(1) (a) Suppose that \( f \) is uniformly continuous on a bounded set \( S \). Prove that \( f \) is bounded on \( S \) (i.e. there exists a number \( M \) such that \( |f(x)| \leq M \) for all \( x \in S \)).

*Hint:* Suppose \( f \) isn't bounded, then use Bolzano-Weierstrass to construct a Cauchy sequence \( (a_n) \) such that \( \lim f(a_n) = \pm \infty \), then derive a contradiction.

(b) Use (a) to prove that \( f(x) = 1/x^2 \) is not uniformly continuous on \((0, 1)\).

(2) A function \( f : X \to \mathbb{R} \) is called *Lipschitz* if there exists a number \( M > 0 \) such that

\[
\left| \frac{f(x) - f(y)}{x - y} \right| \leq M \quad \text{for all } x, y \in X.
\]

Show that if \( f : X \to \mathbb{R} \) is Lipschitz then it is uniformly continuous on \( X \).

(3) Suppose that \( f \) is continuous on \([0, \infty)\). Prove that if \( f \) is uniformly continuous on \([k, \infty)\) for some \( k \geq 0 \) then \( f \) is uniformly continuous on \([0, \infty)\).

(4) Let \( f(x) = \sqrt{x} \). Prove that \( f \) is uniformly continuous on \([0, \infty)\).

*Hint:* Prove that \( f \) is uniformly continuous on \([1, \infty)\) and use (3).

(5) Suppose that \( \lim_{x \to a^+} f(x) = \lim_{x \to a^+} h(x) = L \) and that

\[
f(x) \leq g(x) \leq h(x)
\]

for all \( x \) in \((a, b)\) for some \( b \). Use the \( \varepsilon-\delta \) definition to prove that \( \lim_{x \to a^+} g(x) = L \).

*Hint:* See Exam 1.

(6) Define \( f : \mathbb{R} \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
0 & \text{if } x \in \mathbb{Q}, \\
x^2 & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]

This function is similar to a function we’ve seen before; it is continuous at \( x = 0 \) and discontinuous everywhere else. Prove that \( f \) is differentiable at \( x = 0 \).

*So you can work on this problem before we cover derivatives:* We say \( f \) is differentiable at \( a \) if the limit

\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

exists. *Hint:* I personally find the \( \varepsilon-\delta \) definition easier for this problem.
Extra practice (do not turn these in)

(1) Exercises 19.1, 19.2, 19.5, 19.9, 19.10

(2) Exercises 20.1, 20.2, 20.4, 20.11, 20.16