n \leq M \) for all \( n \geq 1 \) then

\[ \lim_{n \to \infty} a_n \leq M. \]

Similarly, if \( a_n \geq m \) for all \( n \geq 1 \) then

\[ \lim_{n \to \infty} a_n \geq m. \]

**Proof.** Exercise. \( \square \)

## 7. The least upper bound property

We constructed the real numbers by filling in the holes in the rational numbers. But one other advantage to the reals over the rationals is the existence of a least upper bound for any subset of the real numbers. (Your textbook [2] treats this property as an axiom; see §4, p. 23. We will show that it follows from the completeness of the real numbers.)

**Definition 7.1.** Suppose that \( E \subseteq \mathbb{R} \) is a set of real numbers and let \( M, m \in \mathbb{R} \). We say that \( M \) is an upper bound for \( E \) if \( x \leq M \) for every \( x \in E \). Similarly, \( m \) is a lower bound for \( E \) if \( x \geq m \) for every \( x \in E \).
Consider the interval $E = [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. This set has many upper bounds; for example, 1, 2, and 496 are upper bounds. Also \(172893\) is an upper bound. It also has many lower bounds; for example, 0 and \(-5\). But if I asked you to name an upper bound and a lower bound for \(E\), you would likely say 1 and 0, respectively. This is because these are the least upper bound and greatest lower bound (and so they tell you the most information about the set \(E\)).

**Definition 7.2.** Suppose that \(E \subseteq \mathbb{R}\) is a set of real numbers and let \(M, m \in \mathbb{R}\). We say that \(M\) is a least upper bound for \(E\) if (a) \(M\) is an upper bound for \(E\) and (b) if \(M'\) is another upper bound for \(E\), then \(M' \geq M\). Similarly, we say that \(m\) is a greatest lower bound for \(E\) if (a) \(m\) is a lower bound for \(E\) and (b) if \(m'\) is another lower bound for \(E\) then \(m' \leq m\).

It should come as no surprise that least upper bounds are unique.

**Proposition 7.3.** Suppose that \(M\) and \(M'\) are both least upper bounds for \(E\). Then \(M = M'\).

**Proof.** By the definition, since \(M\) is a least upper bound and \(M'\) is another upper bound, we must have \(M' \geq M\). Reversing the roles of \(M\) and \(M'\) (i.e. by symmetry) we find that \(M \geq M'\). Thus \(M = M'\). □

If \(E\) has a least upper bound then we write \(\text{sup}(E)\) for the supremum, or least upper bound, of \(E\). Similarly, if \(E\) has a greatest lower bound then we write \(\text{inf}(E)\) for the infimum, or greatest lower bound.

One of the most important properties of the real numbers is the existence of a least upper bound for every subset of \(\mathbb{R}\) that you would expect to have a least upper bound. That is, if a set \(E\) doesn’t have an upper bound, then we shouldn’t expect it to have a least upper bound. The proof of this theorem is somewhat involved, and I won’t give all of the details; you should fill these in on your own.

**Theorem 7.4.** Let \(E\) be a non-empty subset of \(\mathbb{R}\). If \(E\) has an upper bound, then it has a least upper bound.

**Proof.** Since \(E\) is non-empty, there is some element \(a_1 \in E\), and since \(E\) has an upper bound, we can choose some upper bound \(b_1\). Clearly \(a_1 \leq b_1\). We will define two sequences \((a_i)\) and \((b_i)\) inductively such that \(a_n \leq b_n\) for all \(n\).

For each \(n \geq 1\), suppose that we have constructed \(a_n\) and \(b_n\) such that \(a_n \leq b_n\). Let \(K = \frac{a_n + b_n}{2}\) be the average of \(a_n\) and \(b_n\). Note that \(a_n \leq K \leq b_n\). If \(K\) is an upper bound for \(E\), let \(a_{n+1} = a_n\) and let \(b_{n+1} = K = \frac{a_n + b_n}{2}\). Then \(|b_{n+1} - a_{n+1}| = \frac{a_n + b_n}{2} - a_n| = \frac{b_n - a_n}{2}\). If \(K\) is not an upper bound for \(E\), then there is some element \(x \in E\) such that \(x > K\) (otherwise \(K\) would be an upper bound). In that case, let \(a_{n+1} = x\) and let \(b_{n+1} = b_n\). Then \(|b_{n+1} - a_{n+1}| = b_n - x < b_n - K = \frac{b_n - a_n}{2}\). Note that in either case \(|b_{n+1} - a_{n+1}| \leq \frac{b_n - a_n}{2}\). Also in either case \(a_n\) is increasing (or staying the same) and \(b_n\) is decreasing (or staying the same).

