Aleksandrov-Bakelman-Pucci maximum principles for a class of uniformly elliptic and parabolic integro-PDE

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Abstract
We prove generalized Aleksandrov-Bakelman-Pucci maximum principles for elliptic and parabolic integro-PDEs of Hamilton-Jacobi-Bellman-Isaacs types, whose PDE parts are either uniformly elliptic or uniformly parabolic. The proofs of these results are based on the classical Aleksandrov-Bakelman-Pucci maximum principles for the elliptic and parabolic PDEs and an iteration procedure using solutions of Pucci extremal equations. We also provide proofs of the nonlocal versions of the classical Aleksandrov-Bakelman-Pucci maximum principles for elliptic and parabolic integro-PDEs.

Keywords: integro-PDE, Aleksandrov-Bakelman-Pucci maximum principle.

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1 Introduction
The Aleksandrov-Bakelman-Pucci (ABP) maximum principle, which evolved from the works of Aleksandrov, [1, 2, 3, 4] and his earlier papers on the maximum principle, Bakelman [6] and Pucci [51], is one of the fundamental tools in the theory of elliptic partial differential equations (PDEs). In its slightly simplified version it states that if $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ is a strong subsolution of the Pucci extremal PDE

\begin{equation}
\mathcal{P}^{-}(D^2u) - \gamma |Du| \leq f(x) \quad \text{in } \Omega,
\end{equation}

where $\mathcal{P}^{-}(D^2u)$ is the Pucci extremal operator (see (2.1)), $\gamma \geq 0$ and $f \in L^n(\Omega)$, then

\begin{equation}
\sup_{\Omega} u \leq \sup_{\partial \Omega} u + C(\text{diam}(\Omega)) \|f\|_{L^n(\Omega^+)},
\end{equation}

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where $C$ is a constant depending only on $n, \lambda, \gamma \text{diam}(\Omega)$ and $\Gamma_1^+$ is the so-called upper contact set of $u$. The more general classical version for linear equations, which does not require full uniform ellipticity and allows for the first order coefficients to be in $L^n(\Omega)$, can be found for instance in [34] and the statement given here can be obtained from the classical version in [34] through linearization. Many extensions and versions of the ABP maximum principle have been obtained over the years. In particular it was proved in [28] that there exists an exponent $p_0 = p_0(n, \Lambda/\lambda)$ such that the ABP maximum principle still holds if $u \in W^{2,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ for $p_0 < p < n$ (see also [11, 25, 27, 30]). However, in this generalized version of the ABP maximum principle, estimate (1.2) must be replaced by

$$
\sup_{\Omega} u \leq \sup_{\partial \Omega} u + C(\text{diam}(\Omega))^{2-n} \| f^+ \|_{L^p(\Omega)},
$$

(1.3)

for some constant $C = C(n, p, \gamma \text{diam}(\Omega), \lambda, \Lambda)$. Thus the $L^n$ norm of $f^+$ over the upper contact set is replaced by the $L^p$ norm of $f^+$ over $\Omega$. ABP maximum principle has also been extended to semiconvex subsolutions, viscosity subsolutions and the so-called $L^p$-viscosity subsolutions, see e.g. [12, 13, 14, 25, 29, 30, 40, 41, 42, 49], semiconvex subsolutions [54], degenerate/singular equations [26, 37], unbounded domains, see e.g. [5, 9, 16, 17, 18, 56] and the references there. Pointwise version of the ABP maximum principle for $W^{2,n}$ functions is known as the Bony maximum principle, see [10, 48]. Pointwise version for semiconvex functions is known as Jensen’s lemma, see [39, 22]. Pointwise versions for $L^p$-viscosity solutions can be found in [23, 53]. We also mention that pointwise maximum principle for classical viscosity solutions known as a “maximum principle for semicontinuous functions” can be found in [21, 22].

The parabolic analogue of the ABP maximum principle was first obtained by Krylov [43, 44] (see also [45]). It was later improved and made more precise by Tso [55]. It is thus often called the Aleksandrov-Bakelman-Pucci-Krylov-Tso maximum principle. We state here a nonlinear and slightly more restrictive version of the result of Tso. If $\Omega$ is a bounded domain in $\mathbb{R}^n, T > 0, Q = (-T, 0] \times \Omega$ and $u \in W^{1,2,n+1}_{\text{loc}}(Q) \cap C(\overline{Q})$ is a strong subsolution of the Pucci extremal PDE

$$
u_t + \mathcal{P}^-(D^2 u) - \gamma |Du| \leq f(t, x) \quad \text{in } Q,
$$

(1.4)

where $\gamma \geq 0$ and $f \in L^{n+1}(Q)$, then

$$
\sup_{Q} u \leq \sup_{\partial Q} u + C(\text{diam}(\Omega))^{\frac{n}{n+1}} \| f^+ \|_{L^{n+1}(\Gamma_1^+)},
$$

(1.5)

where $C$ is a constant depending only on $n, \lambda, \gamma^{n+1}|Q|/\text{diam}(\Omega)$ and $\Gamma_1^+$ is the so-called parabolic upper contact set of $u$ (see [55]). The statement in [55] is for linear equations, it does not require full uniform ellipticity, allows for the first order coefficients to be in $L^{n+1}(Q)$ and gives a precise value of the constant $C$ in (1.5). Similar results were also obtained independently using slightly different methods in [50, 52]. An improved parabolic ABP maximum principle which holds in more general domains is proved in [47]. For versions of the generalized parabolic ABP maximum principle in the case when $u \in W^{1,2,p}_{\text{loc}}(Q) \cap C(\overline{Q}), f \in L^p(\overline{Q})$ for $p_1 < p < n+1$ for some $p_1 = p_1(n, \Lambda/\lambda)$, see [11, 20, 25, 27, 28]. Extensions of the parabolic maximum principle to $L^p$-viscosity solutions were made in [20, 24, 40, 58]. A pointwise version of the parabolic maximum principle for $W^{1,2,n+1}$ functions is contained in [55] and a pointwise version for $L^p$-viscosity solutions can be found in [24]. A parabolic version of a maximum principle for semicontinuous functions can be found in [21, 22].
Much less is known about maximum principles of ABP type for equations with nonlocal terms. Maximum principle results for classical and strong subsolutions of elliptic Integro-PDE giving estimates with \( \| f^+ \|_{L^{n+1}(\Omega)} \) replaced by \( \| f^+ \|_{L^n(\Omega)} \), are in [33, 35]. Also some maximum principle results based on Green’s functions for nonlocal integro-PDE operators can be found in [32]. It was also remarked in [33] (see Remark 3.1.18 there) that the arguments to prove the classical ABP maximum principle for elliptic PDE from [34] can be adapted to elliptic integro-PDE to obtain an estimate like (1.2). However, to our knowledge, the full proof of such a result has not been written down in the literature. A nonlocal equivalent of the Bony maximum principle was proved in [35] (see also, [33]). Versions of maximum principles for semicontinuous functions for integro-differential equations can be found in [7, 38]. There are also recent results for purely nonlocal equations of elliptic and parabolic types. Among them, estimates of ABP type for such equations can be found in [15, 19]. More detailed study of the ABP maximum principle for uniformly elliptic nonlocal equations is in [36] where several results in this direction are proved. However the purely nonlocal case is very challenging and the area is still largely open.

In this paper we want to present the complete picture of the ABP maximum principle results for a class of uniformly elliptic and parabolic integro-PDEs, where the uniform ellipticity/parabolicity comes from the PDE part of the equation. In this case the integral part of the equation can be considered to be close to a lower-order perturbation of the PDE. To be precise, let us briefly describe the elliptic and parabolic cases we study here.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). We consider a strong subsolution \( u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap C_b(\mathbb{R}^n) \) of the following extremal integro-PDE

\[
\mathcal{P}^- (D^2 u) - \gamma|Du| \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} \left[ u(x+z) - u(x) - \langle Du(x), z \rangle 1_{\{|z|<1\}}(z) \right] N_\alpha(x,z)dz \right\} \leq f(x) \quad \text{in } \Omega,
\]

where \( \gamma \geq 0, f \in L^p(\Omega) \) and the functions \( N_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty) \) are Borel measurable. The index set \( A \) is countable. Additional assumptions about the functions \( N_\alpha \) will be given later. As regards the parabolic case, if \( T > 0 \) and \( Q = (-T,0] \times \Omega \), we consider a strong subsolution \( u \in W^{1,2,p}_{\mathrm{loc}}(\Omega) \cap C_b([-T,0] \times \mathbb{R}^n) \) of the parabolic extremal integro-PDE

\[
u_t + \mathcal{P}^- (D^2 u) - \gamma|Du| \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} \left[ u(t,x+z) - u(t,x) - \langle Du(t,x), z \rangle 1_{\{|z|<1\}}(z) \right] N_\alpha(t,x,z)dz \right\} \leq f(t,x) \quad \text{in } Q,
\]

where \( f \in L^p(Q) \).

We prove four theorems. Theorem 3.1, which is the nonlocal version of the classical ABP estimate, contains the precise statement of a result mentioned in Remark 3.1.18 of [33]. We provide its full proof. Theorem 4.1 is the nonlocal version of Theorem 2.1 of [55] and is the parabolic version of Theorem 3.1. We also equip it with the complete proof. The key difference between these results and the classical elliptic and parabolic ABP estimates is in the definition of the upper contact sets, which are defined using larger domains to account for the nonlocal terms. Our main results are contained in Theorems 3.2 and 4.2 which are nonlocal versions of the generalized APB maximum principles for the elliptic case when \( p_0 < p < n \) and for the parabolic case when \( p_1 < p < n+1 \). The proofs of these theorems rely on the generalized ABP maximum principles for the elliptic and parabolic PDE and the use a perturbation-iteration procedure based on solving Pucci extremal equations to improve the right hand side of the extremal equations.
integro-PDE inequality to a function where having enough integrability so that we can use the standard versions of the ABP estimates. Arguments using solutions of Pucci extremal equations and employing iteration arguments to prove ABP maximum principles were used before in [25] and [40]. However here the procedure is much more complicated since the iteration has infinitely many steps and the desired results is only achieved in the limit. We did not include zero order terms in (1.6) and (1.7), however the proofs can be easily modified to accommodate such terms which would involve standard modifications of the statements of Theorems 3.2 and 4.2. Finally we remark that the types of arguments in the proofs could potentially be adapted to prove ABP type estimates for other classes of extremal inequalities where the integral term might be replaced by a different term with similar properties.

