An infinity of Ramanujan graphs
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Outline

1. Introduction and setup
2. Prerequisites
3. Proof of the main theorem
We will prove the following breakthrough result of the paper *Interlacing Families I: Bipartite Ramanujan graphs of all degrees*:

**Theorem (Marcus-Spielman-Srivastava '13)**

There exists an infinite family of $d$-regular Ramanujan graphs for every $d \geq 3$.

**Note:** Prior to this paper, the existence of such a family was only known for $d$ of the form $1 + q$, where $q$ is a prime power.

**Definition**

Let $G$ be a graph with adjacency matrix $A$. If $G$ is $d$-regular and

$$\sigma(G) := \sigma(A) \subseteq \{\pm d\} \cup [-2\sqrt{d - 1}, 2\sqrt{d - 1}],$$

then we call $G$ a **Ramanujan graph**.

**Note:** All $d$-regular graphs $G$ satisfy $d \in \sigma(G)$. 
Moreover, if $G$ is bipartite then $\sigma(G) = -\sigma(G)$. (Indeed, the adjacency matrix of such a graph is of the form

$$
\begin{pmatrix}
0 & A \\
A^T & 0
\end{pmatrix}.
$$

Thus if $(x, y)^T$ is an eigenvector with eigenvalue $\lambda$, then $(x, -y)^T$ is an eigenvector with eigenvalue $-\lambda$.)

**Note:** The range of eigenvalues in the definition of a Ramanujan graph cannot be shrunk if one hopes to construct an infinite family of such graphs. Indeed it was proved by Alon and Boppana in the mid-80s that for every $\epsilon > 0$, every sufficiently large $d$-regular graph has an eigenvalue $\lambda$ such that $2\sqrt{d-1} - \epsilon \leq |\lambda| < d$. 

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An infinity of Ramanujan graphs
Let $G = (V, E)$ be a graph. A 2-lift of $G$ is a graph

1. that has two vertices $f(v) = \{v_0, v_1\}$ (called the fiber over $v$) for each vertex $v$ of $V$, and

2. that, for each edge $(u, v) \in E$ with $f(v) = \{v_0, v_1\}$ and $f(u) = \{u_0, u_1\}$, contains exactly one of the following two pairs of edges:

   $\{(u_0, v_0), (u_1, v_1)\}$ or $\{(u_0, v_1), (u_1, v_0)\}$

\(\ast\)

**Note:** Given a graph $G$, the 2-lifts of $G$ are in 1–1 correspondence with so-called **signings** of (the edges of) $G$, i.e., maps $E \rightarrow \{\pm 1\}$: The sign of an edge corresponds to the choice \(\ast\).
A lemma of Bilu and Linial

We associate to each signing $s$ the signed adjacency matrix $A_s$.

**Lemma (Bilu-Linial ’06)**

Let $s$ be a signing of a graph $G = (V, E)$. The spectrum of the 2-lift corresponding to $s$ is the union of those of $A$ and $A_s$, counting multiplicities.

**Proof.** The adjacency matrix of the 2-lift $\tilde{G}$ corresponding to $s$ is of the form

$$\tilde{A} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix},$$

where $A_1$ is the adjacency matrix of $(V, s^{-1}(1))$ and $A_2$ is the adjacency matrix of $(V, s^{-1}(-1))$. (Just order the vertices of $\tilde{G}$ as $v_1^1, \ldots, v_n^1, v_1^2, \ldots, v_n^2$, where $\{v_j^1, v_j^2\}$ is the fiber over $v_j \in V$.)
Moreover, $A = A_1 + A_2$ and $A_s = A_1 - A_2$, because $s^{-1}(1)$ and $s^{-1}(-1)$ are disjoint sets of edges. Now, if $v$ is an eigenvector for $A$ corresponding to an eigenvalue $\lambda$, then

$$\tilde{A}(v, v)^T = (Av, Av)^T = \lambda(v, v)^T$$

and, if $v$ is an eigenvector for $A_s$ corresponding to an eigenvalue $\mu$, then

$$\tilde{A}(v, -v)^T = (A_s v, -A_s v)^T = \mu(v, -v)^T.$$ 

By dimension considerations, we get in this way all of the eigenvalues of $\tilde{A}$ with the desired multiplicities. (Note that we are dealing with diagonalizable matrices!) □
Matching polynomials

Given a graph $G$, denote by $m_i$ the number of matchings in $G$ with $i$ edges, i.e., collections of $i$ non-adjacent edges in $G$. Denote the number of vertices by $n$.