Since \(a_n \in E\) for all \(n\) and \(b_n\) is an upper bound for \(E\) for all \(n\), we have \(a_1 \leq a_2 \leq \ldots \leq a_n \leq b_n \leq b_1\).

Using the inequality \(|b_{n+1} - a_{n+1}| \leq \frac{b_n - a_n}{2}\) one can show by induction (check this!) that

\[
|b_{n+1} - a_{n+1}| \leq \frac{b_1 - a_1}{2^n}. \tag{7.1}
\]

We will show that the sequences \((a_n)\) and \((b_n)\) are Cauchy.
Let $\varepsilon > 0$ be given. Choose $N \geq 1$ such that $2^N > \frac{b_1 - a_1}{\varepsilon}$ (why is this possible?). Let $m, n \geq N$. Without loss of generality, assume that $m \geq n$ (otherwise just swap them). Then $a_m \geq a_n$, and we have

$$|a_m - a_n| = a_m - a_n \leq b_n - a_n \leq \frac{b_1 - a_1}{2^n} < \varepsilon.$$ 

Thus $(a_n)$ is Cauchy. A very similar argument shows that $(b_n)$ is Cauchy (check this!).

Using (7.1) one can show that $(a_n)$ and $(b_n)$ are equivalent sequences (check this!). Therefore they have the same limit, say $L$. I claim that $L$ is the least upper bound of $E$. On the one hand, if $x \in E$ then $x \leq b_n$ for all $n$ since every $b_n$ is an upper bound for $E$. Thus $x \leq L$. Since $x$ was an arbitrary element of $E$, we conclude that $L$ is an upper bound for $E$. On the other hand, if $L'$ is any other upper bound, then $L' \geq a_n$ for all $n$ since every $a_n$ is an element of $E$. Thus $L' \geq L$. It follows that $L$ is the least upper bound of $E$. 

There is a version of this for lower bounds as well.

**Corollary 7.5.** Let $E$ be a non-empty subset of $\mathbb{R}$. If $E$ has a lower bound, then it has a greatest lower bound.

**Proof.** Exercise.

We can also talk about the supremum and infimum of a sequence. To each sequence $(a_n)$ we associate the set of numbers $E = \{a_n : n \geq 1\}$, and we define $\sup a_n := \sup E$ and $\inf a_n := \inf E$ if those quantities exist.

For example, let $a_n = (-1)^n$; then $(a_n)$ is the sequence $-1, 1, -1, 1, -1, 1, \ldots$. The associated set is the two-element set $E = \{-1, 1\}$. This has least upper bound 1 and greatest lower bound $-1$. Thus $\sup a_n = 1$ and $\inf a_n = -1$.

As another example, let $a_n = 1/n$. Then the sequence is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ and the set is $E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$. The least upper bound of $E$ is 1 and the greatest lower bound of $E$ is 0. Note that 0 is not in the set, nor is it in the sequence, but it is the largest number less than or equal to every element of the set sequence.

The existence of a least upper bound for bounded sets implies the existence of a least upper bound for bounded sequences. We will use this fact to prove the following proposition.

**Proposition 7.6.** Bounded monotonic sequences are convergent. That is, if $(a_n)$ is a sequence which satisfies $a_n \leq M$ for some $M \in \mathbb{R}$, and if $a_n$ is increasing ($a_{n+1} \geq a_n$ for all $n$) then $(a_n)$ converges.

**Proof.** Suppose that $(a_n)$ is an increasing sequence which is bounded above. Let $E$ be the set of values $\{a_n : n \geq 1\}$. Then $(a_n)$ has a least upper bound which we will call $s = \sup a_n$. We will show that $\lim_{n \to \infty} a_n = s$.

