

# THE QUANTUM FLOW OF WEIGHTS ON A VON NEUMANN ALGEBRA

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*This work is respectfully dedicated to the memory of Y. Misonou*

ABSTRACT. It has long been a part of the “folklore” associated with the theory of Type III von Neumann algebras that the (Type  $\text{II}_\infty$ ) core which arises in the crossed product decomposition of an arbitrary Type III von Neumann algebra  $\mathcal{M}$  is, in fact, a canonical object. This work demonstrates that it is possible to construct the core  $\widetilde{\mathcal{M}}$  in such a way that it no longer depends on one’s choice of weight. Moreover, we demonstrate that the assignment  $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$  is in fact functorial. This, in turn, leads to a natural extension of the Flow of Weights, this time applicable to the whole core (and not just its center), which we call the Quantum Flow of Weights.

Additionally, we derive a notion of  $L^p$ -spaces which can also be associated to each von Neumann algebra. These agree with all the standard definitions of non-commutative  $L^p$ -spaces which appear in the literature. Finally, the naturality of the core construction can be used to enhance and simplify the definition of the characteristic square of a factor, and to introduce the notion of local characteristic square. This enables us to answer completely a longstanding open question, first posed in the original work on the flow of weights by Connes and Takesaki, concerning the extended modular automorphisms.

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### §0. Introduction

In a recent work of Katayama, Sutherland and Takesaki [KtST], it was shown that to every factor  $\mathcal{M}$  there corresponds canonically a nine term exact square of groups, called the *characteristic square* of  $\mathcal{M}$

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathfrak{U}(\mathcal{C}) & \xrightarrow{\partial} & B_{\theta}^1(\mathfrak{U}(\mathcal{C})) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathfrak{U}(\mathcal{M}) & \longrightarrow & \tilde{\mathfrak{U}}(\mathcal{M}) & \xrightarrow{\partial} & Z_{\theta}^1(\mathfrak{U}(\mathcal{C})) \longrightarrow 1 \\
 & & \downarrow \text{Ad} & & \downarrow \tilde{\text{Ad}} & & \downarrow \\
 1 & \longrightarrow & \text{Int}(\mathcal{M}) & \longrightarrow & \text{Cnt}_r(\mathcal{M}) & \xrightarrow{\nu} & H_{\theta}^1(\mathfrak{U}(\mathcal{C})) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

The construction of the characteristic square depends heavily on the functoriality of the core  $\{\tilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\}$  of a factor  $\mathcal{M}$ . Unfortunately, the construction presented therein of a functorial core was rather convoluted. But the beauty of the above characteristic square, and its usefulness, make us wonder if there is a more intrinsic way to associate a core to each factor. The main purpose of this paper is to establish a *natural* construction of the core.

The theory of von Neumann algebras is often viewed as a non-commutative extension of measure theory. A key ingredient of non-commutative integration is modular theory, which yields the modular automorphism groups. The modular condition can then be summarized by means of

$$\varphi(xy) = \varphi(y\sigma_{-i}^{\varphi}(x)) \quad (0.1)$$

for a faithful semi-finite normal weight  $\varphi$  on  $\mathcal{M}$  and sufficiently many  $x$  and  $y$  in  $\mathcal{M}$ . Further, we adopt a notation of physicists by writing  $\langle \varphi x \rangle$  for  $\varphi(x)$ , and accept the convention that  $\varphi^{\alpha}x = \sigma_{-i\alpha}^{\varphi}(x)\varphi^{\alpha}$  for any  $\alpha \in \mathbb{C}$  and for those  $x \in \mathcal{M}$  that the above expression makes sense. Continuing in this manner, we identify  $\varphi^{\alpha}\psi^{-\alpha}$  with  $(D\varphi : D\psi)_{-i\alpha}$  for any pair of faithful semi-finite normal weights  $\varphi$  and  $\psi$  on  $\mathcal{M}$ ; hence, we may “generalize” (0.1):

$$\langle \varphi_1^{\alpha_1} x_1 \varphi_2^{\alpha_2} x_2 \cdots \varphi_n^{\alpha_n} x_n \rangle = \langle \varphi_k^{\alpha_k} x_k \cdots \varphi_n^{\alpha_n} x_n \varphi_1^{\alpha_1} x_1 \cdots \varphi_{k-1}^{\alpha_{k-1}} x_{k-1} \rangle \quad (0.2)$$

Here,  $\sum_{i=1}^n \alpha_i = 1$ ,  $\{\varphi_1, \dots, \varphi_n\}$  is an  $n$ -tuple of faithful semi-finite normal weights on  $\mathcal{M}$  and the  $x_i$  are those elements of  $\mathcal{M}$  such that the both sides of the above “make sense.” In the case where  $\varphi_1 = \varphi_2 = \cdots = \varphi_n = \varphi$ , (0.2) is known as the Araki-Miyata multiple

KMS condition, [AM]. This has motivated us to explore further the algebraic system that  $\mathcal{M}$  and the symbols  $\varphi^\alpha, \alpha \in \mathbb{C}$ , generate, a subject which has also been investigated by S. Yamagami [Y].

In real analysis, it has long been known that, for  $p, q, r \geq 1$  with  $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ , if  $f \in L^p(X, \mu)$  and  $g \in L^q(X, \mu)$  (with  $\{X, \mu\}$  a  $\sigma$ -finite measure space), then  $fg \in L^r(X, \mu)$  and

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

If we set  $\alpha = \frac{1}{p}, \beta = \frac{1}{q}$  and write  $M^\alpha, M^\beta$  and  $M^{\alpha+\beta}$  for  $L^p, L^q$  and  $L^r$ , then the above can be rephrased as

$$M^\alpha M^\beta \subset M^{\alpha+\beta} \text{ and } \|fg\|_{\alpha+\beta} \leq \|f\|_\alpha \|g\|_\beta$$

provided  $0 \leq \alpha, \beta, \alpha + \beta \leq 1$ . This serves as a kind of a “grading” of the algebra. The theory of  $L^p$ -spaces has been successfully generalized to the non-commutative setting by several authors. But the most relevant to us is the canonical construction of  $L^p$ -spaces associated with any ( $\sigma$ -finite) von Neumann algebra  $\mathcal{M}$  due to H. Kosaki, [K]. Pursuing the line suggested by the above observation, we write  $M^{\frac{1}{p}}(\mathcal{M})$  for the canonical  $L^p(\mathcal{M})$  of Kosaki and then consider the complexification of the parameter  $\alpha$  to obtain something to be written as  $\mathcal{M}(t)$  which should correspond to the purely imaginary value  $\alpha = it$ . It turns out that each  $\mathcal{M}(t)$  can be shown to be isometrically isomorphic to  $\mathcal{M}$  as a Banach space and the family  $\{\mathcal{M}(t) : t \in \mathbb{R}\}$  has a multiplicative structure,

$$\mathcal{M}(s)\mathcal{M}(t) \subset \mathcal{M}(s+t), \quad s, t \in \mathbb{R};$$

and a conjugation  $\mathcal{M}(s)^* = \mathcal{M}(-s)$  with  $\mathcal{M}(0) = \mathcal{M}$ . This leads naturally to the construction of an involutive Banach algebra bundle of the kind first introduced by Fell [F]. From this, we can proceed to the “cross-section” von Neumann algebra  $\tilde{\mathcal{M}}$ , i.e., the *core* of  $\mathcal{M}$ . This core turns out to be naturally isomorphic to the crossed product  $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$  of  $\mathcal{M}$  by the modular automorphism group  $\sigma^\varphi$  of any faithful semi-finite normal weight  $\varphi$  on  $\mathcal{M}$ .

In the literature, there have been a number of claims of a canonical construction of the core  $\tilde{\mathcal{M}}$  of a von Neumann algebra  $\mathcal{M}$ . Van Daele, [VD], and Woronowicz, [W], for example, both have made such assertions. However, these have merely been observations which follow from the fact that the crossed product von Neumann algebras  $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R} \cong \mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}$  under a natural isomorphism for any pair  $\varphi, \psi$  of faithful semi-finite normal weights. Every construction in the past had to, at some point, choose a faithful semi-finite normal weight or state. This is *not* the same as the canonical construction. Really, the only canonical construction available so far was the one given by Katayama, Sutherland and Takesaki, and has already been alluded to.

The construction of the canonical core  $\tilde{\mathcal{M}}$  involves the set  $\mathfrak{W}_0(\mathcal{M})$  of all faithful semi-finite normal weights on  $\mathcal{M}$ , as does the flow of weights. The dual action  $\{\theta_t : t \in \mathbb{R}\}$  is then simply the one corresponding to multiplication by the positive scalar  $e^{-s}$ ,  $s \in \mathbb{R}$ , applied to each semi-finite normal weight  $\varphi$  on  $\mathcal{M}$ ; this is instantly recognized as the flow

of weights. The difference is that in the flow of weights the carrier algebra of the flow corresponds to equivalence classes of semi-finite normal weights, i.e., to orbits of semi-finite normal weights under the inner automorphism group  $\text{Int}(\mathcal{M})$ , while in the present case two semi-finite normal weights are not identified but rather connected by the Connes cocycle derivative. This motivates us to call the system  $\{\tilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\}$  the *quantum flow of weights*.

The advantage of the quantum flow of weights construction over the crossed product  $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$  comes from the fact that each faithful semi-finite normal weight appears on the equal footing in the core, i.e., there is nothing involved in switching attention from one semi-finite normal weight to another; all that appears is the notational change from  $\varphi^{it}$  to  $\psi^{it}$ . The information carried by the weight  $\varphi$  is then encoded in the one parameter unitary group  $\{\varphi^{it} : t \in \mathbb{R}\}$  and therefore in the abelian subalgebra  $\mathcal{D}^\varphi = \{\varphi^{it} : t \in \mathbb{R}\}'' \vee \mathcal{C}$ , with  $\mathcal{C}$  the center of the core (i.e., the ‘‘classical’’ carrier algebra of the flow of weights). This in turn gives rise to another (smaller) nine term exact square of abelian groups,  $LC Sq_\varphi$ , to be called the *local characteristic square of  $\varphi$* ; this local square is equivariant for  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  relative to the action of  $\text{Aut}(\mathcal{M})$ . Particularly satisfying is the fact that the middle horizontal short exact sequence splits equivariantly. This allows us to prove, for instance, that the extended modular automorphism  $\sigma_c^\varphi$  can be defined *canonically* for every cocycle  $c \in Z_\theta^1(\mathcal{U}(\mathcal{C}))$  and every  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ , not solely for smooth cocycles or dominant weights (see Theorem 4.2).

### §1. The Bundle Algebra $\mathcal{A}$

We begin by introducing notation. Let  $\mathcal{M}$  be a von Neumann algebra, and let us denote by  $\mathfrak{W}_0(\mathcal{M})$  the set of faithful, normal and semi-finite (fns) weights on  $\mathcal{M}$ . Fix a  $t \in \mathbb{R}$ ; for any  $x \in \mathcal{M}$  and  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ , we consider the expression  $(x, \varphi)_t$ .

Let us define

$$(x, \varphi)_t \sim (y, \psi)_t \iff y = x(D\varphi : D\psi)_t, \quad (1.1)$$

where, of course,  $(D\varphi : D\psi)_t$  means the cocycle derivative. Then (1.1) really does define an equivalence relation amongst such ‘‘symbols’’; the transitivity follows from the ‘‘chain rule’’ for the cocycle derivative. We shall denote a single equivalence class by  $x\varphi^{it}$ , and the set of all such by  $\mathcal{M}(t)$ .

**Proposition 1.1.** *For each  $t \in \mathbb{R}$ ,  $\mathcal{M}(t)$  is a dual Banach space, when equipped with the following structure:*

$$\begin{aligned} (i) \quad & x\varphi^{it} + y\varphi^{it} := (x + y)\varphi^{it} \\ (ii) \quad & \alpha(x\varphi^{it}) := (\alpha x)\varphi^{it} \\ (iii) \quad & \|x\varphi^{it}\| := \|x\|. \end{aligned}$$

Here,  $x, y$  are arbitrary elements from  $\mathcal{M}$ ,  $\alpha \in \mathbb{C}$ , and  $\varphi$  is any element of  $\mathfrak{W}_0(\mathcal{M})$ .

*Proof.* For each  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ , define  $\Psi_{\varphi,t}: \mathcal{M} \times \mathfrak{W}_0(\mathcal{M}) \rightarrow \mathcal{M}$  by

$$\Psi_{\varphi,t}(x, \psi) := x(D\psi : D\varphi)_t.$$

It is clear that  $\Psi_{\varphi,t}(a, \psi_1) = \Psi_{\varphi,t}(b, \psi_2)$  if and only if  $(a, \psi_1)_t \sim (b, \psi_2)_t$ , i.e., if and only if  $a\psi_1^{it} = b\psi_2^{it}$ . Hence,  $\Psi_{\varphi,t}$  induces a bijection between  $\mathcal{M}(t)$  and  $\mathcal{M}$ ; we will write  $\hat{\Psi}_{\varphi,t}: \mathcal{M}(t) \rightarrow \mathcal{M}$  to denote this bijection. In addition, we may use this to “pull back” the dual Banach space structure of  $\mathcal{M}$  onto  $\mathcal{M}(t)$ . However, we need to show that this structure (in particular, the vector space structure) is independent of the choice of  $\varphi$ .

To see this, we first note that

$$\hat{\Psi}_{\varphi,t}^{-1}(\hat{\Psi}_{\varphi,t}(x\varphi^{it}) + \hat{\Psi}_{\varphi,t}(y\varphi^{it})) = (x + y)\varphi^{it},$$

and this agrees with the definition given in the statement of the Proposition. Now, we must verify that this sum did not, in fact, depend upon the choice of the map  $\Psi_{\varphi,t}$ . So, now choose any other  $\psi \in \mathfrak{W}_0(\mathcal{M})$ ; again we compute

$$\begin{aligned} \hat{\Psi}_{\psi,t}^{-1}(\hat{\Psi}_{\psi,t}(x\varphi^{it}) + \hat{\Psi}_{\psi,t}(y\varphi^{it})) &= \hat{\Psi}_{\psi,t}^{-1}(x(D\varphi : D\psi)_t + y(D\varphi : D\psi)_t) \\ &= \hat{\Psi}_{\psi,t}^{-1}((x + y)(D\varphi : D\psi)_t) \\ &= ((x + y)(D\varphi : D\psi)_t)\psi^{it} \\ &= (x + y)\varphi^{it}. \end{aligned}$$

Hence, we see that the induced dual Banach space structure is independent of the choice of map  $\Psi_{\varphi,t}$ , and indeed does agree with the structure introduced in the Proposition’s statement.  $\square$

Now, we will consider the interaction of elements from (possibly) different  $\mathcal{M}(t)$ ’s.

**Proposition 1.2.**

(i) *There exists a  $\mathbb{C}$ -bilinear map  $\mathcal{M}(s) \times \mathcal{M}(t) \rightarrow \mathcal{M}(s + t)$  given by*

$$(x\varphi^{is}, y\varphi^{it}) \mapsto x\sigma_s^\varphi(y)\varphi^{i(s+t)}.$$

*(This map can and should be thought of as multiplication.)*

(ii) *There exists a conjugate-linear map  $\mathcal{M}(t) \rightarrow \mathcal{M}(-t)$  given by*

$$x\varphi^{it} \mapsto \sigma_{-t}^\varphi(x)^*\varphi^{-it}.$$

*(This map can and should be thought of as conjugation — not involution per se, as it does not map  $\mathcal{M}(t)$  back into itself.)*

We will indicate the multiplication map by merely juxtaposing elements, while we will use the standard  $(\cdot)^*$  to indicate conjugation. Notice that we have stated the preceding in the form of a Proposition rather than a Definition; indeed, it is (once again) necessary to check that such operations are well-defined, i.e., independent of the choice of “representative” of  $x\varphi^{it}$ . Most such verifications are done by formulaic manipulation, involving the interaction of the modular automorphism group and the cocycle derivative. These will, for the most part, be omitted — yet, when deemed appropriate, a calculation will be made explicit.

