(1) Prove Lemma 4.2 from the course notes: For all $x \in \mathbb{Q}$ we have $-|x| \leq x \leq |x|$. 

(2) Prove Lemma 4.3 from the course notes: For all $x, y \in \mathbb{Q}$ we have $|x| \leq |y|$ if and only if $-|y| \leq x \leq |y|$. Note: this requires you to prove two things: first, the left statement implies the right statement, then the other way.

(3) Prove that the sequence $(a_n)$ defined by $a_n = \frac{n}{n+1}$ is Cauchy.

(4) Prove that the sequence $(a_n)$ defined by $a_n = \log n$ is not Cauchy. (For this exercise you will need to travel to the future where we know what $\log n$ means. You are welcome to use any properties of log that you know.)

(5) Prove Lemma 5.3 from the course notes: If $x, y, z$ are real numbers then (a) $x = x$, (b) $x = y$ implies $y = x$, and (c) $x = y$ and $y = z$ implies $x = z$.

(6) Prove Lemma 5.17 from the course notes: If $x = \lim a_n$ then $|x| = \lim |a_n|$.

(7) The real number which has decimal expansion .999\ldots is represented by the sequence $(a_n)$ where $a_n = 1 - 10^{-n}$. That is, .999\ldots = \lim a_n. Prove that .999\ldots = 1.

(8) (a) Suppose $x$ is a positive real number. Prove that there exists a natural number $N$ such that $0 < 1/N < x$. Hint: $x^{-1}$ is represented by a Cauchy sequence.

(b) Suppose that $x, y$ are real numbers such that $y - x > 1$. Prove that there exists an integer $m$ such that $x < m < y$. Hint: Suppose not and find a contradiction.

(c) Given any two real numbers $x$ and $y$ with $x < y$, prove that there is a rational number $\frac{m}{N}$ such that $x < \frac{m}{N} < y$. Hint: use (a) and (b).