Let $\varepsilon > 0$ be given. Since $s$ is the least upper bound, $s - \varepsilon$ is not an upper bound for $E$. So there exists some $N \geq 1$ such that $a_N > s - \varepsilon$. Since $(a_n)$ is increasing, it follows that $a_n \geq a_N > s - \varepsilon$ for all $n \geq N$. Thus, for all $n \geq N$ we have

$$|a_n - s| = s - a_n \leq s - a_N < s - (s - \varepsilon) = \varepsilon.$$ 

Therefore $(a_n)$ converges to $s$. 

There is a version of this for decreasing sequences which are bounded below.

**Proposition 7.7.** If $(a_n)$ is a sequence which satisfies $a_n \geq m$ for some $m \in \mathbb{R}$, and if $a_n$ is decreasing ($a_{n+1} \leq a_n$ for all $n$) then $(a_n)$ converges.
8. Subsequences and the Bolzano-Weierstrass theorem

Consider the sequence \((a_n)\) defined by \(a_n = (-1)^n\). The terms of this sequence are 
\[-1, 1, -1, 1, -1, 1, -1, 1, \ldots.\]

It should be clear that this is not a convergent sequence. But it’s kind of close to being convergent in the sense that it seems like the sequence wants to converge to two different limits: 1 and \(-1\). In this case we say that there is a subsequence (say, take all of the \(-1\) terms) that converges.

Definition 8.1. A subsequence of a sequence \((a_n)\) is a sequence of the form \((a_{n_k})\), where \(n_k\) is a strictly increasing sequence of natural numbers.

In plain English, this says that a subsequence is a selection of some (maybe all) of the terms of \((a_n)\) taken in order. So in our example above, say we take \(n_1 = 2\), \(n_2 = 3\), \(n_3 = 5\), \(n_4 = 6\), \(n_5 = 8\), and \(n_k = 2k\) for all \(k \geq 6\). Then the terms of the subsequence \((a_{n_k})\) is 
\[1, -1, 1, 1, 1, \ldots.\]

Since the terms are all 1 after the fourth term, this subsequence is convergent. See §11 of the textbook for more examples of subsequences. The purpose of this section is to prove the Bolzano-Weierstrass theorem, which states that every bounded sequence has a convergent subsequence. We will prove this in a few steps.

Note: we have not explicitly defined what it means for \(\lim_{n \to \infty} a_n = \pm \infty\), so if you are not already familiar with this idea, please see Definition 9.8 on p. 50 of your textbook.

Proposition 8.2. Suppose that \((a_n)\) is a sequence.

1. Let \(t \in \mathbb{R}\). There is a subsequence of \((a_n)\) converging to \(t\) if and only if the set 
\[\{n \in \mathbb{N} : |a_n - t| < \varepsilon\}\]  
\[
\text{is infinite for each } \varepsilon > 0.
\]

2. If \((a_n)\) is unbounded above then it has a subsequence with limit \(+\infty\).

3. If \((a_n)\) is unbounded below then it has a subsequence with limit \(-\infty\).

In each case, we can take the subsequence to be monotonic.

Proof. (1) \((\Rightarrow)\) Suppose that there is a subsequence \((a_{n_k})\) converging to \(t\). Then for each \(\varepsilon > 0\) there exists a number \(N\) such that 
\[|a_{n_k} - t| < \varepsilon\]
for every \(n_k \geq N\). Since \(N\) is a finite number, there are infinitely many \(n_k \geq N\), so the set (8.1) is infinite.

\((\Leftarrow)\) Suppose that, for each \(\varepsilon > 0\), the set (8.1) is infinite. There is one special case that we should consider first. It might be that \(a_n = t\) for infinitely many \(t\) (then certainly (8.1) would be infinite for any \(\varepsilon > 0\)). But then we can just take the subsequence consisting of terms with \(a_{n_k} = t\) and that’s a monotonic convergent subsequence. It’s more interesting when that’s not the case.

So suppose that \(a_n = t\) for only finitely many \(n\). Then, for each \(\varepsilon > 0\), the set 
\[\{n \in \mathbb{N} : 0 < |a_n - t| < \varepsilon\}\]
is infinite. Let’s unpack what this means:
\[ 0 < |a_n - t| < \varepsilon \quad \text{if and only if} \quad 0 < a_n - t < \varepsilon \quad \text{or} \quad -\varepsilon < a_n - t < 0. \]
The last two inequalities can be written
\[ t < a_n < t + \varepsilon \quad \text{or} \quad t - \varepsilon < a_n < t. \]

Let
\[ S_-(\varepsilon) := \{ n \in \mathbb{N} : t - \varepsilon < a_n < t \} \quad \text{and} \quad S_+(\varepsilon) := \{ n \in \mathbb{N} : t < a_n < t + \varepsilon \}. \]
Note that if \( \varepsilon_2 \leq \varepsilon_1 \) then \( S_{\pm}(\varepsilon_2) \subseteq S_{\pm}(\varepsilon_1) \) (think about this carefully until you are convinced).