**Definition (Heilmann-Lieb ’72)**

The matching polynomial $\mu_G$ is the polynomial

$$\mu_G(x) = \sum_{i \geq 0} (-1)^i m_i x^{n-2i}.$$ 

The authors use the following theorem, whose proof we skip:

**Theorem (Heilmann-Lieb ’72)**

*Let $G$ be a graph. Then $\mu_G(x)$ has only real roots. Moreover, if $G$ has maximum degree $d$, then all roots of $\mu_G(x)$ have absolute value at most $2\sqrt{d - 1}$.***
Overview

Let $G$ be a $d$-regular bipartite Ramanujan graph with adjacency matrix $A$.

1° We will first prove that

$$E_s[f_s(x)] = \mu_G(x),$$

where $f_s(x)$ is the characteristic polynomial of $A_s$ and $s$ varies through all possible signings of $G$.

2° We next prove that $\{f_s(x)\}_s$ is an interlacing family. It will follow that there exists a signing $s$ such that every root of $f_s(x)$ is at most the largest root of $E_s[f_s(x)]$.

3° By the result of Heilmann-Lieb and step 1°, there exists a 2-lift of $G$ that is also a $d$-regular bipartite Ramanujan graph. Finally, if we take appropriate successive 2-lifts of a complete $d$-regular bipartite graph, we get an infinite family of $d$-regular (bipartite) Ramanujan graphs.
Step 1: Expected characteristic polynomial

We are given a finite graph $G = (V, E)$ and number the vertices $1, \ldots, n$ and the edges $1, \ldots, m$. We also identify edges with pairs $(i, j)$ of vertices.

**Theorem**

$$E_{s \in \{\pm 1\}^m}[f_s(x)] = \mu_G(x).$$

**Proof.** The theorem will follow from three simple observations:

$1^\circ$. The number of $k$-matchings (i.e., matchings with $k$ edges) equals the number of permutations of the vertices that are products of $k$ disjoint 2-cycles, where each 2-cycle corresponds to an edge.
Step 1: Expected characteristic polynomial, cont.

2°. In the expression

\[ E_{s \in \{\pm 1\}^m}[f_s(x)] = \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) E_{s \in \{\pm 1\}^m} \left[ \prod_{i=1}^{n} (xI - A_s)_{i,\sigma(i)} \right], \]

we need only sum over products of disjoint 2-cycles. (Indeed, let \( \sigma \in \Sigma_n \) contain a cycle of length at least 3, say mapping \( i_1 \) to \( i_2 \) and \( i_2 \) to \( i_3 \). Then the product in the expression above contains the factors \( s_{i_1,i_2} \) and \( s_{i_2,i_3} \) exactly once. Independence implies

\[ E_{s \in \{\pm 1\}^m} \left[ \prod_{i=1}^{n} (xI - A_s)_{i,\sigma(i)} \right] = 0. ) \]
Step 1: Expected characteristic polynomial, cont.

3°. Given $k \in \mathbb{N}$ and a permutation $\sigma$ that is the product of $k$ disjoint 2-cycles, we have, independently of $s \in \{\pm 1\}^m$, that

$$\prod_{i}(A_s)_{i,\sigma(i)} = \begin{cases} 1 & \text{if all of the 2-cycles correspond to edges} \\ 0 & \text{otherwise} \end{cases},$$

where the product runs over all $i$ that are not fixed by $\sigma$.