We are now in the position to construct what is sometimes called a Fell’s Bundle, i.e., we consider

$$\mathcal{F} := \coprod_{t \in \mathbb{R}} \mathcal{M}(t)$$

together with the multiplication and conjugation described in the previous Proposition. Upon fixing any  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ , the map  $\mathcal{F} \rightarrow \mathbb{R} \times \mathcal{M}$  given by  $x\varphi^{it} \mapsto (t, x)$  is clearly a bijection. Hence, we may “pull back” any topology we may choose to fix on  $\mathbb{R} \times \mathcal{M}$ . However, we cannot be cavalier about such a choice, lest the topology induced on  $\mathcal{F}$  depend on the choice of  $\varphi$ .

So, we proceed as follows: define

$$\mathcal{F}_r := \{x\varphi^{it} \in \mathcal{F} : \|x\varphi^{it}\| \leq r\}, \quad r > 0.$$

Then we have  $\mathcal{F} = \cup_{r>0} \mathcal{F}_r$ . Once again, by fixing a weight  $\varphi$ , we have a bijection, this time between  $\mathcal{F}_r$  and  $\mathbb{R} \times r\mathcal{S}_{\mathcal{M}}$ , where  $\mathcal{S}_{\mathcal{M}}$  is the unit ball in  $\mathcal{M}$ . By considering  $\mathcal{M}$  with its  $\sigma$ -weak topology, and the corresponding product topology on  $\mathbb{R} \times r\mathcal{S}_{\mathcal{M}}$ , we induce a topology on  $\mathcal{F}_r$ . A priori, the topology we have produced on  $\mathcal{F}_r$  appears to depend on our choice of  $\varphi$ , but this is in fact not so. More precisely, we have

**Proposition 1.3.**

- (i) *Fix any two fns weights  $\varphi$  and  $\psi$  on  $\mathcal{M}$ . Then there exist bijections  $\rho_\varphi: \mathcal{F}_r \rightarrow \mathbb{R} \times r\mathcal{S}_{\mathcal{M}}$  and  $\rho_\psi: \mathcal{F}_r \rightarrow \mathbb{R} \times r\mathcal{S}_{\mathcal{M}}$  given by  $\rho_\varphi(x\varphi^{it}) = (t, x)$ , and  $\rho_\psi(y\psi^{it}) = (t, y)$ , respectively. By considering the product topology on  $\mathbb{R} \times r\mathcal{S}_{\mathcal{M}}$ , when  $\mathcal{M}$  is given the  $\sigma$ -weak topology, we may use these bijections to induce topologies  $\mathcal{T}_r(\varphi)$  and  $\mathcal{T}_r(\psi)$  on  $\mathcal{F}_r$ . Then,  $(\mathcal{F}_r, \mathcal{T}_r(\varphi))$  and  $(\mathcal{F}_r, \mathcal{T}_r(\psi))$  are in fact identical as topological spaces. Hence, the topology induced on  $\mathcal{F}_r$  is independent of our choice of weight.*
- (ii) *In addition, the map  $\rho_\varphi \circ \rho_\psi^{-1}: \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M} \times \mathbb{R}$  is a homeomorphism when  $\mathcal{M} \times \mathbb{R}$  is given the product topology with  $\mathcal{M}$  endowed with the Arens-Mackey topology (i.e., the  $\tau(\mathcal{M}, \mathcal{M}_*)$ -topology). Hence, all topologies induced on  $\mathcal{F}$  by various  $\rho_\varphi$ ,  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ , are equivalent.*

*Proof.* (i) To prove our claim, we show that  $(t_\nu, x_\nu) \rightarrow (t, x)$ , then  $(t_\nu, x_\nu u_{t_\nu}) \rightarrow (t, x u_t)$ , where we have written  $u_t$  for  $(D\varphi : D\psi)_t$ . Hence it is sufficient to show  $\langle x_\nu u_{t_\nu} - x u_t, \omega \rangle \rightarrow 0$  for any  $\omega \in \mathcal{M}_*$ . In fact, we may assume that  $\omega \in \mathcal{M}_*^+$ , since any element of  $\mathcal{M}_*$  may be

written as a sum of four such. We calculate

$$\begin{aligned}
 |\langle x_\nu u_{t_\nu} - x u_t, \omega \rangle| &\leq |\langle x_\nu (u_{t_\nu} - u_t), \omega \rangle| + |\langle (x_\nu - x) u_t, \omega \rangle| \\
 &\leq \omega(x_\nu x_\nu^*)^{\frac{1}{2}} \omega((u_{t_\nu} - u_t)^*(u_{t_\nu} - u_t))^{\frac{1}{2}} + |\langle (x_\nu - x), u_t \omega \rangle| \\
 &\leq r \|\omega\|^{\frac{1}{2}} \omega((u_{t_\nu} - u_t)^*(u_{t_\nu} - u_t))^{\frac{1}{2}} + |\langle (x_\nu - x), u_t \omega \rangle|.
 \end{aligned}$$

Above, we used the fact that  $\|x_\nu\| \leq r$ . Now, we know that  $u_{t_\nu} \rightarrow u_t$   $\sigma$ -strongly (from the general theory of the cocycle derivative); hence the first term above goes to 0. The  $\sigma$ -weak convergence of  $x_\nu \rightarrow x$  forces the second term to 0 as well.

(ii) Let's define  $\Phi := \text{pr}_1 \circ (\rho_\varphi \circ \rho_\psi^{-1})$ , with  $\text{pr}_1$  the standard projection onto the first factor. Now, we note that if  $K$  is any balanced, convex and  $\sigma(\mathcal{M}_*, \mathcal{M})$ -compact subset of  $\mathcal{M}_*$ , and  $I$  a closed, bounded interval in  $\mathbb{R}$ , then  $K' := \cup_{t \in I} (D\varphi : D\psi)_t K$  is also a weakly compact subset of  $\mathcal{M}_*$ . This follows from the fact that it is the continuous image of  $K \times I$  under the map  $(\omega, t) \mapsto (D\varphi : D\psi)_t \omega$ . Moreover, the convex closure  $L := \overline{\text{co}}(K')$  is again balanced, convex and weakly compact [DS].

So, assume again that  $x_\nu \rightarrow x$ , this time in the  $\tau(\mathcal{M}, \mathcal{M}_*)$ -topology, while  $t_\nu \rightarrow t$  in  $\mathbb{R}$ . Without loss of generality, we may assume that this net is contained in a bounded interval  $I$ . We compute

$$\begin{aligned}
 \sup_{\omega \in K} |\langle \Phi(x_\nu, t_\nu) - \Phi(x, t), \omega \rangle| &= \sup_{\omega \in K} |\langle x_\nu (D\varphi : D\psi)_{t_\nu} - x (D\varphi : D\psi)_t, \omega \rangle| \\
 &\leq \sup_{\omega \in K} (|\langle (x_\nu - x), (D\varphi : D\psi)_{t_\nu} \omega \rangle| + |\langle x, ((D\varphi : D\psi)_{t_\nu} - (D\varphi : D\psi)_t) \omega \rangle|) \\
 &\leq \sup_{\omega \in K} (|\langle (x_\nu - x), (D\varphi : D\psi)_{t_\nu} \omega \rangle| + |\langle ((D\varphi : D\psi)_{t_\nu} - (D\varphi : D\psi)_t), \omega x \rangle|) \\
 &\leq \sup_{\mu \in L} |\langle x_\nu - x, \mu \rangle| + \sup_{\mu \in Kx} |\langle (D\varphi : D\psi)_{t_\nu} - (D\varphi : D\psi)_t, \mu \rangle|.
 \end{aligned}$$

Note that both terms in the last expression tend to 0: the first due to the fact that  $L$ , like  $K$ , is also a balanced, convex and weakly compact subset of  $\mathcal{M}_*$ , and the second due to the fact that the cocycle derivative is a  $\sigma^*$ -strong-continuous map, and we recall that the  $\sigma^*$ -strong- and the Arens-Mackey topologies agree on bounded subsets of  $\mathcal{M}$  [T1].  $\square$

Similar arguments allow us to conclude that, when we consider, in lieu of the  $\sigma$ -weak topology, the  $\sigma$ -strong or  $\sigma^*$ -strong topology on  $\mathcal{M}$ , the resulting topologies on  $\mathcal{F}_r$  are also independent of the choice of weight. We may use these results to topologize all of  $\mathcal{F}$  in a weight-independent manner: we define a set  $U \subset \mathcal{F}$  to be *limit  $\sigma$ -weakly open* (resp., *limit  $\sigma$ -strongly open*, *limit  $\sigma^*$ -strong open*) if  $U \cap \mathcal{F}_r$  is open for all  $r > 0$ , when  $\mathcal{F}_r$  is given the appropriate topology. These limit topologies on  $\mathcal{F}$  are also clearly independent of any choice of weight. We note that the same kind of construction fails if we try to pull back the product topology on  $\mathbb{R} \times r\mathcal{S}_{\mathcal{M}}$  when  $\mathcal{M}$  is given the norm topology — it is in general not possible to make the resulting topology independent of the choice of weight.

In addition, we note that the fiber “above” 0, viz.,  $\mathcal{M}(0)$ , is isomorphic to  $\mathcal{M}$  (trivially); of course, this fiber is the only one which is naturally an algebra. Moreover, we remark that the  $\mathcal{M}$ -valued inner product is realized as

$$\{x\varphi^{it} \mid y\varphi^{it}\} = (y\varphi^{it})^* x\varphi^{it} = \varphi^{-it} y^* x \varphi^{it} = \sigma_{-t}^\varphi(y^* x),$$

which agrees with our intuition.

We now consider sections of  $\mathcal{F}$ , i.e., maps

$$\mathbb{R} \rightarrow \mathcal{F} \quad , \quad s \mapsto x(s)\varphi^{is}.$$

We denote such a section by  $\tilde{x}$ ; hence,  $\tilde{x}(s) = x(s)\varphi^{is}$ . In the event that we want to stress the dependence on a particular choice of a faithful semi-finite normal weight  $\varphi$ , we write  $\tilde{x}(s) = x_\varphi(s)\varphi^{is}$ . We now want to consider the Banach space  $\Gamma^1(\mathcal{F})$  of  $L^1$ -sections of  $\mathcal{F}$ , that is, those (measurable) sections satisfying

$$\int_{\mathbb{R}} \|\tilde{x}(s)\| ds = \int_{\mathbb{R}} \|x(s)\varphi^{is}\| ds = \int_{\mathbb{R}} \|x(s)\| ds < \infty.$$

By “measurable” we mean the measurability of the following kind: we say a section  $x$  is *measurable* if for any finite interval  $I$  in  $\mathbb{R}$  and  $\varepsilon > 0$  there exists a compact subset  $K \subset I$  such that  $|I \setminus K| < \varepsilon$  and the map:  $s \in K \mapsto \tilde{x}(s) \in \mathcal{F}$  is continuous relative to any of the above topology in  $\mathcal{F}$ . As the norm:  $x \in \mathcal{M} \mapsto \|x\| \in \mathbb{R}_+$  is lower semi-continuous relative to the  $\sigma$  weak operator topology, the standard arguments show that the measurability of cross-sections does not depend on the choice of any operator topology in  $\mathcal{F}$ .

We are now going to turn the space  $\Gamma^1(\mathcal{F})$  of  $L^1$ -integrable sections into an involutive Banach algebra  $\mathcal{A}$ .

**Proposition 1.4.** *When equipped with a multiplication given by*

$$\begin{aligned} (\tilde{x}\tilde{y})(t) &:= \int_{\mathbb{R}} \tilde{x}(r)\tilde{y}(t-r) dr = \int_{\mathbb{R}} (x_\varphi(r)\varphi^{ir})(y_\varphi(t-r)\varphi^{i(t-r)}) dr \\ &= \left( \int_{\mathbb{R}} x_\varphi(r)\sigma_r^\varphi(y_\varphi(t-r)) dr \right) \varphi^{it}, \end{aligned}$$

and an involution defined by

$$\tilde{x}^*(t) = \tilde{x}(-t)^* = \sigma_t^\varphi(x_\varphi(-t))^* \varphi^{it},$$

the space  $\Gamma^1(\mathcal{F})$  of integrable  $\mathcal{F}$ -sections becomes an involutive Banach algebra, denoted henceforth by  $\mathcal{A}$ . (The norm on  $\mathcal{A}$  has already been indicated, viz., we have seen

$$\|\tilde{x}\|_{\mathcal{A}} = \int_{\mathbb{R}} \|x(r)\| dr,$$

when  $\tilde{x}(r) = x(r)\varphi^{ir}$ .)

The proof is routine, involving multiple applications of Fubini’s theorem; we leave it to the reader.

We call the involutive Banach algebra  $\mathcal{A}$  the *Bundle Algebra*.



## §2. Representation of the Bundle Algebra $\mathcal{A}$

In order to construct the "bundle von Neumann algebra"  $\tilde{\mathcal{M}}$  out of  $\mathcal{A}$ , we need to represent the von Neumann algebra  $\mathcal{M}$  on a Hilbert space  $\mathfrak{H}$ . So we begin by fixing the pair  $\{\mathcal{M}, \mathfrak{H}\}$ , and stress that in this context  $\mathfrak{H}$  is to be viewed as a *left  $L^2$ -von Neumann module*. Let us, as usual, denote by  $\mathcal{M}'$  the commutant of  $\mathcal{M}$  in  $\mathcal{L}(\mathfrak{H})$ , and define the von Neumann algebra  $\mathcal{N}$  to be the *opposite* von Neumann algebra,  $(\mathcal{M}')^\circ$ . That is to say,  $\mathcal{N}$  and  $\mathcal{M}'$  have the same structure as  $\mathbb{C}$ -Banach spaces – all that changes is that the multiplication in  $\mathcal{N}$  is reversed: if  $a, b$  are elements of  $\mathcal{M}'$ , and we denote by  $a^\circ, b^\circ$  the corresponding elements of  $\mathcal{N}$ , then  $a^\circ b^\circ = (ba)^\circ$ . (Note also that we have  $(a^\circ)^* = (a^*)^\circ$ .) It is clear that  $\mathcal{N}$  too is a von Neumann algebra. This allows us to consider  $\mathfrak{H}$  as an  *$L^2$ -von Neumann bimodule*, specifically an  $\mathcal{M}$ - $\mathcal{N}$  bimodule, where the right action of  $\mathcal{N}$  on  $\mathfrak{H}$  is given by

$$\xi a^\circ := a\xi$$

for all  $\xi \in \mathfrak{H}$ ,  $a^\circ \in \mathcal{N}$ . Notice that since we have  $\mathcal{N}^\circ = \mathcal{M}'$ , expressions of the form  $x\xi a^\circ$ , where  $x \in \mathcal{M}$ , etc., are unambiguous.

Because we are now dealing with two different von Neumann algebras, we will use  $\mathcal{A}(\mathcal{M})$  and  $\mathcal{A}(\mathcal{N})$  to refer to their respective bundle algebras, whenever there is any possibility of confusion. To construct a bundle of Hilbert spaces on which  $\mathcal{A}(\mathcal{M})$  acts from the left and  $\mathcal{A}(\mathcal{N})$  acts from the right, we consider the (cartesian product)  $X := \mathbb{R} \times \mathfrak{W}_0(\mathcal{M}) \times \mathfrak{H} \times \mathfrak{W}_0(\mathcal{N})$ , and a relation  $\sim_t$  on this set, after fixing  $t \in \mathbb{R}$ , viz.,

$$(r_1, \varphi_1, \xi_1, \psi_1) \sim_t (r_2, \varphi_2, \xi_2, \psi_2), \quad (2.1)$$

whenever

$$\left( \frac{d\varphi_1}{d\psi_1^\circ} \right)^{ir_1} \xi_1 = \left( \frac{d\varphi_2}{d\psi_2^\circ} \right)^{ir_2} \xi_2 (D\psi_2 : D\psi_1)_t. \quad (2.2)$$

(Note that here we have  $(r_1, \varphi_1, \xi_1, \psi_1), (r_2, \varphi_2, \xi_2, \psi_2) \in X$ , and  $\frac{d\varphi_1}{d\psi_1^\circ}$ , etc., represent spatial derivatives.)