2 Definitions and assumptions

The norm in \( \mathbb{R}^n \) is denoted by \(| \cdot |\) and the inner product by \( \langle \cdot, \cdot \rangle \). We denote by \( B_r(x) \) the open ball in \( \mathbb{R}^n \) centered at \( x \) with radius \( r > 0 \) and we will write \( \bar{B}_r \) for the ball centered at 0. For a subset \( A \) of \( \mathbb{R}^n \) we denote its complement by \( A^c \), its diameter by \( \text{diam}(A) \) and its Lebesgue measure (if \( A \) is measurable) by \( |A| \). We let \( w_n := |B_1| \). We denote by \( S^n \) the space of real \( n \times n \) symmetric matrices. If \( \Omega \) is an open subset of \( \mathbb{R}^n \) and \( Q = (-T, 0] \times \Omega \) for some \( T > 0 \), the standard parabolic boundary of \( Q = (-T, 0] \times \Omega \) is denoted by \( \partial_p Q_T, \) i.e.

\[
\partial_p Q := \{ -T \} \times \Omega \cup \{ -T, 0 \} \times \partial \Omega.
\]

The nonlocal parabolic boundary of \( Q \) is denoted by \( \partial_pm Q \) and is defined by

\[
\partial_pm Q := \{ -T \} \times \mathbb{R}^n \cup \{ -T, 0 \} \times \Omega^c.
\]

The parabolic distance between \((t, x)\) and \((s, y)\) in \( \mathbb{R} \times \mathbb{R}^n \) is defined by

\[
d_p((t, x), (s, y)) := (|t - s| + |x - y|^2) \frac{1}{2}.
\]

and the diameter of \( Q, \text{diam}(Q) \) is measured in the parabolic distance. We define, for \( \eta > 0 \),

\[
\Omega_\eta := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \eta \}
\]

and \( \tilde{\Omega}_\eta := \{ x \in \mathbb{R}^n : \text{dist}(x, \Omega) < \eta \}. \) We set \( d := \text{diam}(\Omega) \). For a function \( w : \mathbb{R}^n \to \mathbb{R} \) we define the nonlocal upper contact set

\[
\Gamma_{\Omega}^+(w) := \{ x \in \Omega : w(x) > \sup_{\tilde{\Omega}} w, \exists p \text{ such that } w(y) \leq w(x) + \langle p, y - x \rangle \text{ for } y \in \tilde{\Omega}_d \}.
\]

For a function \( v : [-T, 0] \times \mathbb{R}^n \to \mathbb{R} \) we define the nonlocal parabolic upper contact set

\[
\Gamma_Q^+(v) := \{ (t, x) \in Q : v(t, x) > \sup_{\partial_pm Q} v, \exists p \text{ such that } v(s, y) \leq v(t, x) + \langle p, y - x \rangle \text{ for } (s, y) \in [-T, t] \times \tilde{\Omega}_d \}.
\]

For given ellipticity constants \( 0 < \lambda \leq \Lambda \), the Pucci extremal operator \( P^- \) is defined, for \( X \in \mathbb{S}^n \), by

\[
P^-(X) = \min \{ -\text{Tr}(AX) \mid A \in \mathbb{S}^n, \lambda I \leq A \leq \Lambda I \} \tag{2.1}
\]

The Pucci extremal operator \( P^+ \) is defined by \( P^+(X) = -P^-(X) \).

We denote by \( W^{2,p}(\Omega), W^{2,p}_{\text{loc}}(\Omega), W^{1,2,p}(Q), W^{1,2,p}_{\text{loc}}(Q) \) the standard Sobolev and parabolic Sobolev spaces. We denote by \( C_0(\mathbb{R}^n) \) and \( C_0([-T, 0] \times \mathbb{R}^n) \), respectively, the space of bounded continuous functions on \( \mathbb{R}^n \) and the space of bounded continuous functions on \( [-T, 0] \times \mathbb{R}^n \).

The result below is the generalized ABP maximum principle taken from [25], see Theorem 1.1 there.
Theorem 2.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $0 < \lambda \leq \Lambda$ be given ellipticity constants and $\gamma \geq 0$. There exists $p_0 = p_0(n, \Lambda/\lambda)$, $n/2 \leq p_0 < n$, such that if $p_0 < p < n$, $f \in L^p(\Omega)$ and $u \in W^{2,p}_\text{loc}(\Omega) \cap C(\overline{\Omega})$ satisfies
\[ \mathcal{P}^-(D^2u) - \gamma |Du| \leq f(x) \quad \text{in } \Omega, \]
then
\[ \sup_{\Omega} u \leq \sup_{\partial \Omega} u + C(\text{diam}(\Omega))^{2 - \frac{n}{p}} \|f^+\|_{L^p(\Omega)}, \quad (2.2) \]
for some constant $C = C(n, p, \gamma \text{diam}(\Omega), \lambda, \Lambda)$.

We also recall a well known result about solvability of extremal equations, see e.g. [14] or Lemma 1.8 of [53].

Theorem 2.2. For every $f \in L^p(B_2)$, $p > p_0$ there exists a unique strong solution $u \in W^{2,p}_\text{loc}(B_2) \cap C(\overline{B}_2)$ of
\[ \begin{cases} \mathcal{P}^+(D^2u) + \gamma |Du| = f(x) & \text{in } B_2, \\ u = 0 & \text{on } \partial B_2. \end{cases} \quad (2.3) \]
Moreover there exist constants $C_1 = C_1(n, p, \lambda, \Lambda, \gamma)$, $C_2 = C_2(n, p, \lambda, \Lambda, \gamma)$ such that
\[ \|u\|_{W^{2,p}(B_2)} \leq C_1\|f\|_{L^p(B_2)} \quad (2.4) \]
and
\[ \sup_{B_2} |u| \leq C_2\|f\|_{L^p(B_2)}. \quad (2.5) \]
If $p \geq n$, the constant $C_2$ only depends on $n, p, \lambda, \gamma$.

We remark that, using results of [58], one can obtain $u \in W^{2,p}(B_2)$. However the version of Theorem 2.2 stated here will be enough for our purposes.

The generalized parabolic ABP maximum principle below is taken from [25], Theorem 1.1 (see also [20], Theorem 0.2).

Theorem 2.3. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $T > 0$ and denote $Q = (-T, 0) \times \Omega$. Let $0 < \lambda \leq \Lambda$ be given ellipticity constants and $\gamma \geq 0$. There exists $p_1 = p_1(n, \Lambda/\lambda)$, $(n+1)/2 \leq p_1 < n+1$, such that if $p_1 < p < n+1$, $f \in L^p(Q)$ and $u \in W^{1,2,p}_\text{loc}(Q) \cap C(\overline{Q})$ satisfies
\[ u_t + \mathcal{P}^-(D^2u) - \gamma |Du| \leq f(t, x) \quad \text{in } Q, \]
then
\[ \sup_Q u \leq \sup_{\partial_p Q} u + C(Q)^{2 - \frac{n+2}{p}} \|f^+\|_{L^p(\Omega)}, \quad (2.6) \]
for some constant $C = C(n, p, \gamma \text{diam}(Q), \lambda, \Lambda)$.

Theorem 2.4 below is a standard result about solvability of parabolic extremal equations, see e.g. Theorems 2.8 and 9.1 of [24] or Theorem 2.1 of [20] (see also [57]).

Theorem 2.4. For every $f \in L^p((-2, 0] \times B_2), p > p_1$ there exists a unique strong solution $u \in W^{1,2,p}_\text{loc}((-2, 0] \times B_2) \cap C([-2, 0] \times \overline{B}_2)$ of
\[ \begin{cases} u_t + \mathcal{P}^+(D^2u) + \gamma |Du| = f(t, x) & \text{in } (-2, 0] \times B_2, \\ u = 0 & \text{on } \partial_p ((-2, 0] \times B_2). \end{cases} \quad (2.7) \]
Moreover there exist constants $\overline{C}_1 = \overline{C}_1(n,p,\lambda,\gamma)$, $\overline{C}_2 = \overline{C}_2(n,p,\lambda,\gamma)$ such that

$$
\|u\|_{W^{1,2,p}((-\frac{3}{2},0) \times B_2)} \leq \overline{C}_1 \|f\|_{L^p((-2,0) \times B_2)} \quad (2.8)
$$

and

$$
\sup_{(-2,0) \times B_2} |u| \leq \overline{C}_2 \|f\|_{L^p((-2,0) \times B_2)}. \quad (2.9)
$$

If $p \geq n+1$, the constant $C_2$ only depends on $n, p, \lambda, \gamma$.

We will also be using the following calculus lemma and the well known Minkowski’s inequality for integrals, see e.g. [31], Theorem 6.19.

**Lemma 2.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $p \geq n/2, \delta > 0$. If $f \in W^{2,p}(\Omega)$ then for almost every $x \in \Omega_{2\delta}$, we have

$$
f(x + z) - f(x) - (Df(x), z) = \int_0^1 \int_0^1 (D^2 f(x + tsz)z, z) tdsdt \quad (2.10)
$$

for almost every $z \in B_{\delta}$.

**Proof.** Let $f_n \in C^2(\Omega)$ be such that $\|f_n - f\|_{W^{2,p}(\Omega_{\delta})} \to 0$ as $n \to \infty$. Then for every $x \in \Omega_{2\delta}$ and $z \in B_{\delta}$

$$
f_n(x + z) - f_n(x) - (Df_n(x), z) = \int_0^1 \int_0^1 (D^2 f_n(x + tsz)z, z) tdsdt. \quad (2.11)
$$

Obviously, up to a subsequence, for almost every $x \in \Omega_{2\delta}$ the left hand side of (2.11) converges to the left hand side of (2.10) as $n \to \infty$. Moreover

$$
\left| \int_{B_{\delta}} \left| \int_0^1 \int_0^1 (D^2 f_n(x + tsz)z, z) tdsdt - \int_0^1 \int_0^1 (D^2 f(x + tsz)z, z) tdsdt \right| dz \right|

\leq \delta^2 \int_0^1 \int_{B_{\delta}} \left| D^2 f_n(x + tsz) - D^2 f(x + tsz) \right| dz ds dt \leq C \|f_n - f\|_{W^{2,p}(\Omega_{\delta})} \to 0.
$$

Therefore, up to a further subsequence, the right hand side of (2.11) converges to the right hand side of (2.10) for almost every $z \in B_{\delta}$ which proves the claim. \hfill \Box

**Lemma 2.2.** Let $(X, \mathcal{M}, \nu_1)$ and $(Y, \mathcal{N}, \nu_2)$ be $\sigma$-finite measure spaces and let $1 \leq p < \infty$. If $f$ is an $\mathcal{M} \otimes \mathcal{N}$ measurable function on $X \times Y$, then

$$
\left( \int_Y \left( \int_X |f(x,y)| \nu_1(dx) \right)^p \nu_2(dy) \right)^{\frac{1}{p}} \leq \int_X \left( \int_Y |f(x,y)|^p \nu_2(dy) \right)^{\frac{1}{p}} \nu_1(dx).
$$

We make the following assumption.

(A1) The functions $N_\alpha : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty), \alpha \in \mathcal{A}$, are measurable and there exists a measurable function $K : \mathbb{R}^n \to [0, +\infty)$ such that for every $\alpha \in \mathcal{A}, x \in \Omega, t \in [0,T]$, $N_\alpha(t,x,\cdot) \leq K(\cdot)$ and $K$ satisfies

$$
\int_{\mathbb{R}^n} \min(|z|^2,1)K(z)dz < +\infty. \quad (2.12)
$$
3 Maximum principle: elliptic case

This section contains two versions of the ABP maximum principle for uniformly elliptic integro-PDE. Theorem 3.1 is a nonlocal version of the classical ABP maximum principle, while Theorem 3.2 is a version of the generalized ABP maximum principle which extends it to exponents $p_0 < p < n$.