Finally, with $m_k = \#\{k\text{-matchings}\}$, these observations imply that

$$\mathbb{E}_{s \in \{\pm 1\}^m}[f_s(x)] = \sum_k (-1)^k x^{n-2k} \sum_{\sigma} \mathbb{E}_{s \in \{\pm 1\}^m} \left[ \prod_{i}(A_s)_{i,\sigma(i)} \right]$$

$$= \sum_k (-1)^k x^{n-2k} m_k = \mu_G(x). \qed$$
Method of interlacing polynomials

We will now describe the general technique that allowed the authors to prove their result.

**Definition**

We say that a polynomial $g(x) = a \prod_{i=1}^{n-1} (x - \alpha_i)$ interlaces a polynomial $f(x) = b \prod_{i=1}^{n} (x - \beta_i)$ if $\alpha_i, \beta_i, a, b \in \mathbb{R}$ and

\[
\beta_1 \leq \alpha_1 \leq \beta_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n.
\]

**Lemma**

Let $f_1, \ldots, f_k \in \mathbb{R}[x]$ be polynomials of the same degree with positive leading coefficients. If they have a common interlacing, then there exists an $i$ such that the largest root of $f_i$ is at most the largest (real) root of $\sum_{j=1}^{k} f_j$. 
Method of interlacing polynomials, cont.

**Proof of the lemma.** Let $g$ be a polynomial that interlaces each $f_i$, and denote the largest root of $g$ by $\alpha$.

As $f_j$ has positive leading coefficient and any root of $f_j$ strictly to the right of $\alpha$ has multiplicity 1, $f_j(\alpha) \leq 0$ for all $j$. Thus

$$
\sum_{j=1}^{k} f_j(\alpha) \leq 0.
$$

It follows that the largest real root $\beta$ of $\sum_{j=1}^{k} f_j$ satisfies $\beta \geq \alpha$. Hence $f_i(\beta) \geq 0$ for some $i$ so that the largest root of $f_i$ lies in $[\alpha, \beta]$, because $f_i$ has exactly one root that is at least $\alpha$. □
Method of interlacing polynomials, cont.

**Definition**

Let $S_1, \ldots, S_m$ be finite sets and let, for every $(s_1, \ldots, s_m) \in S_1 \times \cdots \times S_m$, $f_{s_1,\ldots,s_m}(x)$ be a real-rooted polynomial of degree $n$ with positive leading coefficient. Given $(s_1, \ldots, s_k) \in S_1 \times \cdots \times S_k$, $s_{k+1}, \ldots, s_m$.

\[
f_{s_1,\ldots,s_k} := \sum_{(s_{k+1},\ldots,s_m)\in S_{k+1} \times \cdots \times S_m} f_{s_1,\ldots,s_m}.
\]

\[
f_{\emptyset} := \sum_{(s_1,\ldots,s_m)\in S_1 \times \cdots \times S_m} f_{s_1,\ldots,s_m}.
\]

We say that the family \( \{f_{s_1,\ldots,s_m}(x)\}_{(s_1,\ldots,s_m)\in S_1 \times \cdots \times S_m} \) is interlacing if, given $0 \leq k \leq m - 1$ and $(s_1, \ldots, s_k) \in S_1 \times \cdots \times S_k$, the polynomials \( \{f_{s_1,\ldots,s_k,t}(x)\}_{t\in S_{k+1}} \) have a common interlacing.
The following theorem follows immediately from the lemma.

**Theorem**

If \( \{f_{s_1,\ldots,s_m}(x)\}_{(s_1,\ldots,s_m)\in S_1\times\cdots\times S_m} \) is an interlacing family, then there exists \((s_1,\ldots,s_m)\in S_1\times\cdots\times S_m\) such that the largest root of \(f_{s_1,\ldots,s_m}(x)\) is at most the largest (real) root of \(f_\emptyset\).

In the proof that the family of characteristic polynomials \(f_s(x)\) is an interlacing family, we will use the following lemma (cf. Fisk ’08).

**Lemma**

Let \(f(x)\) and \(g(x)\) be (monic) real-rooted polynomials of degree \(n\) such that, for all \(\lambda \in [0,1]\), the polynomial \(\lambda f + (1-\lambda)g\) has \(n\) real roots. Then \(f(x)\) and \(g(x)\) have a common interlacing.
Real-rooted polynomials: Towards Step 2 of the proof

We will next prove the following theorem, from which the main result will follow easily.