It follows that one of the following two statements is true (both could be true):
\[ S_+(\varepsilon) \quad \text{is infinite for every} \quad \varepsilon > 0 \quad \text{(8.2)} \]
\[ \text{or} \]
\[ S_-(\varepsilon) \quad \text{is infinite for every} \quad \varepsilon > 0. \quad \text{(8.3)} \]

Let us assume that (8.2) is true. The proof in the case that (8.3) holds is quite similar (and is the case your book covers). We will construct a sequence \( n_1, n_2, n_3, n_4, \ldots \) in a similar way as in the proof of Theorem 7.4. Since \( S_+(1) \) is infinite, there exists some \( n_1 \in S_+(1) \).

Now suppose we have chosen \( n_1 \) through \( n_k \) such that \( n_1 < n_2 < n_3 < \cdots < n_k \) and such that
\[ t < a_{n_j} \leq \min\{ t + \frac{1}{j}, a_{n_{j-1}} \} \quad \text{for} \quad j = 1, 2, 3, \ldots, k. \]
(We use the min so that the subsequence \( a_{n_k} \) is decreasing.) With \( \varepsilon_1 = \frac{1}{k+1} \) we can use that \( S_+(\varepsilon_1) \) is infinite to find the next index \( n_{k+1} \). But we also want this minimum property to hold. So write \( a_{n_k} = t + (a_{n_k} - t) \). Since \( a_{n_k} > t \) the number \( \varepsilon_2 = a_{n_k} - t \) is positive. So the set \( S_+(\varepsilon_2) \) is infinite. Taking \( \varepsilon \) to be the smaller of \( \varepsilon_1, \varepsilon_2 \) we can find some number \( n_{k+1} \in S_+(\varepsilon) \) such that \( n_{k+1} > n_k \) and
\[ t < a_{n_{k+1}} \leq \min\{ t + \frac{1}{k+1}, a_{n_k} \}. \]
Thus we have constructed a subsequence \( (a_{n_k}) \) which is decreasing and bounded below (and therefore convergent).

The proofs of (2) and (3) are much easier; see the proof of Theorem 11.2 in your book if you are interested. \( \square \)

As an example of what we can do with this proposition, let’s prove that every sequence of positive numbers with \( \inf a_n = 0 \) has a subsequence converging to 0. (Can you write down an example of a sequence such that \( \inf a_n = 0 \) but \( (a_n) \) is not convergent?) By the proposition, it suffices to show that the sets \( \{ n \in \mathbb{N} : a_n < \varepsilon \} \) are infinite for each \( \varepsilon > 0 \). Suppose, by way of contradiction, that there is some \( \varepsilon_0 > 0 \) such that \( a_n < \varepsilon_0 \) for only finitely many \( n \). Then \( \inf a_n = \min\{ a_n : a_n < \varepsilon_0 \} \) (think carefully about why this is). But each \( a_n \) is positive, so this means that \( \inf a_n > 0 \), a contradiction. So there must exist a subsequence \( (a_{n_k}) \) converging to 0.

**Proposition 8.3.** If the sequence \( (a_n) \) converges, then every subsequence converges to the same limit.
Proof. This is pretty straightforward. Let \((a_{n_k})\) be a subsequence of \((a_n)\). Note that \(n_k \geq k\) for all \(k \geq 1\) (convince yourself that this is true or look in Theorem 11.3 in your book for an inductive justification). Let \(L = \lim a_n\). Then for each \(\varepsilon > 0\) there is some \(N \geq 1\) such that \(|a_n - L| < \varepsilon\). Thus, for \(k \geq N\) we have \(n_k \geq k \geq N\), which implies that \(|a_{n_k} - L| < \varepsilon\). Thus \((a_{n_k})\) converges to \(L\). □

We are almost ready to prove Bolzano-Weierstrass, but first we need one more intermediate result.