It is easily verified that the relation  $\sim_t$  is an equivalence relation on the set  $X$ . We denote the quotient set  $X / \sim_t$  by  $\mathfrak{H}(t)$  and the class  $[r, \varphi, \xi, \psi] \in \mathfrak{H}(t)$  of  $(r, \varphi, \xi, \psi) \in X$  by  $\varphi^{ir} \xi \psi^{i(t-r)}$ . So in  $\mathfrak{H}(t)$  we have

$$\varphi^{it} \xi = \left( \frac{d\varphi}{d\psi^\circ} \right)^{it} \xi \psi^{it}, \quad \varphi \in \mathfrak{W}_0(\mathcal{M}), \psi \in \mathfrak{W}_0(\mathcal{N}), \xi \in \mathfrak{H}, t \in \mathbb{R} \quad (2.3)$$

or equivalently

$$\varphi^{it} \xi \psi^{-it} = \left( \frac{d\varphi}{d\psi^\circ} \right)^{it} \xi. \quad (2.3')$$

Observe that the relation  $\sim_t$  is generated by subrelations:  $\varphi_1^{it} \varphi_2^{-it} \sim (D\varphi_1 : D\varphi_2)_t$ ,  $\varphi_1, \varphi_2 \in \mathfrak{W}_0(\mathcal{M})$ ,  $\psi_1^{it} \psi_2^{-it} \sim (D\psi_1 : D\psi_2)_t$ ,  $\psi_1, \psi_2 \in \mathfrak{W}_0(\mathcal{N})$  and the relation (2.3').

**Lemma 2.1.** *In the set  $\mathfrak{H}(t), t \in \mathbb{R}$ , the linear structure and the inner product defined by:*

$$\begin{cases} \lambda(\varphi^{ir}\xi\psi^{i(t-r)}) + \mu(\varphi^{it}\eta\psi^{i(t-r)}) = \varphi^{ir}(\lambda\xi + \mu\eta)\psi^{i(t-r)}; \\ (\varphi^{ir}\xi\psi^{i(t-r)} | \varphi^{ir}\eta\psi^{i(t-r)}) = (\xi | \eta) \end{cases}$$

*makes  $\mathfrak{H}(t)$  a Hilbert space which does not depend on the choice of  $\varphi \in \mathfrak{W}_0(\mathcal{M}), \psi \in \mathfrak{W}_0(\mathcal{N})$  nor on  $r \in \mathbb{R}$ .*

*Proof.* Suppose  $\varphi_1^{ir_1}\xi_1\psi_1^{i(t-r_1)} = \varphi_2^{ir_2}\xi_2\psi_2^{i(t-r_2)}$  and  $\varphi_1^{ir_1}\eta_1\psi_1^{i(t-r_1)} = \varphi_2^{ir_2}\eta_2\psi_2^{i(t-r_2)}$ . This means that

$$\begin{aligned} \xi_1 &= (D\varphi_1 : D\varphi_2)_{-r_1} \left( \frac{d\varphi_2}{d\psi_2^\circ} \right)^{i(r_2-r_1)} \xi_2 (D\psi_2 : D\psi_1)_{t-r_1} \\ \eta_1 &= (D\varphi_1 : D\varphi_2)_{-r_1} \left( \frac{d\varphi_2}{d\psi_2^\circ} \right)^{i(r_2-r_1)} \eta_2 (D\psi_2 : D\psi_1)_{t-r_1}. \end{aligned}$$

Hence we have

$$\lambda\xi_1 + \mu\eta_1 = (D\varphi_1 : D\varphi_2)_{-r_1} \left( \frac{d\varphi_2}{d\psi_2^\circ} \right)^{i(r_2-r_1)} (\lambda\xi_2 + \mu\eta_2) (D\psi_2 : D\psi_1)_{t-r_1},$$

which shows that the linear operation in  $\mathfrak{H}(t)$  is independent of  $\varphi, \psi$  and  $r$ . Also we have  $(\xi_1 | \eta_1) = (\xi_2 | \eta_2)$ . Thus the inner product is also independent of the choice of  $\varphi, \psi$  and  $r$ .  $\square$

Observe that for each  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  and  $\psi \in \mathfrak{W}_0(\mathcal{N})$  the maps:

$$\begin{aligned} U_\varphi(t) : \xi &\mapsto \varphi^{it}\xi = U_\varphi(t)\xi \in \mathfrak{H}(t), \quad \xi \in \mathfrak{H}; \\ V_\psi(t) : \eta &\mapsto \eta\psi^{it} = V_\psi(t)\eta \in \mathfrak{H}(t), \quad \eta \in \mathfrak{H} \end{aligned}$$

are both unitaries satisfying

$$V_\psi(t)^* U_\varphi(t) = \left( \frac{d\varphi}{d\psi^\circ} \right)^{it};$$

Set

$$\mathfrak{G} = \coprod_{t \in \mathbb{R}} \mathfrak{H}(t)$$

to obtain a Hilbert space bundle over  $\mathbb{R}$  which is homeomorphic to the product bundle  $\mathfrak{H} \times \mathbb{R}$  where a homeomorphism is given by fixing either  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  or  $\psi \in \mathfrak{W}_0(\mathcal{N})$  as seen above. When we need to indicate the dependence of  $\mathfrak{G}$  on the original Hilbert space  $\mathfrak{H}$  we write  $\mathfrak{G}(\mathfrak{H})$ .

We now define a multilinear product:  $(x\varphi^{ir}, \varphi^{is}\xi, y\psi^{it}) \in \mathcal{M}(r) \times \mathfrak{H}(s) \times \mathcal{N}(t) \mapsto x\varphi^{ir}\varphi^{is}\xi y\psi^{it} \in \mathfrak{H}(r+s+t)$  as follows:

$$\begin{aligned} x\varphi^{ir}\varphi^{is}\xi y\psi^{it} &= \varphi^{i(r+s)} \sigma_{-(r+s)}^\varphi(x) \xi y \psi^{it} \\ &= x \left( \frac{d\varphi}{d\psi^\circ} \right)^{i(r+s)} \xi \sigma_{(r+s)}^\psi(y) \psi^{i(r+s+t)}. \end{aligned}$$

It is again routine to check that the above product does not depend on the choice of  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  and  $\psi \in \mathfrak{W}_0(\mathcal{N})$  and is associative. We omit the detail.

**Lemma 2.2.** *With  $r, s, t \in \mathbb{R}$  fixed, we have the following statements:*

- i) *If a pair  $(a', b') \in \mathcal{L}(\mathfrak{H}(s), \mathfrak{H}(s+t)) \times \mathcal{L}(\mathfrak{H}(r+s), \mathfrak{H}(r+s+t))$  satisfies  $b'(x\xi) = x(a'\xi)$  for every  $x \in \mathcal{M}(r)$ , i.e. if  $(a', b')$  makes the following diagram commutative:*

$$\begin{array}{ccc} \mathfrak{H}(s) & \xrightarrow{a'} & \mathfrak{H}(s+t) \\ x \downarrow & & x \downarrow \\ \mathfrak{H}(r+s) & \xrightarrow{b'} & \mathfrak{H}(r+s+t) \end{array}$$

*then there exists  $y \in \mathcal{N}(t)$  such that  $a'\xi = \xi y$ ,  $\xi \in \mathfrak{H}(s)$ , and  $b'\eta = \eta y$ ,  $\eta \in \mathfrak{H}(r+s)$ .*

- ii) *If a pair  $(a, b) \in \mathcal{L}(\mathfrak{H}(s), \mathfrak{H}(s+t)) \times \mathcal{L}(\mathfrak{H}(r+s), \mathfrak{H}(r+s+t))$  satisfies  $b(\xi y) = (b\xi)y$  for every  $y \in \mathcal{N}(r)$ , i.e. if  $(a', b')$  makes the following diagram commutative:*

$$\begin{array}{ccc} \mathfrak{H}(s) & \xrightarrow{a} & \mathfrak{H}(s+t) \\ \downarrow y & & \downarrow y \\ \mathfrak{H}(r+s) & \xrightarrow{b} & \mathfrak{H}(r+s+t) \end{array}$$

*then there exists  $x \in \mathcal{M}(t)$  such that  $a\xi = x\xi$ ,  $\xi \in \mathfrak{H}(s)$ , and  $b\eta = x\eta$ ,  $\eta \in \mathfrak{H}(r+s)$ .*

*Proof.* We prove only the assertion (i). The other follows by symmetry. Fix a pair  $(\varphi, \psi) \in \mathfrak{W}_0(\mathcal{M}) \times \mathfrak{W}_0(\mathcal{N})$ . For each  $\xi \in \mathfrak{H}$  and  $x \in \mathcal{M}$ , we have

$$\begin{aligned} \varphi^{-i(r+s)} \left( b' \varphi^{ir} (\varphi^{is} x \xi) \right) \psi^{-it} &= \varphi^{-i(r+s)} \left( b' (\varphi^{i(r+s)} x \xi) \right) \psi^{-it} \\ &= \varphi^{-i(r+s)} \left( b' \sigma_{r+s}^\varphi(x) \varphi^{ir} \varphi^{is} \xi \right) \psi^{-it} \\ &= \varphi^{-i(r+s)} \left( \sigma_{r+s}^\varphi(x) \varphi^{ir} a' \varphi^{is} \xi \right) \psi^{-it} \\ &= \left( x \varphi^{-is} a' \varphi^{is} \xi \right) \psi^{-it}; \end{aligned}$$

equivalently

$$V_\psi(t)^* U_\varphi(r+s)^* b' U_\varphi(r+s) x = x V_\psi(t)^* U_\varphi(s)^* a' U_\varphi(s), \quad x \in \mathcal{M}.$$

Taking  $x = 1$ , we conclude that

$$V_\psi(t)^* U_\varphi(r+s)^* b' U_\varphi(r+s) = V_\psi(t)^* U_\varphi(s)^* a' U_\varphi(s)$$

is an operator in the commutant  $\mathcal{M}'$ , which we denote by  $y^\circ \in \mathcal{M}'$ . Namely, there exists an operator  $y = y^{\circ\circ} \in \mathcal{N}$  such that

$$V_\psi(t)^* U_\varphi(r+s)^* b' U_\varphi(r+s) \xi = V_\psi(t)^* U_\varphi(s)^* a' U_\varphi(s) \xi = \xi y, \quad \xi \in \mathfrak{H}.$$

This means:

$$\begin{aligned} a' \varphi^{is} \xi &= \varphi^{is} \xi y \psi^{it}; \\ b' \varphi^{i(r+s)} \xi &= \varphi^{i(r+s)} \xi y \psi^{it}, \quad \xi \in \mathfrak{H}. \end{aligned}$$

This completes the proof.  $\square$

Let  $\tilde{\mathfrak{H}} = \Gamma^2(\mathcal{G})$  be the Hilbert space of square integrable cross-sections of the bundle  $\mathcal{G}$ . We want to let the bundle algebra  $\mathcal{A}(\mathcal{M})$  (res.  $\mathcal{A}(\mathcal{N})$ ) act on  $\tilde{\mathfrak{H}}$  from the left (resp. from the right). Before doing this, we should establish the correspondence between the bundle algebras  $\mathcal{A}(\mathcal{M}')$  and  $\mathcal{A}(\mathcal{N})$ . We have defined  $\mathcal{N}$  to be the opposite algebra  $(\mathcal{M}')^\circ$  of  $\mathcal{M}'$ . Let us denote the canonical correspondence between  $\mathcal{N}$  and  $\mathcal{M}'$  by  $y \in \mathcal{N} \leftrightarrow y^\circ \in \mathcal{M}'$  and  $\psi \in \mathfrak{W}_0(\mathcal{N}) \leftrightarrow \psi^\circ \in \mathfrak{W}_0(\mathcal{M}')$ , i.e.  $\psi^\circ(y^\circ) = \psi(y)$ ,  $y \in \mathcal{N}$ . Furthermore, we write  $y^{\circ\circ} = y$  and  $\psi^{\circ\circ} = \psi$ . Recall that  $\sigma_t^\psi(y)^\circ = \sigma_{-t}^{\psi^\circ}(y^\circ)$ ,  $y \in \mathcal{N}$ . Therefore the natural extension of the  $\circ$ -operation from  $\mathcal{N}$  and  $\mathcal{M}'$  to the Fell bundles  $\mathcal{F}(\mathcal{N})$  and  $\mathcal{F}(\mathcal{M}')$  is then given by:

$$(y\psi^{it})^\circ = \psi^{\circ-it} y^\circ = \sigma_{-t}^{\psi^\circ}(y^\circ) \psi^{\circ-it}, \quad y \in \mathcal{N}, \psi \in \mathfrak{W}_0(\mathcal{N}). \quad (2.4)$$

Thus the  $\circ$ -operation on  $\mathcal{A}(\mathcal{N})$  and  $\mathcal{A}(\mathcal{M}')$  is give by:

$$\underset{\sim}{y}^\circ(t) = (\underset{\sim}{y}(-t))^\circ, \quad \underset{\sim}{y} \in \mathcal{A}(\mathcal{N}) \cup \mathcal{A}(\mathcal{M}') \quad (2.4')$$

and we get  $\mathcal{A}(\mathcal{N})^\circ = \mathcal{A}(\mathcal{M}')$  and  $\mathcal{A}(\mathcal{M}')^\circ = \mathcal{A}(\mathcal{N})$ .

Define left and right actions of the bundle algebras  $\mathcal{A}(\mathcal{M})$  and  $\mathcal{A}(\mathcal{N})$  on  $\tilde{\mathfrak{H}}$  respectively as follows:

$$\begin{cases} (\underset{\sim}{x}\xi)(t) &= \int_{\mathbb{R}} \underset{\sim}{x}(r) \xi(t-r) dr; \\ (\xi\underset{\sim}{y})(t) &= \int_{\mathbb{R}} \xi(s) \underset{\sim}{y}(t-s) ds. \end{cases} \quad (2.5)$$

Also  $\mathcal{A}(\mathcal{M}')$  acts on  $\tilde{\mathfrak{H}}$  from the left:

$$\underset{\sim}{y}\xi = \xi \underset{\sim}{y}^\circ, \quad \underset{\sim}{y} \in \mathcal{A}(\mathcal{M}'),$$

i.e.

$$\begin{aligned} (\underset{\sim}{y}\xi)(t) &= \int_{\mathbb{R}} \underset{\sim}{y}(s) \xi(t-s) ds \\ &= \int_{\mathbb{R}} \xi(t-s) \underset{\sim}{y}^\circ(s) ds \\ &= \int_{\mathbb{R}} \xi(t+s) (\underset{\sim}{y}(s))^\circ ds, \quad \underset{\sim}{y} \in \mathcal{A}(\mathcal{M}'). \end{aligned} \quad (2.5')$$

Each  $x \in \mathcal{M}(r)$  and  $y \in \mathcal{N}(t)$  act on  $\tilde{\mathfrak{H}}$  from the respective side as follows:

$$\begin{cases} (x\xi)(s) &= x\xi(s-r); \\ (\xi y)(s) &= \xi(s-t)y, \\ (y^\circ\xi)(s) &= y^\circ\xi(s+t), \quad \xi \in \tilde{\mathfrak{H}}. \end{cases} \quad (2.5'')$$

In particular,  $\mathcal{M} = \mathcal{M}(0)$  and  $\mathcal{N}(0)$  both act on  $\tilde{\mathfrak{H}}$  respectively. Also  $\{\varphi^{it} : t \in \mathbb{R}\}$  and  $\{\psi^{-it} : t \in \mathbb{R}\}$  are both one parameter unitary groups on  $\tilde{\mathfrak{H}}$  acting from their respective sides; these will be denoted by  $\{u_\varphi(t)\}$  and  $\{v_\psi(t)\}$  when we view them as one parameter unitary groups acting on  $\tilde{\mathfrak{H}}$  from the left.