**Theorem**

Let $p_1, \ldots, p_m \in [0, 1]$ be given. Then the polynomial

$$
\sum_{s \in \{\pm 1\}^m} \left( \prod_{i: s_i = 1} p_i \right) \left( \prod_{i: s_i = -1} (1 - p_i) \right) f_s(x)
$$

is real-rooted.

**Note:** A polynomial over $\mathbb{R}$ is said to be real-rooted if it has no non-real roots.
Real-rooted polynomials, cont.

We will use the notion of real stability:

**Definition**

Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be given. We say that $f$ is **stable** if $f$ has no zeros in $\mathbb{H}^n$, where $\mathbb{H} \subset \mathbb{C}$ is the open upper half plane. We say that $f$ is **real stable** if $f$ is stable and has real coefficients.

**Remarks:**

1° If $f$ and $g$ are (real) stable, then so is $f \otimes g$.

2° By Hurwitz’ theorem from multivariate complex analysis, if $f \in \mathbb{C}[z_1, \ldots, z_n]$ is (real) stable and $c \in \mathbb{R}$, then $f(z_1, \ldots, z_{n-1}, c) \in \mathbb{C}[z_1, \ldots, z_{n-1}]$ is (real) stable as well.

3° A univariate real polynomial is real stable if and only if it is real-rooted.
We will need the following result:

**Proposition (Lieb-Sokal ’81)**

Assume that \( f(z_1, \ldots, z_n) + yg(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n, y] \) is real stable and has degree at most 1 in \( z_j \). Then \( f(z_1, \ldots, z_n) - \partial_j g(z_1, \ldots, z_n) \) is real stable.

**Note:** In order to prove this, it suffices to prove the statement with “real stable” replaced by “stable”. We follow Wagner ’11.

**Lemma**

Let \( f, g \in \mathbb{C}[z_1, \ldots, z_m] \) and assume that \( f \) is stable. Then \( g(z) + yf(z) \) is stable if and only if \( \Im(g(z)/f(z)) \geq 0 \) for all \( z \in \mathbb{H}^m \).
Real-rooted polynomials, cont.

Proof of the lemma. Assume that $g(z) + yf(z)$ is stable and let $z \in \mathbb{H}^m$ be given. Then $f(z) \neq 0$ and we can find $z_0 \in \mathbb{C}$ such that $g(z) + z_0 f(z) = 0$. As $g(z) + yf(z)$ is stable,

$$\Im \left( \frac{g(z)}{f(z)} \right) = -\Im(z_0) \geq 0.$$  

Conversely, if $\Im(g(z)/f(z)) \geq 0$ for all $z \in \mathbb{H}^m$ and $g(z) + z_0 f(z) = 0$ for some $z_0 \in \mathbb{C}$ and $z \in \mathbb{H}^m$, then

$$\Im(z_0) = -\Im \left( \frac{g(z)}{f(z)} \right) \leq 0.$$  

Thus $g(z) + yf(z)$ has no zeros in $\mathbb{H}^{m+1}$, i.e., is stable. \qed
Real-rooted polynomials, cont.

**Proof of the proposition.** Assume that $f(z) + yg(z)$ is stable and has degree at most 1 in $z_1$. Writing $f(z) = \sum a_iz^i$, we get

$$
yf(z_1 - y^{-1}, z_2, \ldots, z_m) = \sum_{i_1=0} a_1yz^i + \sum_{i_1=1} a_1(yz^i - z_2^{i_2} \cdots z_m^{i_m}) = -\partial_1 f(z) + yf(z).
$$

As $-y^{-1} \in \mathbb{H}$ whenever $y \in \mathbb{H}$, this polynomial is stable. By the lemma,

$$\Im \left( \frac{g(z) - \partial_1 f(z)}{f(z)} \right) = \Im \left( \frac{g(z)}{f(z)} \right) + \Im \left( \frac{-\partial_1 f(z)}{f(z)} \right) \geq 0$$

for all $z \in \mathbb{H}^m$. Hence $g - \partial_1 f + yf$ is stable. The proof is completed by setting $y = 0$. □
Real-rooted polynomials, cont.

