MATH 131A HOMEWORK 9

(1) Using the limit definition of the derivative, prove that the function

\[ f(x) = \begin{cases} 
  x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0, \\
  0 & \text{if } x = 0, 
\end{cases} \]

is differentiable at \( x = 0 \). Hint: you should first guess what \( f'(0) \) is.

(2) Suppose that \( f \) is defined on an open interval \( I \) containing \( a \). Prove that \( f'(a) \) exists if and only if there is a function \( \varepsilon : I \to \mathbb{R} \) such that

\[ f(x) - f(a) = (x - a)[f'(a) + \varepsilon(x)] \quad \text{and} \quad \lim_{x \to a} \varepsilon(x) = 0. \]

(For the \( \Leftarrow \) direction, you don’t know that \( f'(a) \) exists so this problem is saying: Assume that there is a function \( \varepsilon : I \to \mathbb{R} \) and a number \( m \) such that \( \lim_{x \to a} \varepsilon(x) = 0 \) and \( f(x) - f(a) = (x - a)[m + \varepsilon(x)] \). Show that \( f'(a) \) exists and that it equals \( m \).)

(3) Using the fact that \((\cos x)' = -\sin x\), prove that

\[ |\cos x - \cos y| \leq |x - y| \]

for all \( x, y \in \mathbb{R} \). Hint: use the MVT.

(4) Suppose that \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( |f(x) - f(y)| \leq (x - y)^2 \) for all \( x, y \in \mathbb{R} \). Prove that \( f \) is a constant function. Hint: show that \( f'(y) = 0 \) for all \( y \in \mathbb{R} \) by using the squeeze theorem and the definition of the derivative.

(5) Suppose that \( f \) and \( g \) are differentiable on \( \mathbb{R} \) and that \( f(0) = g(0) \) and \( f'(x) \leq g'(x) \) for all \( x \geq 0 \). Prove that \( f(x) \leq g(x) \) for all \( x \geq 0 \). Hint: consider \( h = g - f \).

EXTRA PRACTICE (DO NOT TURN THESE IN)

(1) Exercises 28.3, 28.7
(2) Exercises 29.1, 29.4, 29.7, 29.10, 29.11, 29.14, 29.17

THIS PROBLEM WON’T BE GRADED BUT I THINK IT’S COOL

Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable on \( \mathbb{R} \) and suppose that

\[ s = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1. \]

We will prove that there exists a point \( a \in \mathbb{R} \) such that \( f(a) = a \) (i.e. \( f \) has a fixed point).

(i) Pick any point \( a_1 \in \mathbb{R} \). For each \( n \geq 1 \) define \( a_{n+1} = f(a_n) \). So

\[ a_2 = f(a_1), \quad a_3 = f(a_2) = f(f(a_1)), \ldots. \]

Prove that \( (a_n) \) is a Cauchy sequence. Hint: use the MVT to show that

\[ |a_{n+1} - a_n| \leq s|a_n - a_{n-1}|, \]

where \( s \) is the sup defined above.

(ii) Prove that there exists some \( a \in \mathbb{R} \) such that \( f(a) = a \). Hint: since \( f \) is differentiable on \( \mathbb{R} \), it is continuous on \( \mathbb{R} \).