**Theorem 3.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Suppose that (A1) is satisfied and $f \in L^p(\Omega)$. Let $R$ be a number such that $\text{diam}(\Omega) \leq R$. Then there exists a constant $C = C(n, \lambda, \gamma, R, K(\cdot))$ such that if $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C_b(\mathbb{R}^n)$ is a strong subsolution of the extremal integro-PDE (1.6) then

$$\sup_{\Omega} u \leq \sup_{\Omega^c} u + C\text{diam}(\Omega)\|f^+\|_{L^p(\Omega^c)}.$$ \hspace{1cm} (3.1)

**Proof.** The proof is an adaptation of the proof from [34], Section 9.1 and the proof of Proposition 2.3 of [14].

Without loss of generality we assume that $0 \in \Omega$. We begin by assuming that $u \in C^2_{\text{loc}}(\Omega) \cap C_b(\mathbb{R}^n)$. We will later remove this assumption via approximations. Recall that we defined $d = \text{diam}(\Omega)$. We set

$$r_0 = r_0(u) := \frac{\sup_{\Omega} u - \sup_{\Omega^c} u}{2d}. \hspace{1cm} (3.2)$$

If $r_0 \leq 0$, we are done. Otherwise, for any $0 < r < r_0$ we define

$$\Gamma_{\Omega,r}^{n,+}(u) := \{ x \in \Omega : \exists p, |p| \leq r, \text{ such that } u(y) \leq u(x) + \langle p, y - x \rangle \text{ for } y \in \Omega_d \}.$$ 

If $p \in B_r$, the function $u(x) - \langle p, x \rangle$ has a maximum over $\Omega_d$ at some point $\hat{x} \in \Omega$, i.e.

$$u(x) - \langle p, x \rangle \leq u(\hat{x}) - \langle p, \hat{x} \rangle$$

for any $x \in \Omega_d$. It follows that

$$\sup_{\Omega} u - u(\hat{x}) \leq 2rd < 2r_0d = \sup_{\Omega} u - \sup_{\Omega^c} u.$$ 

In particular, $\hat{x} \in \Omega$ and

$$u(\hat{x}) \geq \sup_{\Omega^c} u + 2(r_0 - r)d > \sup_{\Omega^c} u.$$ \hspace{1cm} (3.3)

Since $Du(\hat{x}) = p$ and $D^2u(\hat{x}) \leq 0$, we conclude that, for $0 < r < r_0$, $\Gamma_{\Omega,r}^{n,+}(u)$ is a subset of $\Omega$ and

$$B_r = Du(\Gamma_{\Omega,r}^{n,+}(u)) \text{ and } D^2u(x) \leq 0 \text{ on } \Gamma_{\Omega,r}^{n,+}(u).$$

Hence, for $\kappa \geq 0$, the change of variables $p = Du(x)$ yields

$$\int_{B_r} (|p|^{\frac{n}{n-1}} + \kappa^{\frac{n}{n-1}})^{1-n} dp \leq \int_{\Gamma_{\Omega,r}^{n,+}(u)} (|Du|^{\frac{n}{n-1}} + \kappa^{\frac{n}{n-1}})^{1-n} |\det(D^2u)| dx \leq \int_{\Gamma_{\Omega,r}^{n,+}(u)} (|Du|^{\frac{n}{n-1}} + \kappa^{\frac{n}{n-1}})^{1-n} \left( -\frac{\text{Tr}(D^2u)}{n} \right)^n dx. \hspace{1cm} (3.4)$$

Since $D^2u \leq 0$ on $\Gamma_{\Omega,r}^{n,+}(u)$ and (1.6) holds, we have

$$-\lambda \text{Tr}(D^2u(x)) - \gamma |Du(x)| - \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} [u(x + z) - u(x) - \langle Du(x), z \rangle 1_{\{z < 1\}}(z)] N_\alpha(x, z) dz \right\} \leq f(x) \text{ in } \Gamma_{\Omega,r}^{n,+}(u).$$
Now let us consider the nonlocal term at \( x \in \Gamma_{\tilde{\Omega},r}^{n,+}(u) \).

\[
\int_{\mathbb{R}^n} [u(x + z) - u(x) - \langle Du(x), z \rangle 1_{\{|z| < 1\}}(z)] N_\alpha(x, z) dz
= \int_{B_1 \cap \{z : x + z \in \tilde{\Omega}_d\}} [u(x + z) - u(x) - \langle Du(x), z \rangle] N_\alpha(x, z) dz
+ \int_{B_1 \cap \{z : x + z \in \tilde{\Omega}_d\}} [u(x + z) - u(x) - \langle Du(x), z \rangle] N_\alpha(x, z) dz
+ \int_{\{z : x + z \in \tilde{\Omega}_d^c\}} \langle Du(x), z \rangle N_\alpha(x, z) dz
+ \int_{\{z : x + z \in \tilde{\Omega}_d^c\}} [u(x + z) - u(x) - \langle Du(x), z \rangle 1_{\{|z| < 1\}}(z)] N_\alpha(x, z) dz
=: I_1 + I_2 + I_3 + I_4.
\]

Using that (3.3) holds for \( x \in \Gamma_{\tilde{\Omega},r}^{n,+}(u) \), we obtain \( I_1 \leq 0, I_2 \leq 0 \) and

\[
I_4 \leq - \int_{\{z : x + z \in \tilde{\Omega}_d^c\}} \langle Du(x), z \rangle 1_{\{|z| < 1\}}(z) N_\alpha(x, z) dz.
\]

Recalling that \( d < R \), we notice that, if \( x + z \in \tilde{\Omega}_d \) then \(|z| < 2R\), and if \( x + z \in \tilde{\Omega}_d^c \) then \(|z| \geq d\). Therefore,

\[
\int_{\mathbb{R}^n} [u(x + z) - u(x) - \langle Du(x), z \rangle 1_{\{|z| < 1\}}(z)] N_\alpha(x, z) dz
\leq \int_{B_1 \cap \{z : x + z \in \tilde{\Omega}_d\}} \langle Du(x), z \rangle N_\alpha(x, z) dz
- \int_{B_1 \cap \{z : x + z \in \tilde{\Omega}_d^c\}} \langle Du(x), z \rangle N_\alpha(x, z) dz
\leq \left( \int_{B_1 \cap B_2} |z| K(z) dz + \int_{B_1 \cap B_2^c} \frac{|z|^2}{d} K(z) dz \right) |Du(x)|
\leq C_0 (1 + d^{-1}) |Du(x)|,
\]

(3.5)

where \( C_0 \geq 0 \) depends only on \( R \) and \( K(\cdot) \). We thus have

\[-\lambda \text{Tr}(D^2 u(x)) - [\gamma + C_0 (1 + d^{-1})] |Du(x)| \leq f(x) \quad \text{in } \Gamma_{\tilde{\Omega},r}^{n,+}(u)\]

and (3.4) implies

\[
\int_{B_r} \left( |p|^n \frac{n}{n-1} + \kappa \right)^{-n} \frac{1}{n^\lambda} \int_{\Gamma_{\tilde{\Omega},r}^{n,+}(u)} (|Du|^\frac{n}{n-1} + \kappa)^{1-n} (|\gamma + C_0 (1 + d^{-1})| Du + f^+) n \ dx
\leq \frac{1}{n^\lambda} \int_{\Gamma_{\tilde{\Omega},r}^{n,+}(u)} (|Du|^\frac{n}{n-1} + \kappa)^{1-n} (|\gamma + C_0 (1 + d^{-1})|^n + \left( \frac{f^+}{\kappa} \right)^n) (|Du|^\frac{n}{n-1} + \kappa)^{n-1} \ dx
\leq \frac{1}{n^\lambda} \int_{\Gamma_{\tilde{\Omega},r}^{n,+}(u)} (|\gamma + C_0 (1 + d^{-1})|^n + \left( \frac{f^+}{\kappa} \right)^n) \ dx.
\]

(3.6)
We note that \((a + b)^k \leq 2^{k-1}(a^k + b^k)\) implies

\[
2^{2-n} \frac{1}{|p|^n + \kappa^n} \leq \left( |p|^{\frac{n}{n-1}} + \kappa^{\frac{n}{n-1}} \right)^{1-n} ,
\]

and then it follows from (3.6) that

\[
2^{2-n} w_n \ln \left( \frac{r_n}{\kappa^n} + 1 \right) = 2^{2-n} \int_{B_r} \frac{1}{|p|^n + \kappa^n} dp \\
\leq \int_{B_r} \left( |p|^{\frac{n}{n-1}} + \kappa^{\frac{n}{n-1}} \right)^{1-n} dp \\
\leq \frac{1}{n^2 \lambda n} \int_{\Gamma_{\Omega+,r}^n(u)} \left( |\gamma + C_0(1 + d^{-1})|^n + \frac{(f^+)^n}{\kappa^n} \right) dx.
\]

If \(f^+ \neq 0\) on \(\Gamma_{\Omega+,r}^n(u)\), we let \(\kappa = \frac{\|f^+\|_{L^n(\Gamma_{\Omega+,r}^n(u))}}{\lambda} \neq 0\) to obtain

\[
r \leq \left( \exp \left( \frac{2^{n-2}}{w_n n^n} \left( 1 + \int_{\Gamma_{\Omega+,r}^n(u)} \left( |\gamma + C_0(1 + d^{-1})|^n + \frac{(f^+)^n}{\kappa^n} \right) dx \right) \right) - 1 \right)^{\frac{1}{\lambda^n}} \|f^+\|_{L^n(\Gamma_{\Omega+,r}^n(u))}.
\]

If \(f^+ \equiv 0\), we let \(\kappa \to 0\) so that (3.7) is again satisfied. We notice that the constant in (3.7) depends only on \(n, \lambda, \gamma, R\) and \(K(\cdot)\).

If \(u \in W_{\text{loc}}^{3,n}(\Omega) \cap C_b(\mathbb{R}^n)\) using mollifications it is easy to construct functions \(u_\epsilon, \epsilon > 0\) such that \(u_\epsilon \in C^2(\mathbb{R}^n) \cap C_b(\mathbb{R}^n)\) and they converge, as \(\epsilon \to 0\), to \(u\) in \(W_{\text{loc}}^{3,n}(\Omega)\) and uniformly in \(\mathbb{R}^n\). The functions \(u_\epsilon\) satisfy in every \(\Omega_\eta, \eta > 0\), for sufficiently small \(\epsilon\),

\[
\mathcal{P}^-(D^2 u_\epsilon) - \gamma |Du_\epsilon| - \sup_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \left[ u_\epsilon(x + z) - u_\epsilon(x) - \langle Du_\epsilon(x), z \rangle 1_{\{|z| < 1\}}(z) \right] M_\alpha(x, z) dz \right\} \\
\leq f_\epsilon(x) \quad \text{in } \Omega_\eta,
\]

where \(\|f_\epsilon - f^+\|_{L^n(\Omega_\eta)} \to 0\) as \(\epsilon \to 0\). For details of this we refer to Step 1 of the proof of Theorem 3.2. It is easy to see that if \(r < r_0(u_\epsilon)\), then for sufficiently small \(\eta\) and \(\epsilon\), we have \(r < r_0(u_\epsilon)\) (defined as in (3.2)) and \(\Gamma_{\Omega+,r}^{n+}(u_\epsilon) \subset \Omega_\eta\). We can thus obtain (3.7) for \(u_\epsilon\). We can then pass to the limit, along a subsequence \(\epsilon_j \to 0\), in (3.7) using the fact that

\[
\limsup_{j \to +\infty} \Gamma_{\Omega,\epsilon_j}^{n+}(u_{\epsilon_j}) \subset \Gamma_{\Omega,r}^{n+}(u),
\]

which can be proved exactly the same as a similar statement for the usual upper contact sets in Lemma A.1 of [14]. It then remains to let \(r \to r_0\) to establish (3.1).