**Lemma 2.3.** i) *With  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  fixed, the map  $U_\varphi : \Gamma^2(\mathcal{G}) \mapsto \xi_\varphi \in L^2(\mathbb{R}, \mathfrak{H}) = L^2(\mathbb{R}) \otimes \mathfrak{H}$  defined by:*

$$(U_\varphi \xi)(s) = \xi_\varphi(s) = \varphi^{-is} \xi(s) \in \mathfrak{H}, \quad \xi \in \tilde{\mathfrak{H}} \quad (2.6)$$

is a unitary such that

a) *For each  $x \in \mathcal{M}$ ,*

$$\left( U_\varphi x U_\varphi^* \xi \right)(s) = \sigma_{-s}^\varphi(x) \xi(s), \quad \xi \in L^2(\mathbb{R}, \mathfrak{H});$$

a') *For each  $y \in \mathcal{N}$ ,*

$$\left( U_\varphi y^\circ U_\varphi^* \xi \right)(s) = \xi(s) y;$$

b) *For each  $t \in \mathbb{R}$ ,*

$$\left( U_\varphi u_\varphi(t) U_\varphi^* \xi \right)(s) = \xi(s - t), \quad \xi \in L^2(\mathbb{R}, \mathfrak{H});$$

b') *For each  $t \in \mathbb{R}$ ,*

$$\left( U_\varphi v_\psi(t) U_\varphi^* \xi \right)(s) = \left( \frac{d\varphi}{d\psi^\circ} \right)^{it} \xi(s + t).$$

ii) *The unitary  $U_\varphi$ , with  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  fixed, carries the von Neumann algebra  $\tilde{\mathcal{M}}$  generated by the action of  $\mathcal{A}(\mathcal{M})$  onto the cross product  $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$  isomorphically.*

iii) *The von Neumann algebra  $\tilde{\mathcal{M}}'$  generated by the action of  $\mathcal{A}(\mathcal{M}')$  is the commutant of  $\tilde{\mathcal{M}}$  and is mapped isomorphically by the unitary  $U_\varphi$  to the crossed product  $\mathcal{M}' \rtimes_{\sigma^{\psi^\circ}} \mathbb{R}$  with any  $\psi \in \mathfrak{W}_0(\mathcal{N})$  fixed.*

*Proof.* i) Let  $x \in \mathcal{M}$  and  $\xi \in L^2(\mathbb{R}, \mathfrak{H})$ . Then

$$\begin{aligned} (U_\varphi x U_\varphi^* \xi)(s) &= \varphi^{-is} x (U_\varphi^* \xi)(s) = \varphi^{-is} x \varphi^{is} \xi(s) \\ &= \sigma_{-is}^\varphi(x) \xi(s), \quad s \in \mathbb{R}. \end{aligned}$$

ii) Let us simply compute for  $\xi \in L^2(\mathbb{R}, \mathfrak{H})$ :

$$\begin{aligned} (U_\varphi u_\varphi(t) U_\varphi^* \xi)(s) &= \varphi^{-is} (u_\varphi(t) U_\varphi^* \xi)(s) = \varphi^{-is} \varphi^{it} (U_\varphi^* \xi)(s - t) \\ &= \varphi^{-is} \varphi^{it} \varphi^{i(s-t)} \xi(s - t) = \xi(s - t), \quad s, t \in \mathbb{R}. \end{aligned}$$

iii) With  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  fixed, for each  $x \in \mathcal{A}(\mathcal{M})$  we compute:

$$\begin{aligned}
(U_\varphi x U_\varphi^* \xi)(s) &= \varphi^{-is} \int_{\mathbb{R}} x(r) (U_\varphi^* \xi)(s-r) dr = \varphi^{-is} \int_{\mathbb{R}} x(r) \varphi^{i(s-r)} \xi(s-r) dr \\
&= \varphi^{-is} \int_{\mathbb{R}} x_\varphi(r) \varphi^{ir} \varphi^{i(s-r)} \xi(s-r) dr \\
&= \int_{\mathbb{R}} \sigma_{-s}^\varphi(x_\varphi(r)) \xi(s-r) dr \\
&= \left( \left( \int_{\mathbb{R}} x_\varphi(r) (\lambda(r) \otimes 1) dr \right) \xi \right)(s),
\end{aligned}$$

where  $x_\varphi(r) = x(r) \varphi^{-ir}$  as seen before and  $\lambda(\cdot)$  is the regular representation of  $\mathbb{R}$  on  $L^2(\mathbb{R})$ . From these calculations, the assertion follows easily.  $\square$

We are now ready summarize our conclusions:

**Theorem 2.4.** i) *The association:  $\{\mathcal{M}, \mathfrak{H}\} \longrightarrow \{\tilde{\mathcal{M}}, \tilde{\mathfrak{H}}\}$  is a functor from the category **SvNA** of von Neumann algebras with spatial isomorphism as morphisms into the category **SFSvNA** of semi-finite von Neumann algebras with spatial isomorphisms as morphisms.*

ii) *If we fix  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ , then the unitary  $U_\varphi$  defined by (2.6) gives a natural spatial isomorphism of  $\{\tilde{\mathcal{M}}, \tilde{\mathfrak{H}}\}$  onto  $\{\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, L^2(\mathbb{R}, \mathfrak{H})\}$ .*

iii) *Choosing the canonical Hilbert space  $L^2(\mathcal{M})$  attached to every von Neumann algebra  $\mathcal{M}$  as a representing Hilbert space  $\mathfrak{H}$  of  $\mathcal{M}$ , we get a functor  $\mathcal{M} \mapsto \tilde{\mathcal{M}}$  from the category **vNA** of von Neumann algebras with isomorphisms as morphisms into the category **SFvNA** of semi-finite von Neumann algebras with isomorphisms as morphisms.*

Although the proof is by now routine, we are going to give a brief outline in order to establish some notation.

*Proof.* i) Let  $U$  be a unitary which implements a spatial isomorphism  $\alpha = \alpha_U$  of  $\{\mathcal{M}_1, \mathfrak{H}_1\}$  onto  $\{\mathcal{M}_2, \mathfrak{H}_2\}$ . For each  $\varphi \in \mathfrak{W}_0(\mathcal{M}_1)$ , set  $\alpha(\varphi) = \varphi \circ \alpha \in \mathfrak{W}_0(\mathcal{M}_2)$ . Clearly,  $\alpha$  maps  $\mathfrak{W}_0(\mathcal{M}_1)$  onto  $\mathfrak{W}_0(\mathcal{M}_2)$  bijectively. Then the map  $\tilde{\alpha}$  defined by:

$$\tilde{\alpha}(x\varphi^{it}) = \alpha(x)(\alpha(\varphi))^{it}, \quad x\varphi^{it} \in \mathcal{F}(\mathcal{M}_1),$$

gives an isomorphism of  $\mathcal{F}_1 = \mathcal{F}(\mathcal{M}_1)$  onto  $\mathcal{F}_2 = \mathcal{F}(\mathcal{M}_2)$ . We also define a map  $\tilde{U}$  of the Hilbert space bundle  $\mathcal{G}_1 = \mathcal{G}(\mathcal{M}_1, \mathfrak{H}_1)$  onto the other  $\mathcal{G}_2 = \mathcal{G}(\mathcal{M}_2, \mathfrak{H}_2)$  by:

$$\tilde{U}(\varphi^{it}\xi) = (\alpha(\varphi))^{it}U\xi, \quad \varphi^{it}\xi \in \mathcal{G}_1,$$

which can be easily seen to be a bundle isomorphism of  $\mathcal{F}_1$  onto  $\mathcal{F}_2$  and to carry the action of  $\mathcal{F}_1$  on  $\mathcal{G}_1$  onto that of  $\mathcal{F}_2$  on  $\mathcal{G}_2$ . Hence it gives rise to a unitary  $\tilde{U}$  of the Hilbert space  $\tilde{\mathfrak{H}}_1 = \Gamma^2(\mathcal{G}_1)$  of  $L^2$ -cross sections of  $\mathcal{G}_1$  onto the Hilbert space  $\tilde{\mathfrak{H}}_2 = \Gamma^2(\mathcal{G}_2)$  of

$L^2$ -cross sections of  $\mathfrak{G}_2$ , which conjugates the von Neumann algebra  $\{\tilde{\mathcal{M}}_1, \tilde{\mathfrak{H}}_1\}$  onto the other  $\{\tilde{\mathcal{M}}_2, \tilde{\mathfrak{H}}_2\}$ . We denote by  $\tilde{\alpha}$  the spatial isomorphism of  $\tilde{\mathcal{M}}_1$  onto  $\tilde{\mathcal{M}}_2$  implemented by  $\tilde{U}$  which extends the original isomorphism  $\alpha$  of  $\mathcal{M}_1$  onto  $\mathcal{M}_2$ . It is now not difficult to see that if  $U : \mathfrak{H}_1 \mapsto \mathfrak{H}_2$  and  $V : \mathfrak{H}_2 \mapsto \mathfrak{H}_3$  are unitaries implementing respectively spatial isomorphisms  $\alpha = \alpha_U : \{\mathcal{M}_1, \mathfrak{H}_1\} \mapsto \{\mathcal{M}_2, \mathfrak{H}_2\}$  and  $\beta = \alpha_V : \{\mathcal{M}_2, \mathfrak{H}_2\} \mapsto \{\mathcal{M}_3, \mathfrak{H}_3\}$  then  $\widetilde{VU} = \tilde{V}\tilde{U}$  and  $\widetilde{\beta \circ \alpha} = \tilde{\beta} \circ \tilde{\alpha}$ . This completes the proof of (i).

The assertions in (ii) and (iii) have already been established.  $\square$

**Definition 2.5.** *The von Neumann algebra  $\tilde{\mathcal{M}}$  is called the core of a von Neumann algebra  $\mathcal{M}$ . The isomorphism  $\tilde{\alpha}$  appearing in the proof will be called the canonical extension of the given isomorphism  $\alpha$ .*

### §3. The Dual Action and The Trace

Recall that the flow of weights of Connes-Takesaki is a mathematical structure coming from the trivial action of  $\mathbb{R}$  on weights,  $\Theta_s : \varphi \in \mathfrak{W}(\mathcal{M}) \mapsto e^{-s}\varphi \in \mathfrak{W}(\mathcal{M})$ . Let us examine what happens in our context if we consider the same trivial action of  $\mathbb{R}$  on  $\mathfrak{W}_0(\mathcal{M})$ . First, we observe that for each  $t \in \mathbb{R}$

$$(x, \varphi) \sim_t (y, \psi) \iff (x, e^{-s}\varphi) \sim_t (y, e^{-s}\psi), \quad (x, \varphi), (y, \psi) \in \mathfrak{W}_0(\mathcal{M}).$$

Therefore, the corresponding one parameter group of transformations on the Fell's bundle  $\mathcal{F}(\mathcal{M})$  is given by:

$$\theta_t(x\varphi^{is}) = e^{-ist}x\varphi^{is}, \quad x\varphi^{is} \in \mathcal{M}(s).$$

This is easily seen to be a one parameter automorphism group of  $\mathcal{F}(\mathcal{M})$ .

**Lemma 3.1.** *The one parameter automorphism group  $\{\theta_t : t \in \mathbb{R}\}$  of the Fell's bundle  $\mathcal{F}(\mathcal{M})$  can be extended to the von Neumann algebra  $\tilde{\mathcal{M}}$ , which will be denoted by  $\{\theta_t : t \in \mathbb{R}\}$  again. Furthermore, the one parameter automorphism group  $\{\theta_t : t \in \mathbb{R}\}$  is conjugate to the dual action  $\{\widehat{\sigma}_t^\varphi : t \in \mathbb{R}\}$  on  $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$  under the spatial isomorphism given by the unitary  $U_\varphi$  of Theorem 2.11.*

*Proof.* Define a one parameter unitary group  $\{V(t) : t \in \mathbb{R}\}$  on  $\tilde{\mathfrak{H}}$  by

$$(V(t)\xi)(s) = e^{-ist}\xi(s), \quad s \in \mathbb{R}, \quad \xi \in \tilde{\mathfrak{H}}. \tag{3.1'}$$

It then follows easily that  $V(t)xV(t)^* = \theta_t(x)$ ,  $x \in \mathcal{A}(\mathcal{M})$  and  $V(t)\tilde{\mathcal{M}}V(t)^* = \tilde{\mathcal{M}}$ . We leave the details to the reader.  $\square$

As a consequence, the action  $\theta$  of  $\mathbb{R}$  on  $\mathcal{M}$  is integrable. Hence the integral

$$I_\theta(x) = \int_{\mathbb{R}} \theta_s(x) ds, \quad x \in \tilde{\mathcal{M}}_+, \tag{3.2}$$

is an operator valued weight from  $\tilde{\mathcal{M}}$  to  $\tilde{\mathcal{M}}^\theta$  which is canonically identified with the original von Neumann algebra  $\mathcal{M}$ . Thus, we get a semi-finite normal weight  $\tilde{\varphi}$  on  $\tilde{\mathcal{M}}$ , to be called the *dual* semi-finite normal weight, by the formula:

$$\tilde{\varphi}(x) = \varphi \circ I_\theta(x) = \varphi\left(\int_{\mathbb{R}} \theta_s(x) ds\right), \quad x \in \tilde{\mathcal{M}}_+. \quad (3.3)$$

The dual semi-finite normal weight  $\tilde{\varphi}$  is faithful if the original semi-finite normal weight  $\varphi$  is.

**Lemma 3.2.** i) *The modular automorphism group of  $\tilde{\varphi}$  is given by:*

$$\sigma_t^{\tilde{\varphi}} = \text{Ad}(\varphi^{it}), t \in \mathbb{R}.$$

ii) *The weight  $\tau_\varphi$  on  $\tilde{\mathcal{M}}$  defined by:*

$$\tau_\varphi(x) = \lim_{\varepsilon \searrow 0} \tilde{\varphi}\left(\varphi^{-\frac{1}{2}}(1 + \varepsilon\varphi^{-1})^{-\frac{1}{2}}x\varphi^{-\frac{1}{2}}(1 + \varepsilon\varphi^{-1})^{-\frac{1}{2}}\right), x \in \mathcal{M}_+, \quad (3.4)$$

*is a faithful semi-finite normal trace such that*

$$\tau_\varphi \circ \theta_s = e^{-s}\tau_\varphi, \quad s \in \mathbb{R}. \quad (3.5)$$

iii) *The trace  $\tau_\varphi$  does not depend on  $\varphi$ , i.e.*

$$\tau_\varphi = \tau_\psi, \quad \varphi, \psi \in \mathfrak{W}_0(\mathcal{M}).$$

*We will denote this common trace by  $\tau$ .*

*Proof.* i) The one parameter automorphism group  $\{\text{Ad}(\varphi^{it})\}$  and  $\{\theta_s\}$  commute as  $\theta_s(\varphi^{it}) = e^{-ist}\varphi^{it}$ . Hence we have  $I_\theta \circ \text{Ad}(\varphi^{it}) = \text{Ad}(\varphi^{it}) \circ I_\theta$ , which means that  $\tilde{\varphi} \circ \text{Ad}(\varphi^{it}) = \tilde{\varphi}, t \in \mathbb{R}$ . Thus we have  $\varphi^{it} \in \tilde{\mathcal{M}}_{\tilde{\varphi}}$ , i.e.  $\sigma_t^{\tilde{\varphi}}(\varphi^{is}) = \varphi^{is}, s, t \in \mathbb{R}$ . From the general theory of operator valued weights, it follows that  $\sigma_t^{\varphi \circ I_\theta}(x) = \sigma_t^\varphi(x)$  for every  $x \in \mathcal{M}$ . Hence  $\text{Ad}(\varphi^{it})$  and  $\sigma_t^{\tilde{\varphi}}$  agree on  $\mathcal{M}$  and  $\{\varphi^{is}\}$  which together generate  $\tilde{\mathcal{M}}$ . Hence, (i) follows.

ii) The general theory of weights of Pedersen–Takesaki [PT] yields that the weight  $\tau_\varphi$  is a faithful semi-finite normal trace on  $\tilde{\mathcal{M}}$ . For the trace scaling property (3.6), we simply compute informally for  $s \in \mathbb{R}$  and  $x \in \tilde{\mathcal{M}}$  (using a computation which can easily be made rigorous)

$$\begin{aligned} \tau_\varphi(\theta_s(x)) &= \tilde{\varphi}\left(\varphi^{-1}\theta_s(x)\right) = \tilde{\varphi}\left(\theta_s(\theta_{-s}(\varphi^{-1})x)\right) \\ &= \tilde{\varphi}(e^{-s}\varphi^{-1}x) = e^{-s}\tau_\varphi(x). \end{aligned}$$

iii) Take  $\varphi, \psi \in \mathfrak{W}_0(\mathcal{M})$  and compute the Connes cocycle derivative:

$$\begin{aligned} (D\tau_\varphi : D\tau_\psi)_t &= (D\tau_\varphi : D\tilde{\varphi})_t (D\tilde{\varphi} : D\tilde{\psi})_t (D\tilde{\psi} : D\tau_\psi)_t \\ &= \varphi^{-it}(D(\varphi \circ I_\theta) : D(\psi \circ I_\theta))_t \psi^{it} \\ &= \varphi^{-it}(D\varphi : D\psi)_t \psi^{it} = \varphi^{-it}(\varphi^{it}\psi^{-it})\psi^{it} \\ &= 1. \end{aligned}$$

This completes the proof.  $\square$

We can therefore conclude the following:



**Theorem 3.3.** *To every von Neumann algebra  $\mathcal{M}$  there corresponds canonically a covariant system  $\{\tilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\}$  of a semi-finite von Neumann algebra  $\tilde{\mathcal{M}}$  equipped with a faithful semi-finite normal trace  $\tau$  scaling one parameter automorphism group in such a way that the original von Neumann algebra  $\mathcal{M}$  naturally identified with the fixed point algebra  $\tilde{\mathcal{M}}^\theta$ . The covariant system  $\{\mathcal{C}_{\tilde{\mathcal{M}}}, \mathbb{R}, \theta\}$  obtained by restricting  $\theta$  to the center  $\mathcal{C}_{\tilde{\mathcal{M}}}$  of  $\tilde{\mathcal{M}}$  is precisely the flow of weights on  $\mathcal{M}$  of Connes-Takesaki.*

**Definition 3.4.** *We call the covariant system  $\{\tilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\}$  the quantum flow of weights on  $\mathcal{M}$ .*

**Theorem 3.5.** *The quantum flow of weights  $\{\tilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\}$  on a von Neumann algebra  $\mathcal{M}$  is a functor from the category  $\mathbf{vNA}$  of von Neumann algebras  $\mathcal{M}$  with isomorphisms as morphisms onto the category  $\mathbf{SFvNA}$  of semi-finite von Neumann algebras  $\{\tilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\}$  equipped with a one parameter automorphism group  $\theta$  which scales a faithful semi-finite normal trace  $\tau$  in such a way that  $\tau \circ \theta_s = e^{-s}\tau$ ,  $s \in \mathbb{R}$ , where morphisms of  $\mathbf{SFvNA}$  are those isomorphisms which conjugate the one parameter automorphisms and the traces.*

*Proof.* This theorem is merely an extension of Theorem 2.4iii; all that needs to be verified is the naturality of the conjugation.

Let  $\alpha: \mathcal{M}_1 \rightarrow \mathcal{M}_2$  be an isomorphism of von Neumann algebras, and  $\tilde{\alpha}: \tilde{\mathcal{M}}_1 \rightarrow \tilde{\mathcal{M}}_2$  be the corresponding isomorphism of their (respective) cores. We need to verify that

$$(\tilde{\alpha} \circ \theta_s \circ \tilde{\alpha}^{-1})(x(t)\varphi^{it}) = \theta_s(x(t)\varphi^{it}),$$

for all  $s, t$  in  $\mathbb{R}$ . But straightforward calculation yields

$$\begin{aligned} (\tilde{\alpha} \circ \theta_s \circ \tilde{\alpha}^{-1})(x(t)\varphi^{it}) &= (\tilde{\alpha} \circ \theta_s)(\alpha^{-1}(x(t))(\varphi \circ \alpha^{-1})^{it}) \\ &= \tilde{\alpha}(\alpha^{-1}(x(t-s))(\varphi \circ \alpha)^{it}) \\ &= x(t-s)\varphi^{it} = \theta_s(x(t)\varphi^{it}). \end{aligned}$$

Hence, we are done.  $\square$

In order to continue our study of the quantum flow of weights on  $\mathcal{M}$ , we introduce some notation. We set  $\{\mathcal{A}, \mathbb{R}, \rho\}$  to be the covariant system  $\{L^\infty(\mathbb{R}), \text{Translation}\}$ , i.e.

$$(\rho_t(f))(s) = f(s+t), \quad f \in \mathcal{A}, \quad s, t \in \mathbb{R}. \quad (3.6)$$

In  $\{\mathcal{A}, \mathbb{R}, \rho\}$ , we fix the following one parameter unitary group  $\{V(t) : t \in \mathbb{R}\}$  which generates  $\mathcal{A}$  as a von Neumann algebra, and its analytic generator  $H$ , which is affiliated with  $\mathcal{A}$ :

$$\begin{cases} (V(t))(s) = e^{-ist}, & s, t \in \mathbb{R}; \\ H(t) = e^{-t}. \end{cases} \quad (3.7)$$

It then follows that:

$$\rho_t(V(s)) = e^{-ist}V(t), \quad s, t \in \mathbb{R}. \quad (3.8)$$

**Theorem 3.6.** i) Each  $\varphi \in \mathfrak{W}(\mathcal{M})$  gives rise uniquely to an equivariant isomorphism  $\pi^\varphi$  from  $\{\mathcal{A}, \mathbb{R}, \rho\}$  into  $\{\tilde{\mathcal{M}}_{s(\varphi)}, \mathbb{R}, \theta\}$ , where  $s(\varphi)$  is the support projection of  $\varphi$ , such that

$$\pi^\varphi(V(t)) = \varphi^{it}, \quad t \in \mathbb{R}.$$

ii) If  $\pi$  is an equivariant isomorphism of  $\{\mathcal{A}, \mathbb{R}, \theta\}$  into  $\{\tilde{\mathcal{M}}_e, \mathbb{R}, \theta\}$  with  $e \in \text{Proj}(\mathcal{M})$ , then there exists  $\varphi = \varphi_\pi \in \mathfrak{W}(\mathcal{M})$  such that  $\pi = \pi^\varphi$ .

*Proof.* The assertion (i) for faithful semi-finite normal weights has been proven already in the process of constructing  $\tilde{\mathcal{M}}$ . For a non-faithful  $\varphi \in \mathfrak{W}(\mathcal{M})$ , what one needs do is simply consider the reduced algebra  $\mathcal{M}_{s(\varphi)}$  of  $\mathcal{M}$  by the support  $s(\varphi)$  of  $\varphi$  and to apply the assertion for faithful ones.

ii) According to the considerations in the case (i), we may and do assume  $e = 1$ . Set  $v(t) = \pi(V(t)), t \in \mathbb{R}$ . Then we have  $\theta_s(v(t)) = e^{-ist}v(t), s, t \in \mathbb{R}$ . Choose  $\psi \in \mathfrak{W}_0(\mathcal{M})$  and put  $u_t = v(t)\psi^{-it}$ . Then it follows that  $\{u_t\}$  is a  $\sigma^\psi$ -cocycle, so that there exists a  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  by the converse of the Connes cocycle derivative theorem such that  $u_t = (D\varphi : D\psi)_t$ , i.e.  $u_t = \varphi^{it}\psi^{-it}$ . This means that  $v(t) = \varphi^{it}, t \in \mathbb{R}$ . The isomorphism  $\pi$  is determined by the image  $\{v(t) : t \in \mathbb{R}\}$  of the one parameter unitary group  $\{V(t)\}$ .  $\square$

Because of this result, a natural question is how to compute  $\varphi_\pi$  from the embedding  $\pi$  of  $\{\mathcal{A}, \mathbb{R}, \rho\}$ , or from  $\{v(t)\} = \{\pi(V(t))\}$ . The following proposition answers this question:

**Proposition 3.7.** Suppose  $\pi = \pi^\varphi$  for a fixed  $\varphi \in \mathfrak{W}(\mathcal{M})$ . Let  $h = \pi(H)$  be the analytic generator of the one parameter unitary group  $\{v(t)\} = \{\pi(V(t))\}$  in  $\tilde{\mathcal{M}}_{s(\varphi)}$ . For any  $f \in L^\infty(\mathbb{R})_+$  with  $\int_{\mathbb{R}} f(t)dt = 1$  we have

$$\varphi(x) = \lim_{\varepsilon \searrow 0} \tau(h^{\frac{1}{2}}(1 + \varepsilon h^{\frac{1}{2}})^{-\frac{1}{2}} x^{\frac{1}{2}} \pi(f) x^{\frac{1}{2}} h^{\frac{1}{2}}(1 + \varepsilon h^{\frac{1}{2}})^{-\frac{1}{2}}), \quad x \in \tilde{\mathcal{M}}_+. \quad (3.9)$$

Symbolically, we can write:

$$\varphi(x) = \tau(h^{\frac{1}{2}} x^{\frac{1}{2}} \pi(f) x^{\frac{1}{2}} h^{\frac{1}{2}}), \quad x \in \tilde{\mathcal{M}}_+. \quad (3.9')$$

*Proof.* First, for any positive self adjoint operator  $K$  affiliated with  $\tilde{\mathcal{M}}$ , we will write  $\tau_K$  for the semi-finite normal weight on  $\tilde{\mathcal{M}}$  given by

$$\tau_K(x) := \lim_{\varepsilon \searrow 0} \tau(K^{\frac{1}{2}}(1 + \varepsilon K^{\frac{1}{2}})^{-\frac{1}{2}} x K^{\frac{1}{2}}(1 + \varepsilon K^{\frac{1}{2}})^{-\frac{1}{2}}), \quad x \in \tilde{\mathcal{M}}_+.$$

From Lemma 3.2, it follows that  $\tilde{\varphi} = \tau_K$ . Now, we compute for  $x \in \mathcal{M}_+$

$$\begin{aligned} \tau_K(x^{\frac{1}{2}} \pi(f) x^{\frac{1}{2}}) &= \tilde{\varphi}(x^{\frac{1}{2}} \pi(f) x^{\frac{1}{2}}) = \varphi(I_\theta(x^{\frac{1}{2}} \pi(f) x^{\frac{1}{2}})) \\ &= \varphi\left(x^{\frac{1}{2}} I_\theta(\pi(f)) x^{\frac{1}{2}}\right) \\ &= \varphi(x) \end{aligned}$$

as

$$I_\theta(\pi(f)) = \int_{\mathbb{R}} \theta_t(\pi(f)) dt = \pi\left(\int_{\mathbb{R}} (\rho_t(f)) dt\right) = \pi(1) = s(\varphi). \quad \square$$

**Lemma 3.8.** *If  $\mu$  is a normal weight on  $\mathcal{A}$  such that  $\mu \circ \rho_s = e^{-s}\mu$  and  $0 < \mu(f_0) < \infty$  for some  $f_0 \in \mathcal{A}_+$ , then the weight  $\mu$  is of the following form with some constant  $C > 0$*

$$\mu(f) = C \int_{\mathbb{R}} e^s f(s) ds, \quad f \in \mathcal{A} = L^\infty(\mathbb{R}), \quad (3.10)$$

and it is therefore faithful and semi-finite.

*Proof.* Let  $\mathfrak{m}_\mu$  be the definition ideal of the weight  $\rho$ . Then  $f_0 \in \mathfrak{m}_\mu$ , so that  $\mathfrak{m}_\mu \neq \{0\}$ . By the relative invariance of  $\mu$  under the action  $\rho$ , the ideal  $\mathfrak{m}_\mu$  is also invariant. Therefore,  $\mathfrak{m}_\mu$  is  $\sigma$ -weakly dense in  $\mathcal{A}$ . Similarly, the left kernel  $N_\mu = \{x \in \mathcal{A} : \mu(x^*x) = 0\}$  of  $\mu$  is  $\rho$ -invariant and  $N_\mu \neq \mathcal{A}$  as  $f_0 \notin N_\mu$ . Thus  $N_\mu = \{0\}$ . This means that the weight  $\mu$  is semi-finite and faithful. Consider the function  $H$  defined by (3.7) and observe that  $\rho_s(H) = e^{-s}H, s \in \mathbb{R}$ . Thus the new normal weight  $\mu'$  given by  $\mu'(f) = \mu(Hf), f \in \mathcal{A}_+$ , is invariant under the action  $\rho$  and semi-finite and faithful. The uniqueness of translation invariant regular Borel measure on  $\mathbb{R}$  implies the existence of a constant  $C > 0$  such that  $\mu'(f) = C \int_{\mathbb{R}} f(s) ds, f \in \mathcal{A}_+$ . Thus our assertion follows.  $\square$

The following is a counterpart of a result of U. Haagerup, [H: Theorem X], recast in the context of quantum flow of weights.

**Theorem 3.9.** *The following statements for  $\varphi \in \mathfrak{W}(\mathcal{M})$  are equivalent:*

- i) *The weight  $\varphi$  is finite, i.e.  $\varphi(1) < \infty$ ;*
- ii)  *$\pi^\varphi(\mathcal{A}) \cap \mathfrak{m}_\tau \neq \{0\}$ ;*
- iii) *The operator  $\varphi$  in  $\tilde{\mathcal{M}}$  is  $\tau$ -measurable in the sense of I. E. Segal, [Seg].*

*Proof.* i)  $\Rightarrow$  ii): Suppose  $\varphi$  is finite. Let  $p = \chi_{[0,1]} \in \mathcal{A}_+$ , the characteristic function of the unit interval. We have  $I_\theta(p) = 1, \|H^{\pm 1}p\| = e^{\pm 1} < \infty$  and

$$\tau(\pi^\varphi(p)) = \tilde{\varphi}(\varphi^{-1}p) = \varphi(I_\theta(\varphi^{-1}p)) \leq e\varphi(I_\theta(p)) = e\|\varphi\| < \infty.$$

Hence  $\pi^\varphi(\mathcal{A}) \cap \mathfrak{m}_\tau \neq \{0\}$ .

ii)  $\Rightarrow$  iii): The assertion (ii) implies the existence of a function  $f_0 \in \mathcal{A}_+$  with  $0 < \tau(\pi^\varphi(f_0)) < \infty$ . By Lemma 3.8 the weight  $\tau$  on  $\pi^\varphi(\mathcal{A})$  is semi-finite and given by (3.10). Since  $\chi_{[\lambda,+\infty)}(H) = \chi_{(-\infty, -\log \lambda]}$  for any  $\lambda > 0$ , we have

$$\tau(\chi_{[\lambda,+\infty)}(\varphi)) = C \int_{-\infty}^{-\log \lambda} e^s ds = \frac{C}{\lambda} < \infty.$$

iii)  $\Rightarrow$  ii): This implication is an immediate consequence of the definition of  $\tau$ -measurability.

ii)  $\Rightarrow$  i): The  $\theta$ -invariance of  $\mathfrak{m}_\tau \cap \pi^\varphi(\mathcal{A})$  implies the  $\sigma$ -weak density of the ideal  $\mathfrak{m}_\tau \cap \pi^\varphi(\mathcal{A})$  in  $\pi^\varphi(\mathcal{A})$ . This means that  $\tau$  is semi-finite on  $\pi^\varphi(\mathcal{A})$ , so that Lemma 3.16 applies to  $\mu = \tau \circ \pi^\varphi$ . With  $p$  as above, we have

$$\begin{aligned} \varphi(1) &= \varphi(I_\theta(\pi^\varphi(p))) = \tilde{\varphi}(\pi^\varphi(p)) = \tau(\varphi\pi^\varphi(p)) \\ &= \mu(Hp) = C \int_0^1 ds = C < \infty. \quad \square \end{aligned}$$

Motivated by the term "density" in [C9], we introduce

**Definition 3.10.** For each  $\alpha \in \mathbb{C}$ , a closed, densely-defined operator  $T$  affiliated with  $\tilde{\mathcal{M}}$  is said to be of grade  $\alpha$  if

$$\theta_s(T) = e^{-\alpha s}T, \quad s \in \mathbb{R}; \quad (3.11)$$

we will use the notation  $\text{grad}(T)$  to refer to  $\alpha$ .

**Proposition 3.11.** Suppose that  $S$  and  $T$  are closed, densely-defined operators affiliated with  $\tilde{\mathcal{M}}$ . Then,

- i)  $\text{grad}(S^*) = \overline{\text{grad}(S)}$ .
- ii) If the product  $ST$  is densely defined and preclosed, then  $\text{grad}(\overline{ST}) = \text{grad}(S) + \text{grad}(T)$  where  $\overline{ST}$  is the closure of  $ST$ .
- iii)  $\text{grad}(|T|) = \Re(\text{grad}(T)) = \frac{\text{grad}(T) + \overline{\text{grad}(T)}}{2}$ .
- iv) For  $T \neq 0$ , if  $\text{grad}(T)$  is not pure imaginary, then  $T$  must be unbounded and the spectrum of  $|T|$  is absolutely continuous relative to the Lebesgue measure.
- v) If  $T \geq 0$  and  $p = \text{grad}(T) \neq 0$ , then there exists uniquely  $\varphi \in \mathfrak{W}(\mathcal{M})$  such that  $T^{\frac{1}{p}} = h_\varphi$  where  $h_\varphi$  is the operator of Proposition 3.8 corresponding to  $\varphi$ . In other words, after identifying  $\varphi$  with  $h_\varphi$ , we have  $T^{\frac{1}{p}} \in \mathfrak{W}(\mathcal{M})$ .

*Proof.* The claims (i), (ii) and (iii) are trivial, so we omit their proofs. We will prove (iv) and (v) together. The unboundedness of  $T$  is obvious because  $\theta_s$  is an isometry of  $\tilde{\mathcal{M}}$  so that  $\|T\|$  cannot be finite. Let  $e$  be the support projection of  $T \geq 0$ , i.e. the range projection of  $T$ . Consider the one parameter unitary group  $\{U(t) : t \in \mathbb{R}\}$  in  $\tilde{\mathcal{M}}_e$  given by  $U(t) = T^{\frac{it}{p}}$ . Then we have  $\theta_s(U(t)) = e^{-ist}U(t), s, t \in \mathbb{R}$ . Therefore there exists a equivariant isomorphism  $\pi$  from  $\{\mathcal{A}, \mathbb{R}, \rho\}$  into  $\tilde{\mathcal{M}}_e$  such that  $\pi(V(t)) = U(t), t \in \mathbb{R}$ . Thus by Theorem 3.6.ii, there exists  $\varphi \in \mathfrak{W}(\mathcal{M})$  such that  $U(t) = \varphi^{it}, t \in \mathbb{R}$ .  $\square$

Hence, the set  $\mathfrak{W}(\mathcal{M})$  of semi-finite normal weights on  $\mathcal{M}$  is identified with the set of all densely defined positive self-adjoint operator of grade 1 affiliated with  $\tilde{\mathcal{M}}$ . Naturally, we want next to identify the predual  $\mathcal{M}_*$  as a subset of the set of operators with grade one. Recall the polar decomposition  $\omega = u|\omega|$  for  $\omega \in \mathcal{M}_*$  in the predual  $\mathcal{M}_*$ . Theorem 3.9 gives the criteria for  $|\omega|$  to be finite. Thus, we have the following characterization of operators corresponding to elements in the predual:

**Theorem 3.12.** Let  $\mathfrak{M}^1$  be the set of all  $\tau$ -measurable operators affiliated with  $\tilde{\mathcal{M}}$  of grade one. Then there exists a natural bijection:

$$\omega \in \mathcal{M}_* \leftrightarrow T(\omega) \in \mathfrak{M}^1$$

such that

- i)  $T(\omega) \geq 0 \Leftrightarrow \omega \geq 0$ ;
- ii)  $T(\varphi) = \varphi$  if  $\varphi \in \mathcal{M}_*^+$ ;
- iii)  $T(a\omega b) = aT(\omega)b, a, b \in \mathcal{M}, \omega \in \mathcal{M}_*$ ;

iv) if  $a \in \mathfrak{m}_{I_\theta}^+$  has  $I_\theta(a) = 1$ , then  $a^{\frac{1}{2}}xT(\omega)a^{\frac{1}{2}} \in L^1(\tilde{\mathcal{M}}, \tau)$ ,  $x \in \mathcal{M}$ , and

$$\omega(x) = \tau(a^{\frac{1}{2}}xT(\omega)a^{\frac{1}{2}}), \quad x \in \mathcal{M}. \quad (3.12)$$

Consequently, the value of the right hand side does not depend on the choice of  $a$ . Namely, the notation

$$\int T = \tau(a^{\frac{1}{2}}Ta^{\frac{1}{2}}), \quad T \in \mathfrak{M}^1, \quad (3.12')$$

is justified, and the bilinear form:

$$\langle x, T \rangle = \int xT, \quad (3.12'')$$

gives the pairing between  $\mathcal{M}$  and  $\mathfrak{M}^1$  which identifies  $\mathfrak{M}^1$  with  $\mathcal{M}_*$ .

*Proof.* i) To avoid possible confusion, let us write  $T(\varphi)$  for  $\varphi \in \mathfrak{W}(\mathcal{M})$  when we consider it as an operator affiliated with  $\tilde{\mathcal{M}}$ . Theorem 3.9 asserts that  $T(\varphi) \in \mathfrak{M}^1$  if and only if  $\varphi \in \mathcal{M}_*^+$ . For a general  $\omega \in \mathcal{M}_*$  set

$$T(\omega) = uT(|\omega|) \quad (3.13)$$

with  $\omega = u|\omega|$  the polar decomposition of  $\omega$ . Observe that if  $\omega \in \mathcal{M}_*^+$ , then the support projection  $s(T(\omega))$  of  $T(\omega)$ , the range projection, is precisely  $s(\omega)$  the support of  $\omega$ . This means that the decomposition of  $T(\omega)$  for a general  $\omega \in \mathcal{M}_*$  of (3.13) is precisely the polar decomposition. Thus (i) follows.

ii) Trivial.

iii) If  $\omega = u|\omega|$  is the polar decomposition of  $\omega \in \mathcal{M}_*$ , then for all unitaries  $a, b \in \mathcal{U}(\mathcal{M})$  we have the polar decomposition:

$$a\omega b^* = (aub^*)b|\omega|b^*$$

of  $a\omega b^*$ . Thus,

$$T(a\omega b^*) = (aub^*)T(b|\omega|b^*) = auT(|\omega|)b^* = aT(\omega)b^*.$$

The assertion follows by linearity.

iv) Suppose  $T = T(\omega)$ ,  $\omega \in \mathcal{M}_*$ . Let  $T = UK$  be the polar decomposition. By Theorem 3.18,  $K = T(|\omega|)$  is  $\tau$ -measurable. Hence  $T$  is  $\tau$ -measurable.

Conversely, suppose that  $T$  is  $\tau$ -measurable. Let  $T = UK$  be the polar decomposition. Then  $U$  is fixed under  $\theta$ , so that  $U \in \mathcal{M}$ . Since  $K$  is  $\tau$ -measurable and of grade one, there exists  $\varphi \in \mathcal{M}_*^+$  such that  $K = T(\varphi)$ . Now we have  $T = T(\omega)$  with  $\omega = U\varphi \in \mathcal{M}_*$ .

Now we fix  $a \in \mathfrak{m}_+$  with  $I_\theta(a) = 1$  and  $\omega \in \mathcal{M}_*$ . Let  $\omega = u\varphi$  be the polar decomposition. Set  $T = T(\omega)$  and  $K = T(\varphi)$ . Then  $K = |T|$  and  $\tau$  is semi-finite on  $\pi^\varphi(\mathcal{A})$ . Therefore, there exists a conditional expectation  $\mathcal{E}$  from  $\tilde{\mathcal{M}}$  onto  $\pi^\varphi(\mathcal{A})$  such that  $\tau(exe) = \tau \circ \mathcal{E}(x)$ ,  $x \in \tilde{\mathcal{M}}$ , with  $e = s(\varphi)$ . We claim  $\mathcal{E} \circ \theta_s = \theta_s \circ \mathcal{E}$ ,  $s \in \mathbb{R}$ . In fact, for any  $f \in \mathcal{A}$  and  $x \in \tilde{\mathcal{M}}$ , we have

$$\begin{aligned} \tau(\mathcal{E}(\theta_s(x))\pi^\varphi(f)) &= \tau(\theta_s(x)\pi^\varphi(f)) = \tau(\theta_s(x)\theta_{-s}(\pi^\varphi(f))) \\ &= e^{-s}\tau(x\pi^\varphi(\rho_{-s}(f))) = e^{-s}\tau(\mathcal{E}(x)\pi^\varphi(\rho_{-s}(f))) \\ &= \tau(\theta_s(\mathcal{E}(x))\pi^\varphi(f)). \end{aligned}$$

Therefore, we have  $\mathcal{E}(\mathfrak{m}_{I_\theta}) = \pi^\varphi(L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}))$ . We next claim that  $K^{\frac{1}{2}}a^{\frac{1}{2}} \in L^2(\tilde{\mathcal{M}}, \tau)$ . Indeed,

$$\begin{aligned} \tau((K^{\frac{1}{2}}a^{\frac{1}{2}})^*(K^{\frac{1}{2}}a^{\frac{1}{2}})) &= \tau(a^{\frac{1}{2}}Ka^{\frac{1}{2}}) = \tau(K^{\frac{1}{2}}aK^{\frac{1}{2}}) \\ &= \tilde{\varphi}(a) = \varphi(I_\theta(a)) = \varphi(1) = \|\varphi\| = \|\omega\| < \infty. \end{aligned}$$

Also, we have  $a^{\frac{1}{2}}xuK^{\frac{1}{2}} \in L^2(\tilde{\mathcal{M}}, \tau)$ ,  $x \in \mathcal{M}$ , by the similar computation:

$$\begin{aligned} \tau\left((a^{\frac{1}{2}}xuK^{\frac{1}{2}})^*(a^{\frac{1}{2}}xuK^{\frac{1}{2}})\right) &= \tau(K^{\frac{1}{2}}u^*x^*axuK^{\frac{1}{2}}) = \tilde{\varphi}(u^*x^*axu) \\ &= \varphi(I_\theta(u^*x^*axu)) = \varphi(u^*x^*xu) < \infty. \end{aligned}$$

Thus we get  $a^{\frac{1}{2}}xT(\omega)a^{\frac{1}{2}} = (a^{\frac{1}{2}}xuK^{\frac{1}{2}})(K^{\frac{1}{2}}a^{\frac{1}{2}}) \in L^1(\tilde{\mathcal{M}}, \tau)$  and (3.12) follows.  $\square$

The proof of the above theorem immediately yields the following:

**Corollary 3.13.** *If  $\mathfrak{M}^2$  be the set of all  $\tau$ -measurable operators affiliated to  $\tilde{\mathcal{M}}$  of grade  $\frac{1}{2}$ , then the inner product:*

$$(S|T) = \int T^*S, \quad S, T \in \mathfrak{M}^2, \quad (3.14)$$

*makes  $\mathfrak{M}^2$  a Hilbert space which can be identified with the standard form  $L^2(\mathcal{M})$  of  $\mathcal{M}$ .*

It is now possible to unify the theory of non-commutative  $L^p$ -spaces. For a fixed  $1 \leq p \leq \infty$ , the space  $L^p(\mathcal{M})$  is obtained as the completion of the set  $\mathcal{M}\omega^{\frac{1}{p}}$ ,  $\omega \in \mathcal{M}_*^+$ . This is however identified with  $\mathfrak{M}^p$ , the set of all  $\tau$ -measurable operators of grade  $\frac{1}{p}$  affiliated to  $\tilde{\mathcal{M}}$ , which is a Banach space with the norm:

$$\|T\|_p = \left( \int |T|^p \right)^{\frac{1}{p}}, \quad T \in \mathfrak{M}^p, \quad (3.15)$$

and the pairing of  $\mathfrak{M}^p \times \mathfrak{M}^q$  with  $\frac{1}{p} + \frac{1}{q} = 1$  is given by:

$$\langle S, T \rangle = \int ST, \quad S \in \mathfrak{M}^p, \quad T \in \mathfrak{M}^q. \quad (3.16)$$

We leave the details to the interested reader; see also [H2, Iz, Ks].

**Remark 3.14.** i) If we denote the set of all  $\tau$ -measurable operators of degree  $\alpha$  for  $\alpha \in \mathbb{C}$  by  $\mathfrak{M}^{\frac{1}{\alpha}}$  (a natural extension of the above notation), then we have  $\mathfrak{M}^{\frac{1}{s}} = \mathcal{M}(s)$  for  $s \in \mathbb{R}$ .  
 ii) If  $\Re(\alpha) < 0$ , then  $\mathfrak{M}^{\frac{1}{\alpha}} = \{0\}$ .

The remark (i) follows from the fact that if  $T \in \mathfrak{M}$  and  $\theta_t(T) = T, t \in \mathbb{R}$ , then  $T$  is bounded and therefore  $T \in \mathcal{M}$  because  $\tau(p) = +\infty$  for every non-zero  $p \in \text{Proj}(\mathcal{M})$ . The remark (ii) follows from the fact that if  $T$  is a positive self-adjoint operator of degree  $p < 0$ , then we have  $\tau(\chi_{[\lambda, \infty)}(T)) = \infty$  for every  $\lambda \in \mathbb{R}$ . So if  $T$  is  $\tau$ -measurable, then  $T$  must be bounded. But the grade condition on  $T$  makes boundedness impossible except for any  $T$  other than  $T = 0$ .

**Corollary 3.15.** Let  $\mathfrak{M}$  be the  $*$ -algebra of all  $\tau$ -measurable operators affiliated to  $\tilde{\mathcal{M}}$ . Let  $\mathbb{H}$  be the right half plane in  $\mathbb{C}$ . Then for every  $\alpha, \beta \in \mathbb{H}$  and  $\varphi, \psi \in \mathcal{M}_*^+$  we can add and multiply freely elements of  $\varphi^\alpha \mathcal{M}$  and  $\psi^\beta \mathcal{M}$  inside the  $*$ -algebra  $\mathfrak{M}$ .

#### §4. The Local Characteristic Square – Extended Unitary Group and Modular Automorphism Group

We begin by first citing an important result concerning the structure of the quantum flow of weights from the work of Katayama, Sutherland and Takesaki, [KtST].

**Theorem 4.1.** The relative commutant  $\mathcal{M}' \cap \tilde{\mathcal{M}}$  of the original von Neumann algebra  $\mathcal{M}$  in the core  $\tilde{\mathcal{M}}$  is the center  $\mathcal{C}_{\tilde{\mathcal{M}}}$  of  $\tilde{\mathcal{M}}$ .

Now, let's fix a  $\varphi \in \mathfrak{W}(\mathcal{M})$ , and set  $\mathcal{A}^\varphi = \pi^\varphi(\mathcal{A})$  and  $\mathcal{D}^\varphi = \mathcal{C} \vee \mathcal{A}^\varphi = \mathcal{C} \vee \pi^\varphi(L^\infty(\mathbb{R}))$ . Observe the following easy but important facts:

$$\begin{cases} (\mathcal{A}^\varphi)' \cap \mathcal{M} = \mathcal{M}_\varphi; \\ \mathcal{D}^\varphi \cap \mathcal{M} \subset \mathcal{C}_\varphi, \end{cases} \quad (4.1)$$

where  $\mathcal{M}_\varphi$  is the centerizer of  $\varphi$  in  $\mathcal{M}$  and  $\mathcal{C}_\varphi$  is the center of  $\mathcal{M}_\varphi$ .

To avoid unnecessary complication, let us assume that our von Neumann algebra  $\mathcal{M}$  is a factor.

Recall the characteristic square for a factor  $\mathcal{M}$  cited in the introduction, in particular the middle row for the extended unitary group  $\tilde{\mathcal{U}}(\mathcal{M})$ :

$$1 \longrightarrow \mathcal{U}(\mathcal{M}) \longrightarrow \tilde{\mathcal{U}}(\mathcal{M}) \xrightarrow{\partial} Z_\theta^1(\mathcal{U}(\mathcal{C})) \longrightarrow 1, \quad (4.2)$$

where  $(\partial b)_t = b\theta_t(b^*), t \in \mathbb{R}, b \in \tilde{\mathcal{U}}(\mathcal{M})$ . In [CT], it was shown that each dominant  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  gives rise to a continuous injective homomorphism  $\sigma^\varphi : c \in Z_\theta^1(\mathcal{U}(\mathcal{C})) \mapsto \sigma_c^\varphi \in \text{Aut}(\mathcal{M})$ , called an extended modular automorphism. It was further shown that if  $c \in Z_\theta^1(\mathcal{U}(\mathcal{C}))$  is twice differentiable, then the extended modular automorphism  $\sigma_c^\varphi$  makes sense for arbitrary  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ . A natural question left in [CT] unanswered so far is whether this extended modular automorphism  $\sigma_c^\varphi$  makes sense for general  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  and  $c \in Z_\theta^1(\mathcal{U}(\mathcal{C}))$ . In a joint work of Sutherland and Takesaki [ST], they proved that

every element  $c \in Z_\theta^1(\mathcal{U}(\mathcal{C}))$  is cohomologous to an infinitely differentiable one. We want to explore this question in the context of quantum flow of weights.

First, we state

**Theorem 4.2.** *To each  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  there corresponds a right inverse  $b_\varphi : c \in Z_\theta^1(\mathcal{U}(\mathcal{C})) \mapsto b_\varphi(c) \in \tilde{\mathcal{U}}(\mathcal{M}) \cap \mathcal{D}^\varphi$  of the coboundary map  $\partial$  such that*

i)

$$\partial(b_\varphi(c)) = c, \quad c \in Z_\theta^1(\mathcal{U}(\mathcal{C})); \quad (4.3)$$

- ii)  $b_\varphi$  is a continuous homomorphism of  $Z_\theta^1(\mathcal{U}(\mathcal{C}))$  into  $\tilde{\mathcal{U}}(\mathcal{M}) \cap \mathcal{D}^\varphi$ ;  
 iii) For every  $\alpha \in \text{Aut}(\mathcal{M})$  we have:

$$b_{\varphi \circ \alpha^{-1}} = \tilde{\alpha} \circ b_\varphi \circ \tilde{\alpha}^{-1}; \quad (4.4)$$

iv) If  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  is dominant, then

$$\widetilde{\text{Ad}}(b_\varphi(c^*)) = \sigma_c^\varphi, \quad c \in Z_\theta^1(\mathcal{U}(\mathcal{C})); \quad (4.5)$$

v) For every pair  $\varphi, \psi \in \mathfrak{W}_0(\mathcal{M})$  of dominant weights, we have

$$b_\varphi(c^*) b_\psi(c^*)^* = (D\varphi : D\psi)_c, \quad c \in Z_\theta^1(\mathcal{U}(\mathcal{C})). \quad (4.6)$$

vi) For each  $c^1, c^2 \in Z_\theta^1(\mathcal{U}(\mathcal{C}))$ , we have:

$$(D\varphi : D\psi)_{c^1 c^2} = (D\varphi : D\psi)_{c^1} \sigma_{c^1}^\psi ((D\varphi : D\psi)_{c^2}). \quad (4.7)$$

- vii) Relative to the strong resolvent convergence topology on the space  $\mathfrak{W}_0(\mathcal{M})$  of faithful semi-finite normal weights and the  $\sigma^*$ -strong topology in  $\tilde{\mathcal{U}}(\mathcal{M})$  the correspondence  $b : \varphi \in \mathfrak{W}_0(\mathcal{M}) \mapsto b_\varphi(c) \in \tilde{\mathcal{U}}(\mathcal{M})$  is continuous for each  $c \in Z_\theta^1(\mathbb{R}, \mathcal{U}(\mathcal{C}))$ .  
 viii) In the case  $\mathcal{M}$  is  $\sigma$ -finite, the map:  $\varphi \in \mathfrak{W}_0(\mathcal{M}) \cap \mathcal{M}_* \mapsto b_\varphi(c) \in \tilde{\mathcal{U}}(\mathcal{M})$  is continuous relative to the norm topology in the first space and the  $\sigma^*$ -strong topology on the second space for each fixed  $c \in Z_\theta^1(\mathcal{U}(\mathcal{C}))$ .

Before proceeding with the proof, we establish notation and a suitable realization of the quantum flow  $\{\tilde{\mathcal{M}}, \mathbb{R}, \theta, \tau\}$ . With  $\{\mathcal{M}, \mathfrak{H}\}$  a fixed representation, we define a map  $W_\varphi : L^2(\mathbb{R}) \otimes \mathfrak{H} \mapsto L^2(\mathcal{G}(\mathfrak{H}))$  for each  $\varphi \in \mathfrak{W}(\mathcal{M})$  as follows:

$$(W_\varphi \xi)(s) = \varphi^{is} \xi(s) \in s(\varphi) \mathfrak{H}(t), \quad \xi \in L^2(\mathbb{R}, \mathfrak{H}), \varphi \in \mathfrak{W}(\mathcal{M}) \quad (4.8)$$

which is a partial isometry from  $L^2(\mathbb{R}, \mathfrak{H})$  into  $L^2(\mathcal{G}(\mathfrak{H}))$  such that

$$\begin{aligned} W_\varphi(s(\varphi) \otimes \lambda(t)) W_\varphi^* &= \varphi^{it}, \quad t \in \mathbb{R}, \varphi \in \mathfrak{W}(\mathcal{M}); \\ W_\varphi^* W_\varphi &= s(\varphi) \mathfrak{H} \otimes 1 \quad \text{and} \quad W_\varphi W_\varphi^* = s(\varphi)_{L^2(\mathcal{G}(\mathfrak{H}))} \end{aligned} \quad (4.9)$$



where the indices  $\mathfrak{H}$  and  $L^2(\mathcal{G}(\mathfrak{H}))$  indicate the representation spaces of the projection  $s(\varphi)$  respectively.

With  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  fixed, we know that  $\tilde{\mathcal{M}}$  is naturally identified under the unitary  $W_\varphi : L^2(\mathbb{R}, \mathfrak{H}) \mapsto L^2(\mathcal{G}(\mathfrak{H}))$  with the crossed product  $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R} = W_\varphi^* \tilde{\mathcal{M}} W_\varphi$ . We want to choose the diagonalization of  $\{\varphi^{it}\}$  rather than the dual action  $\{\theta_t\}$ . So let  $F$  be the Fourier transform and the Fourier inverse transform on  $\tilde{\mathfrak{H}} = L^2(\mathbb{R}, \mathfrak{H})$  given by

$$\begin{cases} (F\xi)(p) = \sqrt{\frac{1}{2\pi}} \int_{\mathbb{R}} e^{-isp} \xi(s) ds; \\ (F^*\xi)(s) = \sqrt{\frac{1}{2\pi}} \int_{\mathbb{R}} e^{isp} \xi(p) dp, \end{cases} \quad \xi \in L^2(\mathbb{R}, \mathfrak{H}), \quad (4.10)$$

where the above integral should be understood as the improper integral:

$$\int_{\mathbb{R}} f(t) dt = \lim_{n \rightarrow \infty} \int_{-n}^n f(t) dt \quad (4.11)$$

for those functions  $f$  such that the above limit exists. We then work with the von Neumann algebra  $F(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})F^*$  instead of the original crossed product. We have then:

$$\begin{cases} (F\varphi^{it}F^*\xi)(s) = e^{-ist} \xi(s) = (V(t)\xi)(s); \\ (F\mu(p)F^*\xi)(s) = \xi(s-t) = (\bar{\lambda}(t)\xi)(s). \end{cases} \quad (4.12)$$

The von Neumann algebra  $F(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})F^* = F(\mathcal{M} \overline{\otimes} \mathcal{L}(L^2(\mathbb{R})))^{(\sigma^\varphi \otimes \rho)} F^*$  is then identified with the fixed point algebra  $(\mathcal{M} \overline{\otimes} \mathcal{L}(L^2(\mathbb{R})))^{\sigma^\varphi \otimes \text{Ad}(\mu)}$ . In order to keep track of the identification, let us denote by  $\tilde{\pi}^\varphi$  the isomorphism of  $\tilde{\mathcal{M}}$  onto  $F(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})F^*$ , i.e.

$$\tilde{\pi}^\varphi(x) = FW_\varphi^* x W_\varphi F^*, \quad x \in \tilde{\mathcal{M}}. \quad (4.13)$$

*Proof of Theorem 4.2.* As mentioned above, we identify  $\tilde{\mathcal{M}}$  with  $F(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})F^*$  and  $\varphi^{it}$  with  $V(t)$ . The quantum flow  $\{\theta_t\}$  is then given by the one parameter unitary group  $\{\bar{\rho}(s) : s \in \mathbb{R}\} = \{1 \otimes \rho(s) : s \in \mathbb{R}\}$ . We set:

$$(\check{b}_\varphi(c)\xi)(s) = (\tilde{\pi}^\varphi(c_{-s})\xi)(s), \quad c \in Z_\theta^1(\mathcal{U}(\mathcal{C})), \xi \in \tilde{\mathfrak{H}}, s \in \mathbb{R}. \quad (4.14)$$

We then put  $b_\varphi(c) = \tilde{\pi}^{\varphi^{-1}}(\check{b}_\varphi(c))$ . It is easily seen that  $b_\varphi(c) \in \mathcal{D}^\varphi = \mathcal{C} \vee \mathcal{A}^\varphi$  and that  $b_\varphi(c^1 c^2) = b_\varphi(c^1) b_\varphi(c^2)$  for every  $c^1, c^2 \in Z_\theta^1(\mathcal{U}(\mathcal{C}))$ . Also the map  $b_\varphi : c \in Z_\theta^1(\mathcal{U}(\mathcal{C})) \mapsto$

$b_\varphi(c) \in \mathcal{D}^\varphi$  is continuous. Now with  $b = b_\varphi(c)$  for short, we compute:

$$\begin{aligned}
(\tilde{\pi}^\varphi(b\theta_t(b^*))\xi)(s) &= (\tilde{\pi}^\varphi(c_{-s})\tilde{\pi}^\varphi(\theta_t(b^*)))\xi)(s) = (\tilde{\pi}^\varphi(\theta_t(b^*))\tilde{\pi}^\varphi(c_{-s})\xi)(s) \\
&= (\bar{\rho}(t)\tilde{\pi}^\varphi(b^*)\bar{\rho}(t)^*\tilde{\pi}^\varphi(c_{-s})\xi)(s) \\
&= (\tilde{\pi}^\varphi(b^*)\bar{\rho}(t)^*\tilde{\pi}^\varphi(c_{-s})\xi)(s+t) \\
&= (\tilde{\pi}^\varphi(c_{-(s+t)}^*)\bar{\rho}(t)^*\tilde{\pi}^\varphi(c_{-s})\xi)(s+t) \\
&= (\bar{\rho}(t)^*\tilde{\pi}^\varphi(\theta_t(c_{-(s+t)}^*)c_{-s})\xi)(s+t) \\
&= \left( \tilde{\pi}^\varphi(\theta_t(c_{-t}^*)\theta_{-t}(c_{-s}^*))c_{-s})\xi \right)(s) \\
&= \left( \tilde{\pi}^\varphi(\theta_{-t}(c_{-t}^*)c_{-s}^*)c_{-s})\xi \right)(s) \\
&= (\tilde{\pi}^\varphi(\theta_t(c_{-t}^*))\xi)(s) = (\tilde{\pi}^\varphi(c_t)\xi)(s).
\end{aligned}$$

Hence we get  $c_t = b\theta_t(b^*) = \partial(b)_t, t \in \mathbb{R}$ .

We are now going to prove the equivariance of  $b_\varphi$ . As

$$\sigma_t^{\varphi \circ \alpha^{-1}} = \alpha \circ \sigma_t^\varphi \circ \alpha^{-1}, t \in \mathbb{R}, \alpha \in \text{Aut}(\mathcal{M}),$$

we have

$$\left\{ \begin{array}{l} \pi^{\varphi \circ \alpha^{-1}} = \tilde{\alpha} \circ \pi^\varphi; \\ \tilde{\pi}^{\varphi \circ \alpha^{-1}}(x) = \tilde{U}(\alpha)\tilde{\pi}^\varphi(x)U(\alpha)^*, \quad x \in \tilde{\mathcal{M}}; \\ (\varphi \circ \alpha^{-1})^{it} = \tilde{\alpha}(\varphi^{it}); \\ \mathcal{A}^{\varphi \circ \alpha^{-1}} = \tilde{\alpha}(\mathcal{A}^\varphi); \\ \mathcal{D}^{\varphi \circ \alpha^{-1}} = \tilde{\alpha}(\mathcal{D}^\varphi), \end{array} \right. \quad (4.15)$$

where  $\tilde{U}(\alpha) = U(\alpha) \otimes 1$  on  $\tilde{\mathfrak{H}} = L^2(\mathbb{R}, \mathfrak{H}) = \mathfrak{H} \otimes L^2(\mathbb{R})$  with the standard Hilbert space  $\mathfrak{H}$  and  $U(\alpha)$  the unitary such that

$$\left\{ \begin{array}{l} U(\alpha)\mathfrak{H}_+ = \mathfrak{H}_+; \\ U(\alpha)xU(\alpha)^* = \alpha(x), \quad x \in \mathcal{M}. \end{array} \right.$$

The extended automorphism  $\tilde{\alpha} \in \text{Aut}(\tilde{\mathcal{M}})$  is then implemented by the unitary  $\tilde{U}(\alpha) = U(\alpha) \otimes 1$  on  $\tilde{\mathfrak{H}} = \mathfrak{H} \otimes L^2(\mathbb{R})$ . Now we compute for  $c \in Z_\theta^1(\mathcal{U}(\mathcal{C}))$ :

$$\begin{aligned}
(\tilde{\pi}^\varphi(\tilde{\alpha}(b_\varphi(\tilde{\alpha}^{-1}(c))))\xi)(s) &= (\tilde{U}(\alpha)\tilde{\pi}^\varphi(b_\varphi(\tilde{\alpha}^{-1}(c)))\tilde{U}(\alpha)^*\xi)(s) \\
&= U(\alpha)(\tilde{\pi}^\varphi(\tilde{\alpha}^{-1}(c_{-s}))\tilde{U}(\alpha)^*\xi)(s) \\
&= (\tilde{U}(\alpha)\tilde{\pi}^\varphi(\tilde{\alpha}^{-1}(c_{-s}))\tilde{U}(\alpha)^*\xi)(s) \\
&= (\tilde{\pi}^{\varphi \circ \alpha^{-1}}(c_{-s})\xi)(s) \\
&= (\tilde{\pi}^{\varphi \circ \alpha^{-1}}(b_{\varphi \circ \alpha^{-1}}(c))\xi)(s).
\end{aligned}$$

Thus the equivariance of the maps  $b_\varphi, \varphi \in \mathfrak{W}_0(\mathcal{M})$ , follows.

Before continuing with the proofs of statements (iv) and (v), we need to lay some groundwork. We fix a dominant  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  and set

$$\mathcal{M}_\varphi(t) = \{x \in \mathcal{M} : \sigma_s^\varphi(x) = e^{ist}x, s \in \mathbb{R}\}, \quad t \in \mathbb{R}.$$

**Lemma 4.3.** *If  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  is dominant, then the center  $\mathcal{C}_\varphi$  of the centralizer  $\mathcal{M}_\varphi$  of  $\varphi$  carries a one parameter automorphism group  $\{\theta_t^\varphi : t \in \mathbb{R}\}$  such that*

- i)  $xa = \theta_t^\varphi(a)x, \quad x \in \mathcal{M}_\varphi(t), a \in \mathcal{C}_\varphi;$
- ii) *There exists canonically an isomorphism  $\tilde{\pi}^\varphi$  from  $\mathcal{C}$  onto  $\mathcal{C}_\varphi$  which conjugate two one parameter automorphism groups  $\theta$  and  $\theta^\varphi$ ;*
- iii) *The actions of  $\mathcal{C}$  and  $\mathcal{C}_\varphi$  on  $\tilde{\mathfrak{H}}$  are related in the following way:*

$$(\tilde{\pi}^\varphi(a)\xi)(s) = (\tilde{\pi}^\varphi(\theta_{-s}(a))\xi)(s), \quad a \in \mathcal{C}, s \in \mathbb{R}.$$

*Proof.* This is an immediate consequence of the definition of a dominant weight; see [CT]. We leave the proof to the reader.

*End of the Proof of Theorem 4.2.* iv) Now we assume that  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  is dominant. For each  $x \in \mathcal{M}_\varphi(t)$ , we have

$$(\tilde{\pi}^\varphi(x)\xi)(s) = x\xi(s-t), \xi \in \tilde{\mathfrak{H}}, s, t \in \mathbb{R}.$$

$c \in Z_\theta^1(\mathcal{U}(\mathcal{C}))$ , We write  $b = b_\varphi(c)$ ,  $c \in Z_\theta^1(\mathcal{U}(\mathcal{C}))$ , for short. We now compute for each  $x \in \mathcal{M}_\varphi(t)$

$$\begin{aligned} (\tilde{\pi}^\varphi(bxb^*)\xi)(s) &= (\tilde{\pi}^\varphi(c_{-s})\tilde{\pi}^\varphi(xb^*)\xi)(s) = (\tilde{\pi}^\varphi(xb^*c_{-s})\xi)(s) \\ &= x(\tilde{\pi}^\varphi(b^*c_{-s})\xi)(s-t) = x(\tilde{\pi}^\varphi(c_{t-s}^*c_{-s})\xi)(s-t) \\ &= x(\tilde{\pi}^\varphi(\theta_{-s}(c_t^*))\xi)(s-t) = (\tilde{\pi}^\varphi(x\theta_{-s}(c_t^*))\xi)(s) \\ &= (\tilde{\pi}^\varphi(\theta_{-s}(c_t^*)x)\xi)(s) = (\tilde{\pi}^\varphi(c_t^*)\tilde{\pi}^\varphi(x)\xi)(s). \end{aligned}$$

Therefore, we see  $\text{Ad}(b_\varphi(c))(x) = c_t^*x, x \in \mathcal{M}_\varphi(t)$ . This means precisely that  $\tilde{\sigma}_c^\varphi = \tilde{\text{Ad}}(b_\varphi(c^*))$ . This completes the proof of claim (iv).

v) and vi) To discuss the comparison of weights, we need to investigate the  $2 \times 2$  matrix algebra  $\mathcal{M}_2 = M_2(\mathbb{C}) \otimes \mathcal{M}$ . The quantum flow of weights for  $\mathcal{M}_2$  is given by  $\{\tilde{\mathcal{M}}_2 = M_2(\mathbb{C}) \otimes \tilde{\mathcal{M}}, \mathbb{R}, \text{id} \otimes \theta, \text{Tr} \otimes \tau\}$ . For any pair  $\varphi, \psi \in \mathfrak{W}_0(\mathcal{M})$ , we get the balanced weight  $\varphi \oplus \psi \in \mathfrak{W}_0(\mathcal{M}_2)$  whose "it"-power  $(\varphi \oplus \psi)^{it}$  is given by:

$$(\varphi \oplus \psi)^{it} = \begin{bmatrix} \varphi^{it} & 0 \\ 0 & \psi^{it} \end{bmatrix}, \quad t \in \mathbb{R},$$

and therefore

$$\pi^{\varphi \oplus \psi}(f) = \begin{bmatrix} \pi^\varphi(f) & 0 \\ 0 & \pi^\psi(f) \end{bmatrix}, \quad f \in L^\infty(\mathbb{R}).$$

This means then:

$$b_{\varphi \oplus \psi}(c) = \begin{bmatrix} b_\varphi(c) & 0 \\ 0 & b_\psi(c) \end{bmatrix}, \quad c \in Z_\theta^1(\mathcal{U}(\mathcal{C})), \quad (4.16)$$

and we get:

$$\begin{aligned} \text{Ad}((\varphi \oplus \psi)^{it}) \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) &= \begin{bmatrix} 0 & (D\varphi : D\psi)_t \\ 0 & 0 \end{bmatrix}, \quad t \in \mathbb{R}; \\ \text{Ad}(b_{\varphi \oplus \psi}(c^*)) \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) &= \begin{bmatrix} 0 & (D\varphi : D\psi)_c \\ 0 & 0 \end{bmatrix}, \quad c \in Z_\theta^1(\mathcal{U}(\mathcal{C})). \end{aligned}$$

This together with (4.16) yields (v) and (vi).

(vii) and (viii): As the norm convergence of a sequence  $\{\varphi_n\}$  of faithful normal positive linear functionals to a faithful  $\varphi \in \mathcal{M}_*^+$  implies the strong resolvent convergence of  $\{\varphi_n\}$  to  $\varphi$  as the sequence of self-adjoint closed operators affiliated to  $\tilde{\mathcal{M}}$ , the claim (viii) follows immediately from (vii). The strong resolvent convergence of  $\{\varphi_n\}$  implies the  $\sigma^*$ -strong convergence for the sequence  $\{\varphi^{it}\}$  of one parameter unitary groups to  $\varphi^{it}$  uniformly in  $t$  of any bounded interval. Thus the sequence  $\{\tilde{\pi}^{\varphi_n}\}$  of equivariant embeddings of  $\{\mathcal{A}, \mathbb{R}, \rho\}$  to  $\tilde{\mathcal{M}}$  converges to  $\tilde{\pi}^\varphi$   $\sigma^*$ -strongly pointwise. Thus in the definition of  $\tilde{b}_\varphi$  of (4.15),  $\{\tilde{\pi}^{\varphi_n}(c_{-s})\}$  converges  $\sigma^*$ -strongly. The Lebesgue dominated convergence theorem takes care of the desired convergence of  $\{b_{\varphi_n}(c)\}$  to  $b_\varphi(c)$ .  $\square$

The next result justifies the notation  $\text{Cnt}_r(\mathcal{M})$ .

**Corollary 4.4.** *If  $\mathcal{M}$  is a separable factor, then elements of  $\text{Cnt}_r(\mathcal{M})$  acts trivially on strongly central sequences, i.e. if  $\{x_n\}$  is a sequence such that  $\lim_{n \rightarrow \infty} \|x_n \omega - \omega x_n\| = 0$  for every  $\omega \in \mathcal{M}_*$  then  $\{\alpha(x_n) - x_n\}$  converges to 0  $\sigma^*$ -strongly.*

*Proof.* It suffices to prove that the claim is true for every strongly central sequence  $\{u_n\}$  of unitaries in  $\mathcal{M}$ . The strong centrality of  $\{u_n\}$  is equivalent to the convergence:  $\lim_{n \rightarrow \infty} \|\omega \circ \text{Ad}(u_n) - \omega\| = 0, \omega \in \mathcal{M}_*$ . If  $\alpha \in \text{Cnt}_r(\mathcal{M})$ , then by definition there exists  $c \in Z_\theta^1(\mathbb{R}, \mathcal{U}(\mathcal{C}))$  such that  $\alpha = \text{Ad}(u) \circ \sigma_c^\varphi$  for any fixed faithful  $\varphi \in \mathcal{M}_*^+$  and some  $u \in \mathcal{U}(\mathcal{M})$ . As  $\text{Ad}(u)$  acts trivially on every strongly central sequences, we need to prove that  $\{\sigma_c^\varphi(u_n) - u_n\}$  converges to zero or equivalently  $\{u_n^* \sigma_c^\varphi(u_n) - 1\}$  converges to zero. But observe that

$$u_n^* \sigma_c^\varphi(u_n) = (D\varphi \circ \text{Ad}(u_n) : D\varphi)_c = b_{\varphi \circ \text{Ad}(u_n)}(c^*) b_\varphi(c^*)^*.$$

By Theorem 4.2.viii, if  $\varphi$  is a faithful normal state on  $\mathcal{M}$ , then  $\{b_{\varphi \circ \text{Ad}(u_n)}(c)\}$  converges to  $b_\varphi(c)$   $\sigma^*$ -strongly. Thus,  $\{u_n^* \sigma_c^\varphi(u_n)\}$  converges to 1.  $\square$

**Corollary 4.5.** *Each  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  gives rise to an equivariant splitting of the short exact sequence (4.2) given by the homomorphism  $b_\varphi : c \in Z_\theta^1(\mathcal{U}(\mathcal{C})) \mapsto b_\varphi(c) \in \tilde{\mathcal{U}}(\mathcal{M}) \cap \mathcal{D}^\varphi$ . Therefore, the extended unitary group  $\tilde{\mathcal{U}}(\mathcal{M})$  is a semi-direct product:*

$$\tilde{\mathcal{U}}(\mathcal{M}) = \mathcal{U}(\mathcal{M}) \rtimes_{\widetilde{\text{Ad}}(b_\varphi(\cdot))} Z_\theta^1(\mathcal{U}(\mathcal{C})). \quad (4.17)$$

**Lemma 4.6.** *The covariant system  $\{\mathcal{D}^\varphi, \mathbb{R}, \theta\}$  for every  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  splits canonically:*

$$\{\mathcal{D}^\varphi, \mathbb{R}, \theta\} = \{(\mathcal{M} \cap \mathcal{D}^\varphi) \overline{\otimes} \mathcal{A}^\varphi, \mathbb{R}, \text{id} \otimes \theta\}. \quad (4.18)$$

*Proof.* Since  $\theta_s(\varphi^{it}) = e^{-ist}\varphi^{it}$ ,  $s, t \in \mathbb{R}$ , the characterization of a dual covariant system due to M. Landstad [Land], yields that the covariant system  $\{\mathcal{D}^\varphi, \mathbb{R}, \theta\}$  is dual to the system  $\{(\mathcal{D}^\varphi)^\theta, \{\text{Ad}(\varphi^{it}) : t \in \mathbb{R}\}\}$ . As  $\mathcal{D}^\varphi$  is abelian,  $\text{Ad}(\varphi^{it}), t \in \mathbb{R}$ , acts trivially on  $(\mathcal{D}^\varphi)^\theta = \mathcal{M} \cap \mathcal{D}^\varphi$ , so that our assertion follows.  $\square$

**Definition 4.7.** i) *We call  $\text{Mod}^\varphi(\mathcal{M}) = \{\widetilde{\text{Ad}}(u) : u \in \mathcal{D}^\varphi \cap \tilde{\mathcal{U}}(\mathcal{M})\}$  the modular group of  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ .*

ii) *We set  $\mathcal{D}_\varphi = \mathcal{D}^\varphi \cap \mathcal{M}$  and call it the strong center of the centralizer  $\mathcal{M}_\varphi$  of  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ .*

iii) *We set  $\text{Mod}_\circ^\varphi(\mathcal{M}) = \{\text{Ad}(u) : u \in \mathcal{U}(\mathcal{D}_\varphi)\}$ .*

Summarizing the above results, we obtain:

**Theorem 4.8.** *The above groups form the following commutative  $\text{Aut}_\varphi(\mathcal{M}) \times \mathbb{R}$  equivariant exact square:*

$$\begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathcal{U}(\mathcal{C}) & \xrightarrow{\partial} & B_\theta^1(\mathcal{U}(\mathcal{C})) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathcal{U}(\mathcal{D}_\varphi) & \longrightarrow & \tilde{\mathcal{U}}(\mathcal{M}) \cap \mathcal{D}^\varphi & \xrightarrow{\partial} & Z_\theta^1(\mathcal{U}(\mathcal{C})) \longrightarrow 1 \\ & & \text{Ad} \downarrow & & \widetilde{\text{Ad}} \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{Mod}_\circ^\varphi(\mathcal{M}) & \longrightarrow & \text{Mod}^\varphi(\mathcal{M}) & \xrightarrow{\nu} & H_\theta^1(\mathcal{U}(\mathcal{C})) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 1 \end{array}$$

with  $\tilde{\mathcal{U}}(\mathcal{M}) \cap \mathcal{D}^\varphi \cong \mathcal{U}(\mathcal{D}^\varphi) \times Z_\theta^1(\mathcal{U}(\mathcal{C}))$ .

**Definition 4.9.** *We call the above exact square the local characteristic square of  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ . For a non faithful  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ , the local characteristic square can be also defined considering the resuced algebra  $\mathcal{M}_s(\varphi)$ .*

We have now the following easy but important consequence:

**Theorem 4.10 (Functoriality).** *The association of the local characteristic square to each  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  is a functor in the sense that if  $\alpha$  is an isomorphism:  $\mathcal{M} \mapsto \mathcal{N}$  then the canonical extention  $\tilde{\alpha}$ , which maps the quantum flow  $\{\tilde{\mathcal{M}}, \mathbb{R}, \theta^{\mathcal{M}}, \tau^{\mathcal{M}}\}$  of  $\mathcal{M}$  onto the other  $\{\tilde{\mathcal{N}}, \mathbb{R}, \theta^{\mathcal{N}}, \tau^{\mathcal{N}}\}$ , maps the local characteristic square  $LC Sq_\varphi$  of  $\varphi \in \mathfrak{W}_0(\mathcal{M})$  onto the local characteristicsquare  $LC Sq_{\varphi \circ \alpha^{-1}}$  of  $\varphi \circ \alpha^{-1} \in \mathfrak{W}_0(\mathcal{N})$  isomorphically.*

We leave the proof to the reader.

**Collorary 4.11.** *The automorphism group  $\text{Aut}(\mathcal{M})$  acts on the field  $\{LC Sq_\varphi : \varphi \in \mathfrak{W}_0(\mathcal{M})\}$  of local characteristic squares in the obvious way.*

## §5. Conclusions

It should now be apparent that the quantum flow of weights is given by the same types of ideas which produced the flow of weights of Connes-Takesaki, viz., while the original flow of weights is constructed by identifying two semi-finite normal weights when they are equivalent under the Murray-von Neumann equivalence, the quantum flow of weights is given by relating two weights by the Connes cocycle derivative. In each case, the flow is given simply by multiplication of each semi-finite normal weight by the scalars  $e^{-s}$ .

As was noted in the introduction, we can relate this construction of the quantum flow of weights to the theory of  $L^p$ -spaces. Each space  $\mathcal{M}(t)$  is given by considering the purely imaginary power  $\varphi^{it}$  of  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ . (The canonical  $L^p$ -space  $L^p(\mathcal{M})$  due to Kosaki is constructed by considering positive powers  $\varphi^{\frac{1}{p}}, 1 \leq p \leq \infty$ , of the weight  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ .) As in the case of the Fell bundle  $\{\mathcal{M}(t)\}$ , two different  $L^p(\mathcal{M})$  and  $L^q(\mathcal{M})$ ,  $p \neq q$ , do not intersect. But as soon as one fixes a  $\varphi \in \mathfrak{W}_0(\mathcal{M})$ , one can have  $x\varphi^{-\frac{1}{p}} = y\varphi^{-\frac{1}{q}}$  for some pair  $x \in L^p(\mathcal{M})$  and  $y \in L^q(\mathcal{M})$ , or equivalently  $x = a\varphi^{\frac{1}{p}} \in L^p(\mathcal{M})$  and  $y = a\varphi^{\frac{1}{q}} \in L^q(\mathcal{M})$ ; one then can view these two elements in different spaces as the same elements. The same phenomenon occurs if one identifies  $a\varphi^{it} \in \mathcal{M}(t)$  and  $a\varphi^{is} \in \mathcal{M}(s)$ . However this identification simply brings about confusion. Conversely, to view each  $\mathcal{M}(t)$  as an individual Banach space will also fail to yield the complete picture. Our emphasis throughout has been to view  $L^p$ -theory as a mutiplicative theory, and consider these all at once — not as the theory of a singel isolated Banach space. This point of view allows us to observe that the analytic continuation of  $\mathfrak{M}^{\frac{1}{p}}$  gives precisely  $\{\mathcal{M}(t) : t \in \mathbb{R}\} = \{\mathfrak{M}^{\frac{1}{t}} : t \in \mathbb{R}\}$ .

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