**Proposition 8.4.** Every sequence has a monotonic subsequence.

**Proof.** Let \((a_n)\) be a sequence. For the purposes of this proof, let’s say that a term \(a_n\) is dominant if it is larger than every term after it:

\[a_n > a_m \quad \text{for all } m > n.\]

If there are infinitely many dominant terms, let \((a_{n_k})\) be the subsequence consisting of all the dominant terms. Then \((a_{n_k})\) is decreasing.

Now suppose that there are only finitely many dominant terms. Then we can choose \(n_1\) such that \(a_{n_1}\) is after every dominant term. Then for every \(k \geq 2\), choose \(n_k\) such that \(n_k > n_{k-1}\) and \(a_{n_k} \geq a_{n_{k-1}}\). Why can we always do this? Because if we couldn’t, then there would be another dominant term; but we started after all the dominant terms. So we have constructed an increasing subsequence \((a_{n_k})\). □

Our main theorem now follows quickly.

**Theorem 8.5** (Bolzano-Weierstrass). Every bounded sequence has a convergent subsequence.

**Proof.** Let \((a_n)\) be a bounded sequence. By Proposition 8.4 there is a monotonic subsequence \((a_{n_k})\). This subsequence is clearly also bounded, so it is convergent. □

As an example, consider the sequence \((a_n)\) defined by \(a_n = \cos\left(\frac{\pi n}{3}\right)\). The terms of this sequence are

\[\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \ldots.\]

This is a bounded sequence, so it contains a convergent subsequence. For example, we can take the constant subsequences consisting of all terms of the form \(\frac{1}{2}\), or \(-\frac{1}{2}\), or \(-1\), or \(1\). Actually, this example illustrates an important idea: the values \(\frac{1}{2}, -\frac{1}{2}, 1, -1\) are called the limit points of the sequence \((a_n)\). We will discuss this more in the next section.

9. **lim sup** and **lim inf**

**Definition 9.1.** Let \((a_n)\) be a sequence. A limit point of \((a_n)\) is any real number (or symbol \(\pm \infty\)) that is the limit of some subsequence \((a_{n_k})\). Sometimes limit points are referred to as cluster points or accumulation points or condensation points, and your book calls them subsequential limits. I think that the term limit point is used most throughout mathematics, so we will use it here.

If \((a_n)\) is a convergent sequence, then since all subsequences converge to the same limit (say, \(L\)), the set of all limit points of \((a_n)\) is just \(\{L\}\). There are several examples of limit points on p. 73 of your book.
There are two particularly important limit points for any sequence, called the lim sup and the lim inf. These are defined as

\[
\limsup a_n = \lim_{N \to \infty} \sup \{a_n : n > N\}, \\
\liminf a_n = \lim_{N \to \infty} \inf \{a_n : n > N\}.
\]

We allow \(\limsup a_n = +\infty\) and \(\liminf a_n = -\infty\). Note that \(\limsup a_n\) is not necessarily the supremum of the set \(\{a_n : n \in \mathbb{N}\}\), but it is the largest value that infinitely many of the terms \(a_n\) get close to. The following proposition shows that for convergent sequences, the lim inf and the lim sup are equal. We will not prove it here, but a proof is given in Theorem 10.7 of your book.

**Proposition 9.2.** If \((a_n)\) is a convergent sequence then

\[
\liminf a_n = \lim_{n \to \infty} a_n = \limsup a_n.
\]

Now let’s see how limit points and lim sup/lim inf are related.

**Proposition 9.3.** Let \((a_n)\) be a sequence. There exists a monotonic subsequence whose limit is \(\limsup a_n\) and a monotonic sequence whose limit is \(\liminf a_n\).