**Theorem 3.2.** Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\) and let \(p_0 < p < n\), where \(p_0\) is from Theorem 2.1. Suppose that (A1) is satisfied and \(f \in L^p(\Omega)\). Let \(R\) be a number such that \(\text{diam}(\Omega) \leq R\). Then there exists a constant \(C = C(n, p, \lambda, \Lambda, \gamma, R, K(\cdot))\), such that if \(u \in W_{\text{loc}}^{2,p}(\Omega) \cap C_b(\mathbb{R}^n)\) is a strong subsolution of the extremal integro-PDE (1.6), then

\[
\sup_{\Omega\epsilon} u \leq \sup_{\Omega^c} u + C(\text{diam}(\Omega))^{2^\frac{2}{n}} \|f^+\|_{L^p(\Omega)}.
\]

(3.8)
Proof. Step 1. Rescaling and smoothing. We first rescale equation (1.6) to the equation in a domain with unit diameter. Without loss of generality we assume that $0 \in \Omega$. In this section we denote $r := \text{diam}(\Omega)$ and $\hat{\Omega} := \frac{1}{r} \Omega$. We set

$$v(x) := u(rx)$$

and

$$M_\alpha(x, z) := r^{n+2}N_\alpha(rx, rz).$$

Then

$$0 \leq M_\alpha(x, z) \leq r^{n+2}K(rz) =: K_1(z).$$

It is easy to see that $v \in W^{2,p}_{\text{loc}}(\hat{\Omega}) \cap C_b(\mathbb{R}^n)$ is a strong subsolution of the rescaled extremal integro-PDE

$$\mathcal{P}^{-}(D^2 v) - \gamma R|Dv| - \sup_{\alpha \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \left[ v(x + z) - v(x) - \langle Dv(x), z \rangle 1_{\{|z| < 1/r\}}(z) \right] M_\alpha(x, z) dz \right\}$$

$$\leq g(x) := r^2 f^+(rx) \quad \text{in } \hat{\Omega}. \quad (3.9)$$

We notice that

$$\|g\|_{L^p(\hat{\Omega})} = (\text{diam}(\Omega))^{2 - \frac{2}{p}} \|f^+\|_{L^p(\Omega)}.$$

Define, for $\eta > 0$, $\hat{\Omega}_\eta := \{x \in \hat{\Omega} : \text{dist}(x, \partial \hat{\Omega}) > \eta\}$. Without loss of generality we can assume that $\hat{\Omega}_\eta$ is a domain (i.e. it is connected) as otherwise we would take the connected component of $\hat{\Omega}_\eta$ containing 0.

Using mollifications it is easy to construct functions $v_\epsilon$, $\epsilon > 0$ such that $v_\epsilon \in C^2(\mathbb{R}^n) \cap C_b(\mathbb{R}^n)$ and the $v_\epsilon$ converge, as $\epsilon \to 0$, to $v$ in $W^{2,p}_{\text{loc}}(\hat{\Omega})$ and uniformly in $\mathbb{R}^n$. We claim that, for every $\eta > 0$ and small enough $\epsilon > 0$, $v_\epsilon$ is a classical subsolution of the extremal integro-PDE

$$\mathcal{P}^{-}(D^2 v_\epsilon) - \gamma R|Dv_\epsilon| - \sup_{\alpha \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \left[ v_\epsilon(x + z) - v_\epsilon(x) - \langle Dv_\epsilon(x), z \rangle 1_{\{|z| < 1/r\}}(z) \right] M_\alpha(x, z) dz \right\}$$

$$\leq g_\epsilon(x) \quad \text{in } \hat{\Omega}_\eta, \quad (3.10)$$

where $\|g_\epsilon - g\|_{L^p(\hat{\Omega}_\eta)} \to 0$ as $\epsilon \to 0$. To see this we notice that

$$\left| \sup_{\alpha \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \left[ v(x + z) - v(x) - \langle Dv(x), z \rangle 1_{\{|z| < 1/r\}}(z) \right] M_\alpha(x, z) dz \right\} \right|$$

$$- \left| \sup_{\alpha \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \left[ v_\epsilon(x + z) - v_\epsilon(x) - \langle Dv_\epsilon(x), z \rangle 1_{\{|z| < 1/r\}}(z) \right] M_\alpha(x, z) dz \right\} \right|$$

$$\leq \int_{\mathbb{R}^n} \left[ v(x + z) - v(x) - \langle Dv(x), z \rangle 1_{\{|z| < 1/r\}}(z) \right]$$

$$- \left[ v_\epsilon(x + z) - v_\epsilon(x) - \langle Dv_\epsilon(x), z \rangle 1_{\{|z| < 1/r\}}(z) \right] \mid K_1(z) dz$$

$$= \int_{B_{\frac{r}{2}}} + \int_{B_{\frac{r}{2}}^c} =: I_\epsilon(x) + J_\epsilon(x).$$
Now, using Lemmas 2.1 and 2.2,
\[
\|I_\epsilon\|_{L^p(\tilde{\Omega}_\eta)} \leq \left( \int_{\tilde{\Omega}_\eta} \left( \int_{B^{\eta}_{\frac{1}{2}}} \int_0^1 \int_0^1 \|D^2v(x + tsz) - D^2v_\epsilon(x + tsz)\| |z|^2K_1(z)dsdtdz \right)^p dx \right)^{\frac{1}{p}} \\
\leq \int_{B^{\eta}_{\frac{1}{2}}} \int_0^1 \int_0^1 \left( \int_{\tilde{\Omega}_\eta} \|D^2v(x + tsz) - D^2v_\epsilon(x + tsz)\|^p dx \right) |z|^2K_1(z)dsdtdz \\
\leq \|v - v_\epsilon\|_{W^{2,p}(\tilde{\Omega}_{\frac{1}{2}})} \int_{B^{\eta}_{\frac{1}{2}}} |z|^2K_1(z)dz \to 0 \quad \text{as } \epsilon \to 0.
\]

The convergence \( \|J_\epsilon\|_{L^p(\tilde{\Omega}_\eta)} \to 0 \) is an easy consequence of the uniform convergence of \( v_\epsilon \) to \( v \), dominated convergence theorem, and the fact that \( \|Dv_\epsilon - Dv\|_{L^p(\tilde{\Omega}_\eta)} \to 0 \) as \( \epsilon \to 0 \).

We also observe that \( g_\epsilon \in L^q(\tilde{\Omega}_\eta) \) for every \( q < +\infty \). In the rest of the proof we set \( q = p^* \), where \( p^* = np/(n - p) > n \) is the Sobolev conjugate of \( p \).

It is enough to prove that
\[
\sup_{\tilde{\Omega}_\eta} v_\epsilon \leq \sup_{\tilde{\Omega}_\eta} v_\epsilon + C\|g_\epsilon\|_{L^p(\tilde{\Omega}_\eta)}, \quad (3.11)
\]

The result will then follow by letting \( \epsilon \to 0 \) and then \( \eta \to 0 \).

Let \( 0 < \delta < \min\left(\frac{1}{2}, \frac{1}{R}\right) \) be such that
\[
\eta := \int_{B_{R\delta}} |z|^2K(z)dz \leq \min\left(\frac{1}{2C_1}, \frac{1}{2C_2}\right),
\]
where \( C_1 := \max(C_1(n, p, \lambda, \Lambda, \gamma R), C_1(n, q, \lambda, \Lambda, \gamma R)) \) is the maximum of the two constants from (2.4) for the two exponents \( p \) and \( q \), and \( C_2 = C_2(n, p, \lambda, \Lambda, \gamma R) \) is the constant from (2.5).

Then we have
\[
\int_{B_{\delta}} |z|^2K_1(z)dz = \int_{B_{\delta}} |rz|^2K(rz)r^n dz \leq \int_{B_{R\delta}} |z|^2K(z)dz = \eta, \quad (3.12)
\]
\[
\int_{B_{\delta}} K_1(z)dz = \int_{B_{\delta}} r^2K(z)dz \leq \int_{B_{\delta}} \min\left(\frac{|z|^2}{\delta^2}, R^2\right) K(z)dz \leq C_R \quad (3.13)
\]

and
\[
\int_{B_{\delta}} |z|K_1(z)1_{\{|z|<1/\delta\}}(z)dz = \int_{B_{\delta}} r|z|K(z)1_{\{|z|<1\}}(z)dz \leq \int_{B_{\delta}} \frac{|z|^2}{\delta}K(z)1_{\{|z|<1\}}(z)dz \leq C_3 \quad (3.14)
\]
for some absolute constants \( C_R \) and \( C_3 \) which also depend on \( K(\cdot) \).

**Step 2. Iteration.** Let \( u_1 \in W^{2,q}_{\text{loc}}(B_2) \cap C(\overline{B_2}) \) be the unique strong solution of
\[
\begin{align*}
\mathcal{P}^+(D^2u_1) + \gamma R|Du_1| &= -g_\epsilon(x) \quad \text{in } B_2, \\
u_1 &= 0 \quad \text{on } \partial B_2,
\end{align*}
\]
where we extended \( g_\epsilon \) by 0 outside of \( \tilde{\Omega}_\eta \). By Theorem 2.2 we have
\[
\|u_1\|_{W^{2,p}(\tilde{B}_{\frac{3}{2}})} \leq C_1\|g_\epsilon\|_{L^p(\tilde{\Omega}_\eta)}, \quad (3.15)
\]
\[ \|u_1\|_{W^{2,q}(B_2^c)} \leq C_1 \|g_\epsilon\|_{L^q(\hat{\Omega}_n)} \] (3.16)

and

\[ \sup_{B_2} |u_1| \leq C_2 \|g_\epsilon\|_{L^p(\Omega_n)}. \] (3.17)

We extend \( u_1 \) to a function on \( \mathbb{R}^n \) by setting \( u_1 = 0 \) on \( B_2^c \). Then the function \( v_1 = v_\epsilon + u_1 \in W^{2,q}(B_2^c) \cap C_b(\mathbb{R}^n) \) satisfies in \( \Omega \)

\[
P^{-}(D^2v_1(x)) - \gamma R|Dv_1(x)| - \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} \left[ v_1(x + z) - v_1(x) - \langle Dv_1(x), z \rangle 1_{\{|z| < 1/r\}}(z) \right] M_\alpha(x, z) dz \right\} \leq \int_{\mathbb{R}^n} |u_1(x + z) - u_1(x) - \langle Du_1(x), z \rangle 1_{\{|z| < 1/r\}}(z) | K_1(z) dz \]

\[ = \int_{B_{\delta}} + \int_{B_{\delta}^c} =: g_1(x) + h_1(x). \] (3.18)

Using Lemmas 2.1, 2.2 and (3.15) we now have

\[ \|g_1\|_{L^p(\hat{\Omega}_n)} \leq \left( \int_{\hat{\Omega}_n} \left( \int_{B_\delta} \int_0^1 \left( \int_0^1 \|D^2u_1(x + tsz)||z|^2 K_1(z) dsdt dz \right)^p dx \right)^{1/p} \right) \]

\[ \leq \int_{B_\delta} \int_0^1 \int_0^1 \left( \int_{\hat{\Omega}_n} \|D^2u_1(x + tsz)||^p dx \right)^{1/p} |z|^2 K_1(z) dsdt dz \]

\[ \leq \|u_1\|_{W^{2,p}(B_2^c)} \int_{B_\delta} |z|^2 K_1(z) dz = \eta \|u_1\|_{W^{2,p}(B_2^c)} \leq \frac{1}{2} \|g_\epsilon\|_{L^p(\Omega_n)}, \] (3.19)

while replacing \( p \) by \( q \) above and using (3.16) gives

\[ \|g_1\|_{L^q(\hat{\Omega}_n)} \leq \frac{1}{2} \|g_\epsilon\|_{L^q(\hat{\Omega}_n)}. \] (3.20)

Similar computation using (3.13), (3.14), (3.15), (3.17) and the Sobolev imbedding theorem implies

\[ \|h_1\|_{L^q(\hat{\Omega}_n)} \leq C_4 \|g_\epsilon\|_{L^p(\hat{\Omega}_n)} \] (3.21)

for some absolute constant \( C_4 = C_3(n, p, \lambda, \Lambda, \gamma, R, K(\cdot)) \).

We now extend \( g_1, h_1 \) by 0 outside of \( \hat{\Omega}_n \). Let \( u_2 \in W^{2,q}_{\text{loc}}(B_2) \cap C(\overline{B_2}) \) be the unique strong solution of

\[
\left\{ \begin{array}{l}
P^+(D^2u_2) + \gamma R|Du_2| = -g_1(x) \quad \text{in } B_2, \\
u_2 = 0 \quad \text{on } \partial B_2
\end{array} \right.
\]

from Theorem 2.2. By (2.4), (2.5) and (3.19) we have

\[ \|u_2\|_{W^{2,p}(B_2^c)} \leq C_1 \|g_1\|_{L^p(\hat{\Omega}_n)} \leq \frac{C_1}{2} \|g_\epsilon\|_{L^p(\hat{\Omega}_n)}, \] (3.22)

\[ \|u_2\|_{W^{2,q}(B_2^c)} \leq C_1 \|g_1\|_{L^q(\hat{\Omega}_n)} \leq \frac{C_1}{2} \|g_\epsilon\|_{L^q(\hat{\Omega}_n)} \] (3.23)

and

\[ \sup_{B_2} |u_2| \leq C_2 \|g_1\|_{L^p(\hat{\Omega}_n)} \leq \frac{C_2}{2} \|g_\epsilon\|_{L^p(\hat{\Omega}_n)}, \] (3.24)

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We extend \( u_1 \) to a function on \( \mathbb{R}^n \) by setting \( u_1 = 0 \) on \( B_2^c \). Defining \( v_2 = v_1 + u_2 \) and arguing as for \( v_1 \) we then obtain that \( v_2 \) is a strong subsolution of

\[
P^-(D^2v_2(x)) - \gamma R|Dv_2(x)| - \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} [v_2(x + z) - v_2(x) - (Dv_2(x), z) \mathbf{1}_{\{|z| < 1/r\}}(z)] M_\alpha(x, z)dz \right\}
\leq g_2(x) + h_2(x)
\]

(3.25)

in \( \hat{\Omega}_\eta \), where

\[
\|g_2\|_{L^p(\hat{\Omega}_\eta)} \leq \frac{1}{2}\|g_1\|_{L^p(\hat{\Omega}_\eta)} \leq \frac{1}{4}\|g_e\|_{L^p(\hat{\Omega}_\eta)},
\]

(3.26)

\[
\|g_2\|_{L^q(\hat{\Omega}_\eta)} \leq \frac{1}{2}\|g_1\|_{L^q(\hat{\Omega}_\eta)} \leq \frac{1}{4}\|g_e\|_{L^q(\hat{\Omega}_\eta)}
\]

(3.27)

and

\[
\|h_2\|_{L^q(\hat{\Omega}_\eta)} \leq C_4 \left( 1 + \frac{1}{2} \right) \|g_e\|_{L^p(\hat{\Omega}_\eta)} , \quad \|h_1 - h_2\|_{L^q(\hat{\Omega}_\eta)} \leq C_4 \frac{1}{2} \|g_e\|_{L^p(\hat{\Omega}_\eta)}.
\]

(3.28)

We continue the process inductively for \( m > 2 \). This way we construct a sequence \( u_m \in W^{2,q}_b(B_2) \cap C(\overline{B}_2) \) such that \( u_m \) is the unique strong solution of

\[
\begin{cases}
P^+(D^2u_m) + \gamma R|Du_m| = -g_{m-1}(x) & \text{in } B_2, \\
u_m = 0 & \text{on } \partial B_2,
\end{cases}
\]

(3.29)

\[
\|u_m\|_{W^{2,p}(B_2)} \leq C_1 \frac{2m-1}{2m-\epsilon} \|g_e\|_{L^p(\hat{\Omega}_\eta)},
\]

(3.30)

\[
\|u_m\|_{W^{2,q}(B_2)} \leq C_1 \frac{2m-1}{2m-\epsilon} \|g_e\|_{L^q(\hat{\Omega}_\eta)},
\]

(3.31)

(we extend \( u_m \) to \( \mathbb{R}^n \) by setting \( u_m = 0 \) on \( B_2^c \)) and such that the sequence \( v_m = v_{m-1} + u_m \) is a strong subsolution of

\[
P^-(D^2v_m(x)) - \gamma R|Dv_m(x)|
\leq g_m(x) + h_m(x)
\]

(3.32)

in \( \hat{\Omega}_\eta \), where \( g_m, h_m \in L^q(\hat{\Omega}_\eta), \ g_m = h_m = 0 \) on \( \hat{\Omega}_\eta^c \) and

\[
\|g_m\|_{L^p(\hat{\Omega}_\eta)} \leq \frac{1}{2m-\epsilon} \|g_e\|_{L^p(\hat{\Omega}_\eta)},
\]

(3.33)

\[
\|g_m\|_{L^q(\hat{\Omega}_\eta)} \leq \frac{1}{2m-\epsilon} \|g_e\|_{L^q(\hat{\Omega}_\eta)},
\]

(3.34)

\[
\|h_m\|_{L^q(\hat{\Omega}_\eta)} \leq C_4 \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2m-1} \right) \|g_e\|_{L^p(\hat{\Omega}_\eta)} , \quad \|h_m - h_{m-1}\|_{L^q(\hat{\Omega}_\eta)} \leq \frac{C_4}{2m-1} \|g_e\|_{L^p(\hat{\Omega}_\eta)}.
\]

(3.35)

Step 3. Passage to the limit. It follows from (3.30) and (3.31) that there exists \( w \in W^{2,q}(B_2) \cap C_0(\mathbb{R}^n), \ w = v_\epsilon \) on \( B_2^c \), such that \( v_m \to w \) uniformly in \( \mathbb{R}^n \) as \( m \to +\infty \). Moreover,
\[ \|v_m - w\|_{W^{2,q}(B_{\frac{1}{2}})} \to 0 \text{ as } m \to +\infty. \] In particular this also implies that \( Dv_m \to Dw \) uniformly in \( \tilde{\Omega}_\eta \) as \( m \to +\infty. \)

We need to pass to the limit in (3.32) as \( m \to +\infty. \) Obviously, up to a subsequence, we have \( P^{-}(D^2v_m(x)) - \gamma R|Dv_m(x)| \to P^{-}(D^2w(x)) - \gamma R|Dw(x)| \) and \( g_m(x) \to 0 \) for a.e. \( x \in \tilde{\Omega}_\eta. \)

Also, as in Step 1,

\[
\begin{align*}
&\sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} \left[ v_m(x + z) - v_m(x) - \langle Dv_m(x), z \rangle \mathbf{1}_{\{|z| < 1/r\}}(z) \right] M_\alpha(x, z) dz \right. \\
&\left. - \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} \left[ w(x + z) - w(x) - \langle Dw(x), z \rangle \mathbf{1}_{\{|z| < 1/r\}}(z) \right] M_\alpha(x, z) dz \right. \\
&\leq \int_{\mathbb{R}^n} \left[ v_m(x + z) - v_m(x) - \langle Dv_m(x), z \rangle \mathbf{1}_{\{|z| < 1/r\}}(z) \right] M_\alpha(x, z) dz \\
&\quad - \left[ w(x + z) - w(x) - \langle Dw(x), z \rangle \mathbf{1}_{\{|z| < 1/r\}}(z) \right] M_\alpha(x, z) dz \\
&= \int_{B_\delta} + \int_{B_\delta^C} =: I_m(x) + J_m(x).
\end{align*}
\]

Now, arguing as in Step 1, we see that

\[ \|I_m\|_{L^q(\tilde{\Omega}_\eta)} \leq \|v_m - w\|_{W^{2,q}(B_{\frac{1}{2}})} \int_{B_\delta} |z|^2 K_1(z) dz \to 0 \quad \text{as } m \to +\infty, \]

and hence, up to a further subsequence, \( I_m(x) \to 0 \) for a.e. \( x \in \tilde{\Omega}_\eta. \) Moreover, in light of the uniform convergence of \( v_m \) to \( w \) in \( \mathbb{R}^n \) and the uniform convergence of \( Dv_m \) to \( Dw \) in \( \tilde{\Omega}_\eta, \) we obviously have that \( J_m \) converges uniformly to 0 in \( \tilde{\Omega}_\eta \) as \( m \to +\infty. \)

Also, using (3.34) and (3.35), we see that, up to another subsequence, the right-hand side of (3.32) converges for a.e. \( x \in \tilde{\Omega}_\eta \) to a function \( h \in L^q(\tilde{\Omega}_\eta) \) such that

\[ \|h\|_{L^q(\tilde{\Omega}_\eta)} \leq 2C_4 \|g_\ell\|_{L^p(\tilde{\Omega}_\eta)}. \]  

Therefore, passing to the limit in (3.32) as \( m \to +\infty \) we obtain that \( w \) is a strong subsolution of

\[ P^{-}(D^2w(x)) - \gamma R|Dw(x)| - \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} \left[ w(x + z) - w(x) - \langle Dw(x), z \rangle \mathbf{1}_{\{|z| < 1/r\}}(z) \right] M_\alpha(x, z) dz \right. \]

\[ \leq h(x) \quad \text{in } \tilde{\Omega}_\eta. \]  

**Step 4. Conclusion.** Applying the ABP maximum principle from Theorem 3.1 and using (3.36) we obtain

\[ \sup_{\tilde{\Omega}_\eta} w \leq \sup_{\tilde{\Omega}_\eta} w + C \|h\|_{L^q(\tilde{\Omega}_\eta)} \leq \sup_{\tilde{\Omega}_\eta} w + 2CC_4 \|g_\ell\|_{L^p(\tilde{\Omega}_\eta)}. \]