Corollary

Assume \( f(z_1, \ldots, z_n) \) and \( t(w_1, \ldots, w_m) \) are (real) stable, where \( m \leq n \) and both polynomials have degree at most 1 in the variables \( z_j, w_j \) for \( j = 1, \ldots, m \). Then the polynomial

\[
t (−∂_1, \ldots, −∂_m) f(z_1, \ldots, z_n)
\]

is (real) stable.

Proof. If \( m = 1 \) then \( t(w_1) = \alpha + \beta w_1 \) so that \( \alpha f(z) + y \beta f(z) = t(y)f(z) \) is stable. Thus \( t(−∂_1)f(z) = \alpha f(z) − \beta ∂_1 f(z) \) is stable. If \( 2 \leq m \leq n \), then \( t(−∂_1, \ldots, −∂_{m−1}, y)f(z) = t(−∂_1, \ldots, −∂_{m−1}, 0)f(z) + y(∂_m t)(−∂_1, \ldots, −∂_{m−1}) f(z) \) is stable for every \( y \in \mathbb{H} \) (by induction). Vary \( y \) to finish proof. \( \square \)
Real-rooted polynomials, cont.

**Proposition (Borcea-Brändén ’08)**

Let $A_1, \ldots, A_m$ be positive semidefinite matrices. Then $\det[z_1 A_1 + \cdots + z_m A_m]$ is a real stable polynomial.

**Proof.** By Hurwitz’ theorem, we may assume that $A_1$ is invertible. Moreover, it is clear that the polynomial has real coefficients.

Assume for contradiction that the polynomial has a zero $(z_1, \ldots, z_m) \in \mathbb{H}^m$. Write $z_j = \beta_j + i\lambda_j$, where $\lambda_j > 0$ and $\beta_j \in \mathbb{R}$. Put $P = \lambda_1 A_1 + \cdots + \lambda_m A_m$ and $Q = \beta_1 A_1 + \cdots + \beta_m A_m$. Then

$$0 = \det(Q + iP) = \det(P) \det(P^{-1/2} QP^{-1/2} + il),$$

i.e., $-i \in \sigma(P^{-1/2} QP^{-1/2})$, contradicting the fact that $Q$ is self-adjoint. □
Let $a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathbb{R}^n$ and $p_1, \ldots, p_m \in [0, 1]$ be given. Then the polynomial $P(x) =$

$$
\sum_{S \subseteq [m]} \left( \prod_{i \in S} p_i \right) \left( \prod_{i \notin S} (1 - p_i) \right) \det \left[ xI + \sum_{i \in S} a_i a_i^T + \sum_{i \notin S} b_i b_i^T \right]
$$

is real-rooted.

**Proof.** Define a polynomial $Q$ of $2m + 1$ variables by

$$Q(x, u_1, \ldots, u_m, v_1, \ldots, v_m) = \det \left[ xI + \sum_{i \in S} u_i a_i a_i^T + \sum_{i \notin S} v_i b_i b_i^T \right].$$

It follows from the previous proposition that $Q$ is real stable.
Real-rooted polynomials, cont.

Note that the degree of $Q$ in each of the variables $u_j, v_j$ is at most 1. (It suffices to show this for the variable $u_1$. As $a_1a_1^T$ is a rank 1 symmetric matrix, it follows that, as a function of $u_1$ only, $Q$ is of the form

$$\det[u_1a_1a_1^T + B] = \det[u_1E_{11}(\alpha) + B'],$$

which is a polynomial of degree 1 by definition of $\det$.)