**Proof.** If \((a_n)\) is unbounded above or below, then we just apply Proposition 8.2 part (2) or (3). So suppose that \((a_n)\) is bounded either above or below (actually let’s just pick one since the other case is similar). Suppose that \((a_n)\) is bounded above, then \(t = \limsup a_n\) is a finite number. Let \(\varepsilon > 0\). Then there exists \(N \geq 1\) such that (by definition of lim sup)

\[
|\sup \{a_n : n > N\} - t| < \varepsilon.
\]

But the sup is larger than or equal to \(t\), so we can say that

\[
\sup \{a_n : n > N\} < t + \varepsilon.
\]

In particular, \(a_n < t + \varepsilon\) for all \(n > N\) (since \(a_n \leq \sup \{a_n : n > N\}\)). I claim that the set

\[
\{n \in \mathbb{N} : |a_n - t| < \varepsilon\} \tag{9.1}
\]

is infinite. If not, then for some \(N'\) we have \(a_n \leq t - \varepsilon\) for all \(n \geq N'\). But this means that \(\limsup a_n < t\), a contradiction. So the set (9.1) is infinite; by Proposition 8.2 there is a monotonic subsequence of \((a_n)\) converging to \(t = \limsup a_n\).

As an immediate consequence of the previous proposition, we see that the set of limit points of any sequence \((a_n)\) contains at least one element (both the lim sup and lim inf are in the set, but they might be equal). Not only that, but it turns out that \(\limsup a_n\) is the supremum of the set of limit points (and similarly with the lim inf). See Theorem 11.8 of your book for a proof of this last fact. Also see p. 75 for some examples.

There are more properties of lim sup and lim inf in §12 of your book, but we will not need them here (and if we need them later, we will come back and prove them).

10. Series

When you saw sequences and series in your Calculus courses previously, you likely spent very little time talking about sequences and a lot of time talking about series. Maybe, because of this, you think of series as being “more complicated” than sequences. However, since we have taken such a long time carefully building up the theory of sequences and limits,
it turns out that we can prove many results about series very quickly. But first we need some rigorous definitions.

**Definition 10.1.** An infinite series is an object of the form

\[ \sum_{k=1}^{\infty} a_k \] (10.1)

where \((a_k)\) is a sequence of real numbers. To give a precise meaning to (10.1), for each \(n \geq 1\) let

\[ s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n \]

be the \(n\)th partial sum. We say that (10.1) *converges* if the sequence \((s_n)\) is a convergent sequence. In this case, if \(S = \lim s_n\), we write

\[ \sum_{k=1}^{\infty} a_k = S. \]

If the sequence of partial sums is not convergent, then we say the series *diverges*. Sometimes we can be more precise about how it diverges by saying that it *diverges to \(+\infty* if \(\lim s_n = +\infty\) and that it *diverges to \(-\infty* if \(\lim s_n = -\infty\). Lastly, if \(\sum |a_n|\) is convergent, we say the series \(\sum a_n\) is *absolutely convergent*.

Note: above I have written the series as starting with the index \(k = 1\). In fact we can start at any index \(k = m\) and modify the definition appropriately by defining \(s_n = a_m + a_{m+1} + \ldots + a_n\). Often it will not matter where the index begins, and so I will just write \(\sum a_n\) for a “generic” series.

By transferring what we know about sequences to series, we can prove some very useful results without too much effort. We’ll start by using the fact that a sequence converges if and only if it is Cauchy. We say that a series satisfies the *Cauchy criterion* if its sequence of partial sums is Cauchy, and we immediately get the following proposition.

**Proposition 10.2 (Cauchy criterion).** A series \(\sum a_n\) is convergent if and only if for each \(\varepsilon > 0\) there exists a number \(N \geq 1\) such that

\[ \left| \sum_{k=m}^{n} a_k \right| < \varepsilon \]

whenever \(n \geq m \geq N\).

From this we (almost) immediately get the following familiar corollary.

**Corollary 10.3.** If \(\sum a_n\) is convergent then \(\lim_{n \to \infty} a_n = 0\).

**Proof.** Apply the Cauchy criterion with \(m = n\) to get: for every \(\varepsilon > 0\) there exists a number \(N \geq 1\) such that \(|a_n| < \varepsilon\) whenever \(n \geq N\). \(\square\)

You may be more familiar with the previous corollary under the name “the divergence test.” Indeed, one (usually easy) way to check if a series diverges is to check whether its terms go to zero or not. If not, then the series diverges. Of course, even if the terms do go to zero, that’s not a guarantee that the series converges.
We can also do arithmetic on series (addition and multiplication by a constant) by applying the limit laws for sequences: if \( \sum a_n \) and \( \sum b_n \) are both convergent then
\[
\sum (a_n + b_n) = \sum a_n + \sum b_n
\]
and if \( \sum a_n \) is convergent and \( c \) is any constant then
\[
\sum ca_n = c \sum a_n.
\]

In order to determine the convergence of a given series, it is often useful to compare it to another “simpler” series which you already know converges or diverges. Two such “simple” series are the geometric series and the \( p \)-series.