It now remains to use estimates (3.31) and the construction of \( w \) to conclude that

\[ \sup_{\tilde{\Omega}_\eta} v_\ell \leq \sup_{\tilde{\Omega}_\eta} v_\ell + 2C_2 \|g_\ell\|_{L^p(\tilde{\Omega}_\eta)} \sum_{m=1}^{\infty} \frac{1}{2m-1} + 2CC_4 \|g_\ell\|_{L^p(\tilde{\Omega}_\eta)}. \]
Example 3.1. Consider the following Hamilton-Jacobi-Bellman-Isaacs equation

\[
\sup_{\alpha \in A} \inf_{\beta \in B} \left\{ -\text{Tr}(a_{\alpha,\beta}(x)D^2u) + (b_{\alpha,\beta}(x), Du) + f_{\alpha,\beta}(x) - \int_{\mathbb{R}^n} \left[ u(x+z) - u(x) - \langle Du(x), z \rangle 1_{\{||z|| < 1\}}(z) \right] N_{\alpha,\beta}(x,z)dz \right\} = 0 \quad \text{in } \Omega. \tag{3.38}
\]

Assume that $A, B$ are countable, $a_{\alpha,\beta} : \Omega \to \mathbb{S}^n$, $b_{\alpha,\beta} : \Omega \to \mathbb{R}^n$, $f_{\alpha,\beta} : \Omega \to \mathbb{R}$ are measurable for every $\alpha \in A$, $\beta \in B$, $\lambda I \leq a_{\alpha,\beta}(x) \leq \Lambda I$ in $\Omega$ for some $0 < \lambda \leq \Lambda$, $|b_{\alpha,\beta}(x)| \leq \gamma$ in $\Omega$ for some $\gamma \geq 0$ and $g := \sup_{\alpha \in A, \beta \in B} |f_{\alpha,\beta}| \in L^p(\Omega)$. Suppose moreover that the functions $N_{\alpha,\beta}$ satisfy $(A1)$ uniformly for $\alpha \in A, \beta \in B$. Then if $u \in W^{2,\beta}_{\text{loc}}(\Omega) \cap C_b(\mathbb{R}^n)$ is a strong subsolution of (3.38), $u$ is a strong subsolution of the extremal integro-PDE (1.6) with the index set $A := A \cup B$.

4 Maximum principle: parabolic case

In this section we prove two theorems which contain two nonlocal versions of the ABP maximum principle for uniformly parabolic integro-PDE. The proof of Theorem 4.1 is an adaptation of the proof of Theorem 2.1 of [55] and the proof of Theorem 7.1 of [47]. In fact it follows closer the principle for uniformly parabolic integro-PDE. The proof of Theorem 4.1 is an adaptation of the proof of Theorem 2.1 of [55] and the proof of Theorem 7.1 of [47]. In fact it follows closer the principle for uniformly parabolic integro-PDE.

Theorem 4.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $T > 0$. Suppose that $(A1)$ is satisfied and $f \in L^{n+1}(\Omega)$. Let $R$ be a number such that $\text{diam}(Q) \leq R$. Then there exists a constant $C = C(n, \lambda, \gamma, R, K(\cdot))$ such that if $u \in W^{1,2,n+1}_{\text{loc}}(\Omega) \cap C_b([-T,0] \times \mathbb{R}^n)$ is a strong subsolution of the extremal integro-PDE (1.7) then

\[
\sup_{Q} u \leq \sup_{\partial_p Q} u + C\text{diam}(\Omega)^{\frac{n}{n+1}} \|f^+\|_{L^{n+1}(\Gamma^+_{\text{loc}}(u))}. \tag{4.1}
\]

Proof. We begin by assuming that $u \in C^{1,2}_{\text{loc}}(Q) \cap C_b([-T,0] \times \mathbb{R}^n)$. We will later remove this restriction using approximations. Without loss of generality we can assume that $0 \notin \Omega$. We define for $r > 0$ \[\Gamma_{Q,r}^{n+} := \{(t,x) \in Q : u(t,x) > \sup_{\partial_p Q} u, \exists p, |p| \leq r, \text{ such that } u(s,y) \leq u(t,x) + \langle p, y - x \rangle \text{ for } (s,y) \in [-T,t] \times \tilde{\Omega}_d\}.\]

We set $M := \sup_{Q} u - \sup_{\partial_p Q} u$. By considering $u := u - \sup_{\partial_p Q} u$ we can assume that $\sup_{\partial_p Q} u \leq 0$ and $M = \sup_{Q} u$. We recall that $d = \text{diam}(\Omega)$. Define $r_0 = r_0(u) = \frac{M}{2d}$. If $r_0 \leq 0$ we are done. Otherwise we consider for $0 < r < r_0$ the set \[A^r := \{(p,h) : 2d|p| < h < rd\}\]

and define the function $\Phi(t,x) := (Du(t,x), u(t,x) - \langle x, Du(t,x) \rangle)$.

Arguing as in the proof of Theorem 4.1 it is easy to see that if $(t,x) \in \Gamma_{Q,r}^{n+}(u)$ then

\[u(t,x) \geq 2d(r_0 - r) > 0.\]

and hence $\Gamma_{Q,r}^{n+}(u)$ is a subset of $Q$. Moreover $u_t(t,x) \geq 0$, $Du(t,x) = 0$, $D^2u(t,x) \leq 0$ on $\Gamma_{Q,r}^{n+}(u)$. Now, if $(p,h) \in A^r$ then the function $p(s,y) = h - \langle p, y \rangle$ is positive on $[-T,0] \times \tilde{\Omega}_d$ and
\sup_{(s,y)\in \bar{Q}} p(s,y) < M. Therefore there exists \((t,x) \in Q\) such that \(p(t,x) = u(t,x)\) and \(p(s,y) \geq u(s,y)\) for all \(-T \leq s \leq t, y \in \bar{Q}_d\). Thus we have \(-p = Du(t,x)\) and \(h = u(t,x) - \langle x, Du(t,x) \rangle\), which implies

\[ A_u^r \subset \Phi^r(\Gamma_{Q,r}^+(u)). \]

Moreover, see [47], page 156, the determinant of the Jacobi matrix of \(\Phi\) is equal to \(u_d \det(D^2 u)\).

Hence for \(\kappa \geq 0\), the change of variables \((p, h) = (\Phi(t,x))\) yields

\[
\int_{A_u^r} \left( |p|^{\frac{n+1}{n}} + \kappa \frac{n+1}{n} \right)^{-n} dpdh \leq \int_{\Gamma_{Q,r}^+(u)} \left( |Du|^{\frac{n+1}{n}} + \kappa \frac{n+1}{n} \right)^{-n} u_d \det(-D^2 u) dxdt
\]

\[
\leq \frac{1}{\lambda^n} \int_{\Gamma_{Q,r}^+(u)} \left( |Du|^{\frac{n+1}{n}} + \kappa \frac{n+1}{n} \right)^{-n} \left( u_t - \lambda \text{Tr}(D^2 u)^{n+1} \right) dxdt,
\]

where we have used that \((n+1)\lambda^n u_d \det(-D^2 u) \leq (u_t - \lambda \text{Tr}(D^2 u))^{n+1} \) on \(\Gamma_{Q,r}^+(u)\).

Since \(D^2 u \leq 0\) on \(\Gamma_{Q,r}^+(u)\) and (1.7) holds, we have

\[
u_t(t,x) - \lambda \text{Tr}(D^2 u(t,x)) - \gamma |Du(t,x)|
\]

\[-\sup_{\alpha \in \mathbb{A}} \left\{ \int_{\mathbb{R}^n} |u(t,x+z) - u(t,x) - \langle Du(t,x), z \rangle| \right\} N_\alpha(t,x,z) du \leq f^+(t,x)
\]

in \(\Gamma_{Q,r}^+(u)\). Arguing the same as in the proof of Theorem 3.1 to obtain (3.5), we have

\[
\int_{\mathbb{R}^n} |u(t,x+z) - u(t,x) - \langle Du(t,x), z \rangle| N_\alpha(t,x,z) du \leq C_0(1 + d^{-1}) |Du(t,x)|
\]

where \(C_0\) depends on \(R\) and \(K(\cdot)\). We thus see that

\[
u_t(t,x) - \lambda \text{Tr}(D^2 u(t,x)) - [\gamma + C_0(1 + d^{-1})] |Du(t,x)| \leq f^+(t,x)
\]

in \(\Gamma_{Q,r}^+(u)\).

Now (4.2) implies

\[
\int_{A_u^r} \left( |p|^{\frac{n+1}{n}} + \kappa \frac{n+1}{n} \right)^{-n} dpdh
\]

\[
\leq \frac{1}{(n+1)\lambda^n} \int_{\Gamma_{Q,r}^+(u)} \left( |Du|^{\frac{n+1}{n}} + \kappa \frac{n+1}{n} \right)^{-n} \left\{ \left[ \gamma + C_0(1 + d^{-1}) \right] |Du| + f^+ \right\}^{n+1} \frac{dxdt}{\kappa^{n+1}}
\]

\[
\leq \frac{1}{(n+1)\lambda^n} \int_{\Gamma_{Q,r}^+(u)} \left( |Du|^{\frac{n+1}{n}} + \kappa \frac{n+1}{n} \right)^{-n} \left\{ \left[ \gamma + C_0(1 + d^{-1}) \right]^{n+1} + \frac{(f^+)^{n+1}}{\kappa^{n+1}} \right\} \frac{dxdt}{\kappa^{n+1}}
\]

\[
\leq \frac{1}{(n+1)\lambda^n} \int_{\Gamma_{Q,r}^+(u)} \left\{ \left[ \gamma + C_0(1 + d^{-1}) \right]^{n+1} + \frac{(f^+)^{n+1}}{\kappa^{n+1}} \right\} \frac{dxdt}{\kappa^{n+1}}
\]

Since

\[
\left( |p|^{\frac{n+1}{n}} + \kappa \frac{n+1}{n} \right)^{-n} \geq 2^{1-n} \left( |p|^n + \kappa^n \right)^{-\frac{n+1}{n}},
\]

it follows that

\[
\int_{A_u^r} \left( |p|^{\frac{n+1}{n}} + \kappa \frac{n+1}{n} \right)^{-n} dpdh \geq 2^{1-n} n\omega_n \int_0^r \int_0^{\frac{h}{2d}} \frac{s^{n-1}}{(s^n + \kappa^n)^{\frac{n+1}{n}}} dsdh
\]

\[
\geq 2^{1-n} n\omega_n \int_0^r \left( \frac{1}{\kappa} - \left( \frac{h}{2d} \right)^n + \kappa^n \right)^{-\frac{1}{n}} dh.
\]
If \( r \geq 4\kappa \) then, since \( h > 2d\kappa \) implies \( \left( \frac{h}{2d} \right)^n + \kappa^n \leq 2^{-\frac{1}{n}}\kappa^{-1} \), we get
\[
\int_{A_n} \left( \|p\|^{\frac{n+1}{n}} + \kappa^{\frac{n+1}{n}} \right)^{-n} d\rho dh \geq C(n) \int_{2d\kappa}^{rd} \frac{1}{\kappa} dh \geq C(n) rd\frac{1}{2\kappa},
\]
where \( C(n) \) is some positive constant depending only on \( n \). Therefore we obtain
\[
r \leq \frac{C_1(n)}{\lambda^n} \int_{\Gamma_{Q,n}} \left\{ \frac{\kappa}{d} \left[ \gamma + C_0(1 + d^{-1}) \right]^{n+1} + \frac{(f^+)^{n+1}}{d\kappa^n} \right\} dx dt + 4\kappa.
\]
If \( \|f^+\|_{L^{n+1}(\Gamma_{Q,n}^+)} = 0 \) then we take \( \kappa \) arbitrarily small to conclude that \( r \leq 0 \). Otherwise we take \( \kappa = d^{-\frac{1}{n+1}}\|f^+\|_{L^{n+1}(\Gamma_{Q,n}^+)} \) to obtain
\[
r \leq \frac{C_1(n)}{\lambda^n} d^{-\frac{1}{n+1}} \left[ \int_{\Gamma_{Q,n}^+} \frac{1}{d} \left[ \gamma + C_0(1 + d^{-1}) \right]^{n+1} \right] dx dt + 5 \frac{\|f^+\|_{L^{n+1}(\Gamma_{Q,n}^+)}}. \tag{4.3}
\]
We observe that
\[
\int_{\Gamma_{Q,n}^+} \frac{1}{d} \left[ \gamma + C_0(1 + d^{-1}) \right]^{n+1} dx dt
\]
is bounded by a constant which only depends on \( n, \gamma, R \) and \( K(\cdot) \).

If \( u \in W^{1,2,n+1}_{\text{loc}}(Q) \cap C_b([-T,0] \times \mathbb{R}^n) \) we repeat the approximation argument used in the proof of Theorem 3.1 (see also Step 3 of Theorem 4.2). We take functions \( u_\epsilon \in C^2([-T,0] \times \mathbb{R}^n) \cap C_b([-T,0] \times \mathbb{R}^n) \) converging, as \( \epsilon \to 0 \), to \( u \) in \( W^{1,2,n+1}_{\text{loc}}(Q) \) and uniformly in \([-T,0] \times \mathbb{R}^n \). Arguing similarly as in the elliptic case, the functions \( u_\epsilon \) are classical solutions, in every \((-T + \eta,0] \times \Omega_\eta, \eta > 0\), for sufficiently small \( \epsilon \), of
\[
(u_\epsilon)_{t} + \mathcal{P}^-(D^2 u_\epsilon) - \gamma |Du_\epsilon|
- \sup_{a \in A} \left\{ \int_{\mathbb{R}^n} [u_\epsilon(t,x+z) - u_\epsilon(t,x) - (Du_\epsilon(t,x),z)1_{\{|z|<1\}}(z)] N_\alpha(t,x,z)dz \right\} \leq f_\epsilon(t,x),
\]
where \( \|f_\epsilon - f^+\|_{L^{n+1}(-T+\eta,0] \times \Omega_\eta} \to 0 \) as \( \epsilon \to 0 \). Moreover if \( r < r_0(u) \), then for sufficiently small \( \eta \) and \( \epsilon \), we have \( r < r_0(u_\epsilon) \) and \( \Gamma_{Q,r}^{n+1}(u_\epsilon) \subset (-T + \eta,0] \times \Omega_\eta \). We then get (4.3) for \( u_\epsilon \) and pass to the limit, along a subsequence \( \epsilon_j \to 0 \), in (4.3)
\[
\limsup_{j \to +\infty} \Gamma_{Q,r}^{n+1}(u_{\epsilon_j}) \subset \Gamma_{Q,r}^{n+1}(u),
\]
which can be proved as in the elliptic case. It then remains to let \( r \to r_0 \).

\[\Box\]

**Theorem 4.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n, T > 0 \) and let \( p_1 < p < n + 1 \), where \( p_1 \) is from Theorem 2.3. Suppose that (A1) is satisfied and \( f \in L^p(Q) \). Let \( R \) be a number such that \( \text{diam}(Q) \leq R \). Then there exists a constant \( C = C(n,p,T,\lambda,\Lambda,\gamma,R,K(\cdot)) \) such that if \( u \in W^{1,2,p}_{\text{loc}}(Q) \cap C_b([-T,0] \times \mathbb{R}^n) \) is a strong subsolution of the extremal integro-PDE (1.7) then
\[
\sup_{Q} u \leq \sup_{\partial_p Q} u + C(\text{diam}(Q))^{2 - \frac{n+2}{p}} \|f^+\|_{L^p(Q)} \tag{4.4}
\]
Proof. The proof is similar to the proof of Theorem 3.2. However we present most of the details even though we will often repeat the arguments from the proof of Theorem 3.2.

Step 1. Rescaling. As in the proof for the elliptic case we first rescale equation (1.7) to the equation in a domain with unit diameter. Without loss of generality we assume that $0 \in \Omega$. We denote $r := \text{diam}(Q)$ and $Q_1 := (\frac{-r}{2}, 0) \times \frac{1}{r} \Omega$. We set

$$v(x) := u(r^2t, rx)$$

and

$$M_\alpha(t, x, z) := r^{\alpha + 2} N_\alpha(r^2t, rx, rz).$$

Then

$$0 \leq M_\alpha(t, x, z) \leq r^{\alpha + 2} K(rz) = K_1(z).$$

It is easy to see that $v \in W^{1,2,p}_{\text{loc}}(Q_1) \cap C([\frac{-r}{2}, 0] \times \mathbb{R}^n)$ is a strong subsolution of the rescaled extremal integro-PDE

$$v_t + \mathcal{P}^{-}(D^2v) - \gamma R |Dv| - \sup_{\alpha \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \left[ v(t, x + z) - v(t, x) - (Dv(t, x), z) 1_{\{|z| < 1/r\}}(z) \right] M_\alpha(t, x, z) dz \right\} \leq g(t, x) := r^2 f^+(r^2 t, rx) \quad \text{in } Q_1. \quad (4.5)$$

We notice that

$$\|g\|_{L^p(Q_1)} = \left( \text{diam}(Q) \right)^{2 - \frac{2\alpha + n}{p}} \|f^+\|_{L^p(Q)}.$$ 

By mollification and approximation by interior domains as in Step 1 of the proof of Theorem 3.2, without loss of generality we can assume that $v \in C^{1,2}([\frac{-r}{2}, 0] \times \mathbb{R}^n) \cap C_b([\frac{-r}{2}, 0] \times \mathbb{R}^n)$ and $g \in L^q(Q_1)$ for every $q < +\infty$. In the rest of the proof we set $q = p^*$, where $p^* = (n + 2)p/(n + 2 - p)$ is the parabolic Sobolev conjugate of $p$.

Let $0 < \delta < \min(\frac{1}{2}, \frac{1}{R})$ be such that

$$\eta := \int_{B_{R\delta}} |z|^2 K(z) dz \leq \min\left( \frac{1}{2C_1}, \frac{1}{2C_2} \right),$$

where $C_1 := \max\left( C_1(n, p, \lambda, \gamma R), C_1(n, q, \lambda, \gamma R) \right)$ is the maximum of the two constants from (2.8) for the two exponents $p$ and $q$, and $C_2 = C_2(n, p, \lambda, \gamma R)$ is the constant from (2.9). We also have (3.12), (3.13) and (3.14).

Step 2. Iteration. Let $u_1 \in W^{1,2,q}_{\text{loc}}((-2, 0] \times B_2) \cap C([-2, 0] \times \overline{B_2})$ be the unique strong solution of

$$\begin{cases}
(u_1)_t + \mathcal{P}^+(D^2u_1) + \gamma R |Du_1| = -g(t, x) & \text{in } (-2, 0] \times B_2, \\
u_1 = 0 & \text{on } \partial p((-2, 0] \times B_2),
\end{cases}$$

where we extended $g$ by 0 outside of $Q_1$. By Theorem 2.4 we have

$$\|u_1\|_{W^{1,2,p}((-2, 0] \times B_2)} \leq C_1 \|g\|_{L^p(Q_1)}, \quad (4.6)$$

$$\|u_1\|_{W^{1,2,q}((-2, 0] \times B_2)} \leq C_1 \|g\|_{L^q(Q_1)}, \quad (4.7)$$

and

$$\sup_{(-2, 0] \times B_2} |u_1| \leq C_2 \|g\|_{L^p(Q_1)}. \quad (4.8)$$

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We extend \( u_1 \) to a function on \([-2, 0] \times \mathbb{R}^n \) by setting \( u_1 = 0 \) on \([-2, 0] \times B_2^c \). Then the function \( v_1 = v + u_1 \in W^{1,2,q}((-2, 0] \times B_2^c) \cap C([-2, 0] \times \mathbb{R}^n) \) satisfies in \( Q_1 \)
\[
(v_1)_t(t, x) + \mathcal{P}^- (D^2 v_1(t, x)) - \gamma R |Dv_1(t, x)| - \sup_{a \in A} \left\{ \int_{\mathbb{R}^n} \left[ v_1(t, x + z) - v_1(t, x) - \langle Du_1(t, x), z \rangle 1_{\{|z| < 1 / r\}}(z) \right] M_\alpha(t, x, z) d\nu \right\}
\leq \int_{\mathbb{R}^n} |u_1(t, x + z) - u_1(t, x) - \langle Du_1(t, x), z \rangle 1_{\{|z| < 1 / r\}}(z) | K_1(z) dz
= \int_{B_3} + \int_{B_3^c} =: g_1(t, x) + h_1(t, x). \tag{4.9}
\]

Using Lemmas 2.1, 2.2 and (4.6) we now have
\[
\|g_1\|_{L^p(Q_1)} \leq \left( \int_{Q_1} \left( \int_{B_3} \int_0^1 \int_0^1 \|D^2 u_1(t, x + krz)\| |z|^2 K_1(z) d\nu dr dz \right)^p dx dt \right)^{1/p}
\leq \int_{B_3} \int_0^1 \int_0^1 \left( \int_{Q_1} \|D^2 u_1(t, x + krz)\|^p dx dt \right)^{1/p} |z|^2 K_1(z) d\nu dr dz
\leq \|u_1\|_{W^{1,2,p}((-\frac{3}{2}, 0] \times B_2^c)} \int_{B_3} |z|^2 K_1(z) dz = \eta\|u_1\|_{W^{1,2,p}((-\frac{3}{2}, 0] \times B_2^c)} \leq \frac{1}{2} \|g\|_{L^p(Q_1)}. \tag{4.10}
\]

Replacing \( p \) by \( q \) above and using (4.7) gives
\[
\|g_1\|_{L^q(Q_1)} \leq \frac{1}{2} \|g\|_{L^q(Q_1)}. \tag{4.11}
\]

By (3.13), (3.14), (4.6), (4.8) and the imbedding theorem for anisotropic Sobolev spaces (see [8], Theorem 10.2), which guarantees that \( \|Du_1\|_{L^q(Q_1)} \) is controlled by \( \|u_1\|_{W^{1,2,p}((-\frac{3}{2}, 0] \times B_2^c)} \), it follows that
\[
\|h_1\|_{L^q(Q_1)} \leq \overline{C}_3 \|g\|_{L^p(Q_1)} \tag{4.12}
\]
for some absolute constant \( \overline{C}_3 = \overline{C}_3(n, p, \lambda, \Lambda, \gamma, R, K(\cdot)) \).

We now extend \( g_1, h_1 \) by 0 outside of \( Q_1 \). Let \( u_2 \in W^{1,2,p}_{\text{loc}}((-2, 0] \times B_2) \cap C([-2, 0] \times \overline{B}_2) \) be the unique strong solution of
\[
\begin{cases}
(u_2)_t + \mathcal{P}^+ (D^2 u_2) + \gamma R |Du_2| = -g_1(t, x) & \text{in } (-2, 0] \times B_2, \\
u_2 = 0 & \text{on } \partial_p((-2, 0] \times B_2).
\end{cases}
\]

By (2.8), (2.9), (4.10) and (4.11) we have
\[
\|u_2\|_{W^{1,2,p}((-\frac{3}{2}, 0] \times B_2^c)} \leq \overline{C}_1 \|g_1\|_{L^p(Q_1)} \leq \overline{C}_1 \|g\|_{L^p(Q_1)}, \tag{4.13}
\]
\[
\|u_2\|_{W^{1,2,q}((-\frac{3}{2}, 0] \times B_2^c)} \leq \overline{C}_1 \|g_1\|_{L^q(Q_1)} \leq \frac{\overline{C}_1}{2} \|g\|_{L^q(Q_1)}, \tag{4.14}
\]
and
\[
\sup_{(-2, 0] \times B_2} |u_2| \leq \overline{C}_2 \|g_1\|_{L^p(Q_1)} \leq \frac{\overline{C}_2}{2} \|g\|_{L^p(Q_1)}. \tag{4.15}
\]
We extend \( u_2 \) to a function on \([-2, 0] \times \mathbb{R}^n \) by setting \( u_1 = 0 \) on \([-2, 0] \times B_2^c \). Defining \( v_2 = v_1 + u_2 \) and arguing as for \( v_1 \) we then obtain that \( v_2 \) is a strong subsolution of

\[
(v_2)_t(t, x) + \mathcal{P}^- (D^2 v_2(t, x)) - \gamma R |D v_2(t, x)| \\
- \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} [v_2(t, x + z) - v_2(t, x) - \langle D v_2(t, x), z \rangle 1_{\{z < 1/r\}}(z)] M_\alpha(t, x, z) dz \right\}
\leq g_2(t, x) + h_2(t, x)
\]

(4.16)

in \( Q_1 \), where

\[
\|g_2\|_{L^q(Q_1)} \leq \frac{1}{2}\|g_1\|_{L^q(Q_1)} \leq \frac{1}{4}\|g\|_{L^q(Q_1)},
\]

(4.17)

\[
\|g_2\|_{L^p(Q_1)} \leq \frac{1}{2}\|g_1\|_{L^p(Q_1)} \leq \frac{1}{4}\|g\|_{L^p(Q_1)}
\]

(4.18)

and

\[
\|h_2\|_{L^q(Q_1)} \leq C_3 \left( 1 + \frac{1}{2} \right) \|g\|_{L^p(Q_1)}, \quad \|h_1 - h_2\|_{L^q(Q_1)} \leq \frac{C_3}{2} \|g\|_{L^p(Q_1)}.
\]

(4.19)

We continue the process inductively for \( m > 2 \). This way we construct a sequence \( u_m \in W^{1,2, p}_{loc}([-2, 0] \times B_2) \cap C([-2, 0] \times \overline{B_2}) \) of the unique strong solutions of

\[
\begin{cases}
(u_m)_t + \mathcal{P}^+ (D^2 u_m) + \gamma R |D u_m| = -g_{m-1}(t, x) & \text{in } (-2, 0] \times B_2, \\
u_m = 0 & \text{on } \partial_p ((-2, 0] \times B_2),
\end{cases}
\]

such that

\[
\|u_m\|_{W^{1,2, p}((-2, 0] \times B_2)} \leq \frac{C_1}{2^{m-1}} \|g\|_{L^p(Q_1)},
\]

(4.20)

\[
\|u_m\|_{W^{1,2, q}((-2, 0] \times B_2)} \leq \frac{C_1}{2^{m-1}} \|g\|_{L^q(Q_1)},
\]

(4.21)

\[
\sup_{(-2, 0] \times B_2} |u_m| \leq \frac{C_2}{2^{m-1}} \|g\|_{L^p(Q_1)}
\]

(4.22)

(we extend \( u_m \) to \([-2, 0] \times \mathbb{R}^n \) by setting \( u_m = 0 \) on \([-2, 0] \times B_2^c \) and such that the sequence \( v_m = v_{m-1} + u_m \) is a strong subsolution of

\[
(v_m)_t(t, x) + \mathcal{P}^- (D^2 v_m(t, x)) - \gamma R |D v_m(t, x)| \\
- \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} [v_m(t, x + z) - v_m(t, x) - \langle D v_m(t, x), z \rangle 1_{\{z < 1/r\}}(z)] M_\alpha(t, x, z) dz \right\}
\leq g_m(t, x) + h_m(t, x)
\]

(4.23)

in \( Q_1 \), where \( g_m, h_m \in L^q(Q_1), g_m = h_m = 0 \) outside of \( Q_1 \),

\[
\|g_m\|_{L^p(Q_1)} \leq \frac{1}{2^m}\|g\|_{L^p(Q_1)}
\]

(4.24)

\[
\|g_m\|_{L^q(Q_1)} \leq \frac{1}{2^m}\|g\|_{L^q(Q_1)}
\]

(4.25)

\[
\|h_m\|_{L^q(Q_1)} \leq C_3 \left( 1 + \frac{1}{2} + \ldots + \frac{1}{2^{m-1}} \right) \|g\|_{L^p(Q_1)}, \quad \|h_m - h_{m-1}\|_{L^q(Q_1)} \leq \frac{C_3}{2^{m-1}} \|g\|_{L^p(Q_1)}.
\]

(4.26)
Step 3. Passage to the limit. It follows from (4.21) and (4.22) that there exists \( w \in W^{1,2,q}_{1/2}((-T/2,0) \times B_2) \cap C_b((-T/2,0) \times \mathbb{R}^n) \), \( w = v \) on \( [-T/2,0] \times B_2^c \) such that \( v_m \to w \) uniformly in \( [-T/2,0] \times \mathbb{R}^n \) as \( m \to +\infty \) and \( \|v_m - w\|_{W^{1,2,q}_{1/2}((-T/2,0) \times B_2)} \to 0 \) as \( m \to +\infty \). In particular, by the imbedding theorem, see e.g. [46], Lemma 3.3, p. 80, this implies that \( Dw_m \to Dw \) uniformly in \( Q_1 \) as \( m \to +\infty \).

We need to pass to the limit in (4.23) as \( m \to +\infty \). Obviously, up to a subsequence, we have \( (v_m)_t(t,x) + \mathcal{P}^- (D^2v_m(t,x)) - \gamma R|Dv_m(t,x)| \to w_t(t,x) + \mathcal{P}^- (D^2w(t,x)) - \gamma R|Dw(t,x)| \) for a.e. \( (t,x) \in Q_1 \). Also

\[
\left| \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} \left[ v_m(t,x + z) - v_m(t,x) - \langle Dv_m(t,x), z \rangle 1_{\{|z| < 1/\rho\}}(z) \right] M_\alpha(t,x,z)dz \right\} \right|
\]

\[
- \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} \left[ w(t,x + z) - w(t,x) - \langle Dw(t,x), z \rangle 1_{\{|z| < 1/\rho\}}(z) \right] M_\alpha(t,x,z)dz \right\}
\]

\[
\leq \int_{\mathbb{R}^n} \left| v_m(t,x + z) - v_m(t,x) - \langle Dv_m(t,x), z \rangle 1_{\{|z| < 1/\rho\}}(z) \right| dz
\]

\[
- \left| w(t,x + z) - w(t,x) - \langle Dw(t,x), z \rangle 1_{\{|z| < 1/\rho\}}(z) \right| dz \leq m(t,x) + J_m(t,x).
\]

Now

\[
\|I_m\|_{L^q(Q_1)} \leq \left( \int_{Q_1} \left( \int_{B_\delta} \left( \int_0^1 \left( \int_0^1 \|D^2v_m(t,x + \kappa rz) - D^2w(x + \kappa rz)\|^2 |z| K_1(z) dz \right) dr \right) dt \right) \right)^{\frac{1}{q}}
\]

\[
\leq \int_{B_\delta} \int_0^1 \left( \int_0^1 \left( \int_{Q_1} \|D^2v_m(t,x + \kappa rz) - D^2w(x + \kappa rz)\|^2 |z|^2 K_1(z) dz \right) \right) \frac{1}{q} dz
\]

\[
\leq \|v_m - w\|_{W^{1,2,q}_{1/2}((-T/2,0) \times B_2)} \int_{B_\delta} |z|^2 K_1(z) dz \to 0 \quad \text{as} \quad m \to +\infty,
\]

and hence, up to a further subsequence, \( I_m(t,x) \to 0 \) for a.e. \( (t,x) \in Q_1 \). Moreover, in light of the uniform convergence of \( v_m \) to \( w \) in \( [-T/2,0] \times \mathbb{R}^n \) and the uniform convergence of \( Dw_m \) to \( Dw \) in \( Q_1 \), we obviously have that \( J_m \) converges uniformly to 0 on \( Q_1 \) as \( m \to +\infty \). In addition, using (4.25) and (4.26), there exist a function \( h \in L^q(Q_1) \) such that, up to another subsequence, \( g_m(t,x) + h_m(t,x) \to h(t,x) \) for a.e. \( (t,x) \in Q_1 \) and

\[
\|h\|_{L^q(Q_1)} \leq 2\overline{C}_3 \|g\|_{L^p(Q_1)}.
\]

(4.27)

Therefore, passing to the limit in (4.23) as \( m \to +\infty \) we obtain that \( w \) is a strong subsolution of

\[
w_t(t,x) + \mathcal{P}^- (D^2w(t,x)) - \gamma R|Dw(t,x)|
\]

\[
- \sup_{\alpha \in A} \left\{ \int_{\mathbb{R}^n} \left[ w(t,x + z) - w(t,x) - \langle Dw(t,x), z \rangle 1_{\{|z| < 1/\rho\}}(z) \right] M_\alpha(t,x,z)dz \right\}
\]

\[
\leq h(t,x) \quad \text{in} \quad Q_1.
\]

(4.28)

Step 4. Conclusion. Applying the ABP maximum principle from Theorem 4.1 and using (4.27) we obtain

\[
sup_{Q_1} w \leq \sup_{\partial_{\mu}Q_1} w + C\|h\|_{L^q(Q_1)} \leq \sup_{\partial_{\mu}Q_1} w + 2\overline{C}_3 \|g\|_{L^p(Q_1)}
\]

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which, using estimates (4.22) and the construction of \(w\), yields

\[
\sup_{Q_1} v \leq \sup_{\partial_p Q_1} v + 2C_2\|g\|_{L^p(Q_1)} \sum_{m=1}^{\infty} \frac{1}{2^{m-1}} + 2C_3\|g\|_{L^p(Q_1)}.
\]

References


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