Put $T_i = t_i(-\partial_u^i, -\partial_v^i)$, where $t_i(u_i, v_i) = 1 - p_i u_i - (1 - p_i) v_i$ is real stable. Then the corollary above implies that the operator

$$\prod_{i=1}^m T_i = \sum_{S, T \subseteq [m]} \left( \prod_{i \in S} p_i \right) \left( \prod_{i \in T} (1 - p_i) \right) \partial_u^S \partial_v^T$$

preserves the real stability of $Q$. 
Finally, as substitution of real numbers preserves real stability, we get that the univariate polynomial

\[ \hat{P}(x) = \left( \sum_{S, T \subseteq [m]} \left( \prod_{i \in S} p_i \right) \left( \prod_{i \in T} (1 - p_i) \right) \partial_u^S \partial_v^T Q(x, u, v) \right) \bigg|_{u_1 = \cdots = u_m = 0}^{v_1 = \cdots = v_m = 0} \]

is real stable, hence real-rooted.

We claim that \( P = \hat{P} \). We will show this by comparing coefficients. For this, we will need the formula

\[ \partial_u \det(u a a^T + B) = \det(aa^T + B), \]

valid for any \( a \in \mathbb{R}^m \) and \( B \in \mathbb{M}_m \).
Real-rooted polynomials, cont.

As the only terms that appear in $\hat{P}(x)$ are the ones that contain exactly the variables with respect to which we are differentiating, we get that the coefficient of $x^{d-k}$ in $\hat{P}(x)$ is

$$\sum_{|R|+|W|=k \atop R \cap W = \emptyset} \left( \prod_{i \in R} p_i \right) \left( \prod_{i \in W} (1 - p_i) \right) \det \left[ \sum_{i \in R} a_i a_i^T + \sum_{i \in W} b_i b_i^T \right].$$

On the other hand, for $S \subseteq [m]$, “Cauchy-Binet formula” implies

$$\det \left[ xI + \sum_{i \in S} a_i a_i^T + \sum_{i \notin S} b_i b_i^T \right]$$

$$= \sum_{k=0}^d x^{d-k} \sum_{|T|=k} \det \left[ \sum_{i \in T \cap S} a_i a_i^T + \sum_{i \in T \setminus S} b_i b_i^T \right].$$
Thus

\[ P(x) = \sum_{k=0}^{d} x^{d-k} \sum_{S \subseteq [m]} \left( \prod_{i \in S} p_i \right) \left( \prod_{i \notin S} (1 - p_i) \right) \]

\[ \times \sum_{|T|=k} \det \left[ \sum_{i \in T \cap S} a_i a_i^T + \sum_{i \in T \setminus S} b_i b_i^T \right] . \]

Note also that, e.g. by induction on \([m] \setminus T|,\]

\[ \sum_{Q \subseteq [m] \setminus T} \prod_{i \in Q} p_i \prod_{i \in ([m] \setminus T) \setminus Q} (1 - p_i) = 1. \]
Real-rooted polynomials, cont.

Now we can write the coefficient of $x^{d-k}$ in $P(x)$ as

$$\sum_{|T|=k} \sum_{R \subseteq T} \det \left[ \sum_{i \in R} a_i a_i^T + \sum_{i \in T \setminus R} b_i b_i^T \right]$$

$$\times \sum_{Q \subseteq [m] \setminus T} \left( \prod_{i \in Q \cup R} p_i \right) \left( \prod_{i \in ([m] \setminus Q) \setminus R} (1 - p_i) \right)$$

$$= \sum_{|T|=k} \sum_{R \subseteq T} \det \left[ \sum_{i \in R} a_i a_i^T + \sum_{i \in T \setminus R} b_i b_i^T \right] \left( \prod_{i \in R} p_i \right) \left( \prod_{i \in T \setminus R} (1 - p_i) \right)$$

$$= \sum_{|R| + |W|=k} \left( \prod_{i \in R} p_i \right) \left( \prod_{i \in W} (1 - p_i) \right) \det \left[ \sum_{i \in R} a_i a_i^T + \sum_{i \in W} b_i b_i^T \right].$$

□
Real-rooted polynomials, cont.