A geometric series is a series of the form \( \sum ar^n \) for some constant \( a \) and some constant ratio \( r \). Since we can always pull out the constant \( a \), we will only need to think about geometric series of the form \( \sum r^n \).

**Proposition 10.4.** If \( |r| < 1 \) then the geometric series \( \sum r^n \) is absolutely convergent; furthermore,
\[
\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \tag{10.2}
\]
If \( |r| \geq 1 \) then \( \sum r^n \) diverges.

*Proof.* It is not too hard to show by induction (try it!) that for any \( r \neq 1 \) we have
\[
\sum_{n=0}^{m} r^n = \frac{1 - r^{m+1}}{1-r}.
\]
(If \( r = 1 \) then it is clear that the series diverges.) By the limit laws, it follows that the series converges if and only if \( \lim_{m \to \infty} r^m \) is finite, which is true if and only if \( |r| < 1 \). In that case, we have \( \lim_{m \to \infty} r^m = 0 \), which proves equation (10.2). Finally, if \( |r| < 1 \) then \( \sum |r^n| = \sum |r|^n \) is convergent, so \( \sum r^n \) is absolutely convergent.

A \( p \)-series is a series of the form \( \sum \frac{1}{n^p} \) for some fixed \( p \). We already showed that for \( p = 1 \), this series is divergent (this was way back when we first learned about Cauchy sequences; see Proposition 5.4). Since this case is so important and is used as a counterexample to many theorems in analysis, let’s give it its own proposition.

**Proposition 10.5.** The harmonic series \( \sum \frac{1}{n} \) is divergent.

What happens for other values of \( p \)? It turns out that the harmonic series is the dividing line between convergent \( p \)-series and divergent \( p \)-series, as the next proposition shows.

**Proposition 10.6.** If \( p \leq 1 \) then the \( p \)-series \( \sum \frac{1}{n^p} \) diverges. If \( p > 1 \) then it converges.

This is usually proved using the integral test, but since we haven’t covered integrals yet, we’re going to use a different method. We will, however, need to prove the comparison test first.

**Proposition 10.7** (The comparison test). Let \( \sum a_n \) be a series of nonnegative terms (\( a_n \geq 0 \) for all \( n \)).

1. If \( \sum a_n \) converges and \( |b_n| \leq a_n \) for all \( n \) then \( \sum b_n \) converges.
2. If \( \sum a_n = +\infty \) and \( b_n \geq a_n \) for all \( n \) then \( \sum b_n \) diverges.
Proof. (1) Suppose that \( \sum a_n \) converges. Then it satisfies the Cauchy criterion. Given \( \varepsilon > 0 \) there exists a number \( N \geq 1 \) such that

\[
\sum_{k=n}^{m} a_k = \left| \sum_{k=n}^{m} a_k \right| < \varepsilon
\]

for all \( n \geq m \geq N \) (note \( a_k \geq 0 \) for all \( k \)). So for \( n \geq m \geq N \) we have

\[
\left| \sum_{k=n}^{m} b_k \right| \leq \sum_{k=n}^{m} |b_k| \leq \sum_{k=n}^{m} a_k < \varepsilon,
\]

where we used the Triangle inequality in the first step. Thus \( \sum b_n \) satisfies the Cauchy criterion, and therefore it converges.

(2) Let \((s_n)\) be the sequence of partial sums for \( \sum a_n \) and let \((t_n)\) be the sequence of partial sums for \( \sum b_n \). Then \( t_n \geq s_n \) for all \( n \). Since \( \lim s_n = +\infty \) we have \( \lim t_n = +\infty \) too. \( \square \)

Before we prove Proposition 10.6, I should mention a quick corollary of the comparison test which you are likely familiar with.

Corollary 10.8. Absolutely convergent series are convergent.

Proof. Suppose that \( \sum b_n \) is absolutely convergent. Then \( \sum |b_n| \) converges. Since (trivially) \( |b_n| \leq |b_n| \), the comparison test says that \( \sum b_n \) converges. \( \square \)

Now we can prove Proposition 10.6.

Proof of Proposition 10.6. Suppose first that \( p \leq 1 \). Then \( \frac{1}{n^p} \geq \frac{1}{n} \) for all \( n \geq 1 \). By the comparison test, \( \sum \frac{1}{n^p} \) diverges.

Now suppose that \( p > 1 \). Let \((s_n(p))\) denote the sequence of partial sums, i.e.

\[
s_n(p) = \sum_{k=1}^{n} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}.
\]

Consider the 2n-th partial sum and split into even and odd terms:

\[
s_{2n}(p) = 1 + \left( \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \cdots + \frac{1}{(2n)^p} \right) + \left( \frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \cdots + \frac{1}{(2n-1)^p} \right).
\]

Since \( 3^p > 2^p \) and \( 5^p > 4^p \) and \( 7^p > 6^p \), etc. we have

\[
s_{2n}(p) < 1 + 2 \left( \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \cdots + \frac{1}{(2n)^p} \right) = 1 + \frac{2}{2^p} \left( \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} \right) = 1 + \frac{2}{2^p} s_n(p).
\]

Since \((s_n(p))\) is an increasing sequence (we are adding positive terms every time) we have

\[
s_n(p) < 1 + \frac{2}{2^p} s_n(p).
\]

Solving this inequality for \( s_n(p) \) yields

\[
s_n(p) < \left( 1 - \frac{2}{2^p} \right)^{-1}.
\]
Thus \((s_n(p))\) is an increasing sequence which is bounded above. Therefore it converges. □

We will only talk about one more convergence test before we finish our discussion of series. You probably remember several other convergence tests from previous calculus courses; these are certainly interesting, but won’t help us much in our ultimate goal for this course. If you are interested in learning more about infinite series, I suggest you take a course in complex analysis or analytic number theory.

**Proposition 10.9** (Alternating series test). If \((a_n)\) is a decreasing sequence \((a_{n+1} \leq a_n)\) of nonnegative real numbers \((a_n \geq 0)\) and \(\lim a_n = 0\) then the alternating series \(\sum (-1)^{n+1}a_n\) converges.

**Proof.** Let \((s_n)\) denote the sequence of partial sums. Let \((e_n)\) be the subsequence of even-indexed terms and \((o_n)\) the subsequence of odd-indexed terms. Then \(o_n = s_{2n-1}\) and \(e_k = s_{2n}\):

\[(s_n) = o_1, e_1, o_2, e_2, o_3, e_3, \ldots .\]

The even-indexed terms are increasing since

\[e_{n+1} - e_n = s_{2n+2} - s_{2n} = -a_{2n+2} + a_{2n+1} \geq 0.\]

Similarly, the odd-indexed terms are decreasing since

\[o_{n+1} - o_n = s_{2n+1} - s_{2n-1} = a_{2n+1} - a_{2n} \leq 0.\]

I claim that every even-indexed term is less than or equal to every odd-indexed term:

\[e_m \leq o_n \quad \text{for all } m, n \in \mathbb{N}.\]

First, \(e_n \leq o_{n+1} \leq o_n\) because \(o_{n+1} - e_n = s_{2n+1} - s_{2n} = a_{2n+1} \geq 0\) and \((o_n)\) is decreasing. Now, if \(m \leq n\), then because \((e_n)\) is increasing, \(e_m \leq e_n \leq o_n\). On the other hand, if \(m \geq n\) then \(o_n \geq o_m \geq e_m\). To summarize,

\[e_1 \leq e_2 \leq e_3 \leq \cdots \leq o_3 \leq o_2 \leq o_1.\]

Since \((e_n)\) is increasing and bounded above, it converges, say to \(L_e\). Similarly, \((o_n)\) converges, say to \(L_o\). But

\[L_o - L_e = \lim_{n \to \infty} o_{n+1} - \lim_{n \to \infty} e_n = \lim_{n \to \infty} (o_{n+1} - e_n) = \lim_{n \to \infty} a_{2n+1} = 0.\]

Thus \(L_o = L_e\). It follows that \((s_n)\) converges. □

As an example, the alternating harmonic series

\[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\]

converges since \(\left(\frac{1}{n}\right)\) is a decreasing sequence and \(\lim \frac{1}{n} = 0\).

**References**