**Theorem**

Let \( p_1, \ldots, p_m \in [0, 1] \) be given. Then the polynomial

\[
\sum_{s \in \{\pm 1\}^m} \left( \prod_{i : s_i = 1} p_i \right) \left( \prod_{i : s_i = -1} (1 - p_i) \right) f_s(x)
\]

is real-rooted.

**Proof.** Let \( d \) be the maximum degree of \( G \). We will show that the polynomial

\[
Q(x) = \sum_{s \in \{\pm 1\}^m} \left( \prod_{i : s_i = 1} p_i \right) \left( \prod_{i : s_i = -1} (1 - p_i) \right) \det(xl + dl - A_s)
\]

has only real roots.
Denoting by $e_u$ the standard basis vector indexed by the vertex $u$, 

$$dI - A_s = \sum_{(u,v) \in E} (e_u - e_v)(e_u - e_v)^T + \sum_{(u,v) \in E} (e_u + e_v)(e_u + e_v)^T + D,$$

where $D = \sum_{u \in V} d_u e_u e_u^T$ is the diagonal matrix with entries $d_u = d - \deg(u)$. Setting $a_{uv} = e_u - e_v$ and $b_{uv} = e_u + e_v$,

$$Q(x) = \sum_{s \in \{\pm 1\}^m} \left( \prod_{i : s_i = 1} p_i \right) \left( \prod_{i : s_i = -1} (1 - p_i) \right) \times \det \left[ xI + \sum_{u \in V} d_u e_u e_u^T + \sum_{(u,v) \in E} a_{uv} a_{uv}^T + \sum_{(u,v) \in E} b_{uv} b_{uv}^T \right].$$
Identify each $s \in \{\pm 1\}^m$ with the set $\{i \in [m] : s_i = 1\}$ and put $p_u = 1$ for each $u \in V$. Then it is clear that the aforementioned polynomial is of the same form as the one in the statement of the previous theorem (restated below). Thus it is real-rooted.  

**Theorem**

Let $a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathbb{R}^n$ and $p_1, \ldots, p_m \in [0, 1]$ be given. Then the polynomial $P(x) =$

$$
\sum_{S \subseteq [m]} \left( \prod_{i \in S} p_i \right) \left( \prod_{i \notin S} (1 - p_i) \right) \det \left[ xI + \sum_{i \in S} a_i a_i^T + \sum_{i \notin S} b_i b_i^T \right]
$$

is real-rooted.
Conclusion of Step 2: $\{f_s(x)\}_{s \in \{\pm 1\}^m}$ is interlacing

**Theorem**

The family $\{f_s(x)\}_{s \in \{\pm 1\}^m}$ is interlacing.

**Proof.** Let $0 \leq k \leq m - 1$, $(s_1, \ldots, s_k) \in \{\pm 1\}^k$, and $\lambda \in [0, 1]$ be given. We must show that the polynomial

$$
\lambda f_{s_1, \ldots, s_k, 1}(x) + (1 - \lambda)f_{s_1, \ldots, s_k, -1}(x)
$$

is real-rooted. But this follows easily from the previous theorem with $p_1 = (1 + s_1)/2$, $\ldots$, $p_k = (1 + s_k)/2$, $p_{k+1} = \lambda$, and $p_{k+2} = \cdots = p_m = 1/2$. \qed
Steps 3 and 4: Wrapping up the proof

By applying the fact that \( \{ f_s(x) \}_{s \in \{ \pm 1 \}^m} \) is interlacing, we get immediately from the theorem of Heilmann-Lieb concerning the roots of the matching polynomial \( \mu_G(x) \) that

**Corollary**

Let \( G \) be a \( d \)-regular bipartite graph with adjacency matrix \( A \). Then \( G \) has a signing \( s \) such that every eigenvalue of \( A_s \) is at most \( 2\sqrt{d - 1} \). Thus, if \( G \) is a Ramanujan graph, so is the 2-lift associated to \( s \).

**Corollary**

Let \( d \geq 3 \) be given. Then there is an infinite family of \( d \)-regular bipartite Ramanujan graphs.

**Proof.** Start with a complete graph and take successive 2-lifts.
Main reference:


Other references:


