Page 5. Problem 8. Prove that if $x$ and $y$ are real numbers, then

$$2xy \leq x^2 + y^2.$$ 

Proof. First we prove that if $x$ is a real number, then $x^2 \geq 0$. The product of two positive numbers is always positive, i.e., if $x \geq 0$ and $y \geq 0$, then $xy \geq 0$. In particular if $x \geq 0$ then $x^2 = x \cdot x \geq 0$. If $x$ is negative, then $-x$ is positive, hence $(-x)^2 \geq 0$. But we can conduct the following computation by the associativity and the commutativity of the product of real numbers:

\[
0 \geq (-x)^2 = (-x)(-x) = ((-1)x)((-1)x) = ((-1)x)((-1)x)\]

\[
= ((-1)(x(-1)))x = (((-1)(-1))x) = (1x)x = xx = x^2.
\]

The above change in bracketting can be done in many ways. At any rate, this shows that the square of any real number is non-negative. Now if $x$ and $y$ are real numbers, then so is the difference, $x - y$ which is defined to be $x + (-y)$. Therefore we conclude that $0 \leq (x + (-y))^2$ and compute:

\[
0 \leq (x + (-y))^2 = (x + (-y))(x + (-y)) = x(x + (-y)) + (-y)(x + (-y))
\]

\[
= x^2 + x(-y) + (-y)x + (-y)^2 = x^2 + y^2 + (-xy) + (-xy)
\]

\[
= x^2 + y^2 + 2(-xy);
\]

adding $2xy$ to the both sides,

\[
2xy = 0 + 2xy \leq (x^2 + y^2 + 2(-xy)) + 2xy = (x^2 + y^2) + (2(-xy) + 2xy)
\]

\[
= (x^2 + y^2) + 0 = x^2 + y^2.
\]

Therefore, we conclude the inequality:

$$2xy \leq x^2 + y^2$$

for every pair of real numbers $x$ and $y$. ♥
Page 5. Problem 11. If $a$ and $b$ are real numbers with $a < b$, then there exists a pair of integers $m$ and $n$ such that

$$a < \frac{m}{n} < b, \quad n \neq 0.$$  

Proof. The assumption $a < b$ is equivalent to the inequality $0 < b - a$. By the Archimedian property of the real number field, $\mathbb{R}$, there exists a positive integer $n$ such that

$$n(b - a) > 1.$$  

Of course, $n \neq 0$. Observe that this $n$ can be 1 if $b - a$ happen to be large enough, i.e., if $b - a > 1$. The inequality $n(b - a) > 1$ means that $nb - na > 1$, i.e., we can conclude that

$$na + 1 < nb.$$  

Let $m$ be the smallest integer such that $na < m$. Does there exists such an integer? To answer to the question, we consider the set $A = \{k \in \mathbb{Z} : k > na\}$ of integers. First $A \neq \emptyset$. Because if $na \geq 0$ then 1 $\in A$ and if $na > 0$ then by the Archimedian property of $\mathbb{R}$, there exists $k \in \mathbb{Z}$ such that $k = k \cdot 1 > na$. Hence $A \neq \emptyset$. Choose $\ell \in A$ and consider the following chain:

$$\ell > \ell - 1 > \ell - 2 > \cdots > \ell - k, \quad k \in \mathbb{N}.$$  

This sequence eventually goes down beyond $na$. So let $k$ be the first natural number such that $\ell - k \leq na$, i.e., the natural number $k$ such that $\ell - k \leq na < \ell - k + 1$. Set $m = \ell - k + 1$ and observe that

$$na < m = \ell - k \leq +1 \leq na + 1 < nb.$$  

Therefore, we come to the inequality $na < m < nb$. Since $n$ is a positive integer, we devide the inequility by $n$ withoug changing the direction of the inequality:

$$a = \frac{na}{n} < \frac{m}{n} < \frac{nb}{n} = b.$$  

Page 14, Problem 6. Generate the graph of the following functions on $\mathbb{R}$ and use it to determine the range of the function and whether it is onto and one-to-one:

- a) $f(x) = x^3$.
- b) $f(x) = \sin x$.
- c) $f(x) = e^x$.
- d) $f(x) = \frac{1}{1 + x^2}$.
Solution. a) The function $f$ is bijective since $f(x) < f(y)$ for any pair $x, y \in \mathbb{R}$ with the relation $x < y$ and for every real number $y \in \mathbb{R}$ there exists a real number $x \in \mathbb{R}$ such that $y = f(x)$.

b) The function $f$ is neither injective nor surjective since

$$f(x + 2\pi) = f(x)$$

$x + \pi \neq x, x \in \mathbb{R}$, and if $y > 1$ then there is no $x \in \mathbb{R}$ such that $y = f(x)$.

c) The function $f$ is injective because

$$f(x) < f(y)$$

if $x < y, x, y \in \mathbb{R}$, but not surjective as a map from $\mathbb{R}$ to $\mathbb{R}$, because there exists no $x \in \mathbb{R}$ such that $f(x) = -1$. 
d) The function $f$ is not injective as $f(x) = f(-x)$ and $x \neq -x$ for $x \neq 0$, nor surjective as there is no $x \in \mathbb{R}$ such that $f(x) = -1$.

**Page 14, Problem 8.** Let $\mathcal{P}$ be the set of polynomials of one real variable. If $p(x)$ is such a polynomial, define $I(p)$ to be the function whose value at $x$ is

$$I(p)(x) \equiv \int_0^x p(t)\, dt.$$  

Explain why $I$ is a function from $\mathcal{P}$ to $\mathcal{P}$ and determine whether it is one-to-one and onto.

**Solution.** Every element $p \in \mathcal{P}$ is of the form:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}, \quad x \in \mathbb{R},$$

with $a_0, a_1, \ldots, a_{n-1}$ real numbers. Then we have

$$I(p)(x) = \int_0^x (a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1})\, dt = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_{n-1}}{n} x^n.$$  

Thus $I(p)$ is another polynomial, i.e., an element of $\mathcal{P}$. Thus $I$ is a function from $\mathcal{P}$ to $\mathcal{P}$.

We claim that $I$ is injective: If

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1};$$

$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

have $I(p)(x) = I(q)(x), x \in \mathbb{R}$, i.e.,

$$a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_{m-1}}{m} x^m = b_0 x + \frac{b_1}{2} x^2 + \frac{b_2}{3} x^3 + \cdots + \frac{b_{n-1}}{n} x^n.$$  

Let $P(x) = I(p)(x)$ and $Q(x) = I(q)(x)$. Then the above equality for all $x \in \mathbb{R}$ allows us to differentiate the both sides to obtain

$$P'(x) = Q'(x) \quad \text{for every } x \in \mathbb{R},$$

$$p(x) = q(x), x \in \mathbb{R},$$

and since $p(x) = q(x)$ for every $x \in \mathbb{R}$, $I$ is injective.
in particular \(a_0 = P'(0) = Q'(0) = b_0\). The second differentiation gives
\[
P''(x) = Q''(x) \quad \text{for every } x \in \mathbb{R},
\]
in particular \(a_1 = P''(0) = Q''(0) = b_1\).

Suppose that with \(k \in \mathbb{N}\) we have \(P^{(k)}(x) = Q^{(k)}(x)\) for every \(x \in \mathbb{R}\). Then the differentiation of the both sides gives
\[
P^{(k+1)}(x) = Q^{(k+1)}(x) \quad \text{for ever } x \in \mathbb{R},
\]
in particular \(a_{k+1} = P^{(k+1)}(0) = Q^{(k+1)}(0) = b_{k+1}\). Therefore the mathematical induction gives
\[
a_0 = b_0, a_1 = b_1, \ldots, a_{m-1} = b_{m-1} \quad \text{and } m = n,
\]
i.e., \(p = q\). Hence the function \(I\) is injective.

We claim that \(I\) is not surjective: As \(I(p)(0) = 0\), the constant polynomial \(q(x) = 1\) cannot be of the form \(q(x) = I(p)(x)\) for any \(p \in \mathcal{P}\), i.e., there is no \(p \in \mathcal{P}\) such that \(I(p)(x) = 1\). Hence the constant polynomial \(q\) is not in the image \(I(\mathcal{P})\).

\[\Box\]

**Page 19, Problem 3.** Prove that:

a) The union of two finite sets is finite.

b) The union of a finite set and a countable set is countable.

c) The union of two countable sets is countable.

**Proof.** a) Let \(A\) and \(B\) be two finite sets. Set \(C = A \cap B\) and \(D = A \cup B\). First, let \(a, b\) and \(c\) be the total number of elements of \(A\), \(B\) and \(C\) respectively. As \(C \subset A\) and \(C \subset B\), we know that \(c \leq a\) and \(c \leq b\). We then see that the union:

\[
D = C \cup (A\setminus C) \cup (B\setminus C)
\]
is a disjoint union, i.e., the sets \(C\), \(A\setminus C\) and \(B\setminus C\) are mutually disjoint. Thus the total number \(d\) of elements of \(D\) is precisely \(c + (a - c) + (b - c) = a + b - c\) which is a finite number, i.e., \(D\) is a finite set with the total number \(d\) of elements.

b) Let \(A\) be a finite set and \(B\) a countable set. Set \(C = A \cap B\) and \(D = A \cup B\). Since \(C\) is a subset of the finite set \(A\), \(C\) is finite. Let \(m\) be the total number of elements of \(C\) and \(\{c_1, c_2, \ldots, c_m\}\) be the list of elements of \(C\). Let \(n\) be the total number of elements of \(A\) and let \(\{a_1, a_2, \ldots, a_{n-m}\}\) be the leballing of the set \(A\setminus C\). Arrange an enumeration of the elements of \(B\) in the following fashion:

\[
B = \{c_1, c_2, \ldots, c_m, b_{m+1}, b_{m+2}, \ldots\}.
\]

Arranging the set \(A\) in the following way:

\[
A = \{a_1, a_2, \ldots, a_{n-m}, c_1, c_2, \ldots, c_m\},
\]
we enumerate the elements of $D = A \cup B$ in the following way:

$$d_i = \begin{cases} a_i & \text{for } 1 \leq i \leq n - m; \\ c_{i-n-m} & \text{for } n - m < i \leq n; \\ b_{i-n+m} & \text{for } i > n. \end{cases}$$

This gives an enumeration of the set $D$. Hence $D$ is countable.

c) Let $A$ and $B$ be two countable sets. Let $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$ be enumerations of $A$ and $B$ respectively. Define a map $f$ from the set $\mathbb{N}$ of natural numbers in the following way:

$$f(2n - 1) = a_n, \quad n \in \mathbb{N};$$
$$f(2n) = b_n, \quad n \in \mathbb{N}.$$  

Then $f$ maps $\mathbb{N}$ onto $D = A \cup B$. The surjectivity of the map $f$ guarantees that $f^{-1}(d) \neq \emptyset$ for every $d \in D$. For each $d \in \mathbb{N}$, let $g(d) \in \mathbb{N}$ be the first element of $f^{-1}(d)$. Since $f^{-1}(d) \cap f^{-1}(d') = \emptyset$ for every distinct pair $d, d' \in D$, $g(d) \neq g(d')$ for every distinct pair $d, d' \in D$. Hence the map $g$ is injective. Now we enumerate the set $D$ by making use of $g$. Let $d_1 \in D$ be the element of $D$ such that $g(d_1)$ is the least element of $g(D)$. After $\{d_1, d_2, \ldots, d_n\}$ were chosen, we choose $d_{n+1} \in D$ as the element such that $g(d_{n+1})$ is the least element of $g(D \setminus \{d_1, d_2, \ldots, d_n\})$. By induction, we choose a sequence $\{d_n\}$ of elements of $D$. Observe that $1 \leq g(d_1) < g(d_2) < \cdots < g(d_n) < \cdots$ in $\mathbb{N}$. Hence we have $n \leq g(d_n)$. This means that every $d \in \mathbb{N}$ appears in the list $\{d_1, d_2, \cdots\}$. Hence $D$ is countable.

**Page 24, Problem 1.** Give five examples which show that $P$ implies $Q$ does not necessarily mean that $Q$ implies $P$.

**Examples.** 1) $P$ is the statement that $x = 1$ and $Q$ is the statement that $x^2 = 1$.

2) $P$ is the statement that $x \leq 1$ and $Q$ is the statement that $x \leq 2$.

3) Let $A$ be a subset of a set $B$ with $A \neq B$. $P$ is the statement that $x$ is an element of $A$ and $Q$ is the statement that $x$ is an element of $B$.

4) $P$ is the statement that $x$ is a positive real number and $Q$ is the statement that $x^2$ is a positive real number.

5) $P$ is the statement that $x = 0$ or $x = 1$ and $Q$ is the statement that $x(x-1)(x-2) = 0$.

**Page 24, Problem 3.** Suppose that $a, b, c,$ and $d$ are positive real numbers such that $a/b < c/d$. Prove that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$  

**Proof.** The inequality $a/b < c/d$ is equivalent to the inequality $bc - ad > 0$. We compare two numbers by subtracting one from the other. So we compare the first two of the above
three fractions first and then the second pair of the fractions:

\[
\frac{a + c}{b + d} - \frac{a}{b} = \frac{b(a + c) - a(b + d)}{b(b + d)} = \frac{bc - ad}{b(b + d)} > 0; \\
\frac{c - a + c}{d} - \frac{b + d}{b + d} = \frac{c(b + d) - (a + c)d}{c(b + d)} = \frac{bc - ad}{c(b + d)} > 0.
\]

Therefore the desired inequalities follows.

\begin{align*}
\text{Page 24, Problem 4.} & \quad \text{Suppose that } 0 < a < b. \text{ Prove that} \\
\text{a) } & \quad a < \sqrt{ab} < b. \\
\text{b) } & \quad \sqrt{ab} < \frac{a + b}{2}.
\end{align*}

\textit{Proof.} a) We compute, based on the fact that the inequality } \quad \sqrt{a} < \sqrt{b},
\begin{align*}
b & = \sqrt{b} \sqrt{b} > \sqrt{a} \sqrt{b} = \sqrt{ab} > \sqrt{a} \sqrt{a} = a.
\end{align*}

b) We simply compute:
\begin{align*}
\frac{a + b}{2} - \sqrt{ab} & = \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2 \sqrt{a} \sqrt{b}}{2} \\
& = \frac{(\sqrt{a} - \sqrt{b})^2}{2} > 0,
\end{align*}

where we used the fact that \((x + y)^2 = x^2 - 2xy + y^2\) which follows from the distributive law and the commutativity law in the field of real numbers as seen below:
\begin{align*}
(x - y)^2 = (x - y)x - (x - y)y & \quad \text{by the distributive law} \\
& = x^2 - xy - xy + y^2 = x^2 - 2xy + y^2 & \quad \text{by the commutativity law}.
\end{align*}

\textit{Remark.} In the last computation, is
\begin{align*}
(x - y)^2 = x^2 - 2xy + y^2
\end{align*}

obvious? 
If so, take
\begin{align*}
x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and } y = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}
\end{align*}
and compute
\[ x - y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad (x - y)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \]
\[ x^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , \quad y^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , \quad xy = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} , \quad yx = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}; \]
\[ x^2 - 2xy + y^2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (x - y)^2. \]

Therefore, the formula
\[ (x - y)^2 = x^2 - 2xy + y^2 \]
is not universally true. This is a consequence of the distributive law and the commutative law which governs the field \( \mathbb{R} \) of real numbers as discussed in the very early class.

**Page 24, Problem 5.** Suppose that \( x \) and \( y \) satisfy \( \frac{x}{2} + \frac{y}{3} = 1 \). Prove that \( x^2 + y^2 > 1 \).
Proof. The point \((x, y)\) lies on the line:

\[
\frac{x}{2} + \frac{y}{3} = 1 \quad (L)
\]

which cuts through \(x\)-axis at \((2, 0)\) and \(y\)-axis at \((0, 3)\). The line \(L\) is also described by parameter 
\((2t, 3(1 - t)), t \in \mathbb{R}\). So we compute

\[
x^2 + y^2 = (2t)^2 + (3(1 - t))^2 = 4t^2 + 9(1 - t)^2
\]

\[
= 4t^2 + 9t^2 - 18t + 9
\]

\[
= 13t^2 - 18t + 9 = 13 \left( t^2 - \frac{18}{13}t + \frac{9}{13} \right)
\]

\[
= 13 \left\{ \left( t - \frac{9}{13} \right)^2 + \frac{9}{13} - \left( \frac{9}{13} \right)^2 \right\} > 0
\]

for all \(t \in \mathbb{R}\) as \(\frac{9}{13} < 1\).

Page 25, Problem 8. Prove that for all positive integers \(n\),

\[
1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.
\]

Proof. Suppose \(n = 1\). Then the both sides of the above identity is one. So the formula hold for \(n = 1\).

Suppose that the formula hold for \(n\), i.e.,

\[
1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.
\]

Adding \((n + 1)^3\) to the both sides, we get

\[
1 + 2 + \cdots + n^3 + (n + 1)^3 = (1 + 2 + \cdots + n)^2 + (n + 1)^3
\]

\[
= \left( \frac{n(n + 1)}{2} \right)^2 + (n + 1)^3 \quad \text{by Proposition 1.4.3}
\]

\[
= (n + 1)^2 \left( \frac{n^2 + 4n + 4}{4} \right) = \frac{(n + 1)^2(n + 2)^2}{4}
\]

\[
= (1 + 2 + \cdots + n + n + 1)^2 \quad \text{by Proposition 1.4.3.}
\]

Thus the formula holds for \(n+1\). Therefore mathematical induction assures that the formula holds for every \(n \in \mathbb{N}\).
Page 25, Problem 9. Let \( x > -1 \) and \( n \) be a positive integer. Prove Bernoulli’s inequality:

\[(1 + x)^n \geq 1 + nx.\]

Proof. If \( x \geq 0 \), then the binary expansion theorem for real numbers gives

\[(1 + x)^n = 1 + nx + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \cdots + \binom{n}{n-1} x^{n-1} + x^n \geq 1 + nx\]
as \( x^2, x^3, \ldots, x^{n-1} \) and \( x^n \) are all non-negative. If \(-1 < x < 0\), then the inequality is more delicate. But we can proceed in the following way:

\[(1 + x)^n - 1 = x \left( (1 + x)^{n-1} + (1 + x)^{n-1} + \cdots + (1 + x) + 1 \right);\]

\[n \geq (1 + x)^{n-1} + (1 + x)^{n-1} + \cdots + (1 + x) + 1 \text{ as } 1 + x < 1.\]

As \( x < 0 \), we get

\[nx \leq x \left( (1 + x)^{n-1} + (1 + x)^{n-1} + \cdots + (1 + x) + 1 \right),\]

consequently the desired inequality:

\[ (1 + x)^n - 1 = x \left( (1 + x)^{n-1} + (1 + x)^{n-1} + \cdots + (1 + x) + 1 \right) \geq nx.\]

Page 25, Problem 11. Suppose that \( c < d \).

a) Prove that there is a \( q \in \mathbb{Q} \) so that \( |q - \sqrt{2}| < d - c \).

b) Prove that \( q - \sqrt{2} \) is irrational.

c) Prove that there is an irrational number between \( c \) and \( d \).

Proof. a) Choose \( a_1 = 1 \) and \( b_1 = 2 \) and observe that

\[ a_1^2 = 1 < 2 < 4 = b_1^2 \text{ consequently } a_1 < \sqrt{2} < b_1.\]

Consider \((a_1 + b_1)/2\) and square it to get

\[ \left( \frac{a_1 + b_1}{2} \right)^2 = \frac{9}{4} > 2 \]
and put $a_2 = a_1$ and $b_2 = (a_1 + b_1)/2$. Suppose that $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ were chosen in such a way that

i) if

$$\left(\frac{a_{k-1} + b_{k-1}}{2}\right)^2 < 2,$$

then $a_k = (a_{k-1} + b_{k-1})/2$ and $b_k = b_{k-1}$;

ii) if

$$\left(\frac{a_{k-1} + b_{k-1}}{2}\right)^2 > 2,$$

then $a_k = a_{k-1}$ and $b_k = (a_{k-1} + b_{k-1})/2$.

Thus we obtain sequences $\{a_n\}$ and $\{b_k\}$ of rational numbers such that

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots, b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1$$

and

$$b_k - a_k = \frac{b_{k-1} - a_{k-1}}{2};$$

$$a_n^2 < 2 < b_n^2, \quad n \in \mathbb{N}, \quad \text{hence} \quad a_n < \sqrt{2} < b_n.$$

As $b_1 - a_1 = 1$, we have $b_n - a_n = 1/2^n$. For a large enough $n \in \mathbb{N}$ we have $1/2^n < d - c$. Now we conclude

$$0 < \sqrt{2} - a_n \leq b_n - a_n = \frac{1}{2^n} < d - c \quad \text{and} \quad a_n \in \mathbb{Q}.$$

Thus $q = a_n$ has the required property.

b) Set $p = \sqrt{2} - q$. If $p \in \mathbb{Q}$, then $\sqrt{2} = p + q \in \mathbb{Q}$ which is impossible. Therefore, $p$ cannot be rational.

c) From (b), $p = \sqrt{2} - q$ is an irrational number and $0 < p < d - c$ from (a). Thus we get

$c < c + p < d$. As seen in (b), $c + p$ cannot be rational. Because if $c + p = a$ is rational, then $p = a - c$ has to be rational which was just proven not to be the case. \hfill \heartsuit
Problems.

1) Negate the following statement on a function $f$ on an interval $[a, b], a < b$. The function $f$ has the property:

$$f(x) \geq 0 \quad \text{for every } x \in [a, b].$$

2) Let

$$f(x) = x^2 + bx + c, \quad x \in \mathbb{R}.$$ 

What can you say about the relation on the constants $b$ and $c$ in each of the following cases?

a) $$f(x) \geq 0 \quad \text{for every } x \in \mathbb{R}.$$ 

b) $$f(x) \geq 0 \quad \text{for some } x \in \mathbb{R}.$$ 

3) Let

$$f(x) = x^2 + 4x + 3, \quad x \in \mathbb{R}.$$ 

Which of the following statements on $f$ is true? State the proof of your conclusion.

a) $$f(x) < 0 \quad \text{for some } x \in \mathbb{R};$$

b) $$f(x) > 0 \quad \text{for some } x \in \mathbb{R};$$

c) $$f(x) \geq 0 \quad \text{for every } x \in \mathbb{R}.$$ 

d) $$f(x) < 0 \quad \text{for every } x \in \mathbb{R}.$$ 

Solution. 1) There exists an $x_0 \in [a, b]$ such that

$$f(x_0) < 0.$$ 

2-a) First, we look at the function $f$ closely:

$$f(x) = x^2 + bx + c = x^2 + bx + \frac{b^2}{4} + c - \frac{b^2}{4}$$

$$= \left(x + \frac{b}{2}\right)^2 + \frac{4c - b^2}{4} \geq \frac{4c - b^2}{4} \quad \text{for every } x \in \mathbb{R}.$$
The function $f$ assumes its smallest value

$$f\left(\frac{-b}{2}\right) = \frac{4c - b^2}{4}$$

at $x = -\frac{b}{2}$. Thus $f(x) \geq 0$ for every $x \in \mathbb{R}$ if and only if

$$f\left(\frac{-b}{2}\right) = \frac{4c - b^2}{4} \geq 0 \text{ if and only if } 4c \geq b^2.$$

2-b) If $4c \geq b^2$, then $f(x) \geq 0$ for every $x \in \mathbb{R}$, in particular $f(0) \geq 0$. If $4c < b^2$, then we have

$$f\left(-\frac{b + \sqrt{b^2 - 4c}}{2}\right) = 0.$$ 

Therefore the condition that $f(x) \geq 0$ for some $x \in \mathbb{R}$ holds regardless of the values of $b$ and $c$. So we have no relation between $b$ and $c$.

3) First, we factor the polynomial $f$ and draw the graph:

$$f(x) = x^2 + 4x + 3 = (x + 3)(x + 1).$$

We conclude that $f(x) \leq 0$ is equivalent to the condition that $-3 \leq x \leq -1$. Therefore we conclude that (a) and (b) are both true and that (c) and (d) are both false.

**Page 25, Problem 12.** Prove that the constant

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

is an irrational number.

**Proof.** Set

$$s_k = \sum_{j=0}^{k} \frac{1}{j!}, \quad k \in \mathbb{N}.$$
Clearly we have $2 = s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_k \leq \cdots$, i.e., the sequence $\{s_k\}$ is an increasing sequence. If $k \geq 3$, we have

$$s_k = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdot 3 \cdots k} \leq 1 + 1 + \frac{1}{2 \cdot 2} + \cdots + \frac{1}{2^{k-1}} = 1 + \frac{1 - \frac{1}{2^k}}{1 - \frac{1}{2}} < 1 + 2 = 3.$$ 

Thus the sequence $\{s_k\}$ is bounded. The Bounded Monotone Convergence Axiom (MC) guarantees the convergence of $\{s_k\}$. Thus the limit $e$ of $\{s_k\}$ exists and $e \leq 3$.

a) If $n > k$, then we have

$$s_n - s_k = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \frac{1}{(k+1)!} + \cdots + \frac{1}{n!} - \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!}\right)$$

$$= \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \cdots + \frac{1}{n!}$$

$$= \frac{1}{(k+1)!} \times \left(1 + \frac{1}{k+2} + \frac{1}{(k+2)(k+3)} + \frac{1}{(k+2)(k+3)(k+4)} + \cdots + \frac{1}{(k+1)\cdots n}\right).$$

Since

$$k + 1 < k + 2 < k + 3 < \cdots < n,$$

and

$$\frac{1}{k+1} > \frac{1}{k+2} > \frac{1}{k+3} > \cdots > \frac{1}{n},$$

we have

$$\frac{1}{(k+1)^2} > \frac{1}{(k+1)(k+2)}, \quad \frac{1}{(k+1)^3} > \frac{1}{(k+1)(k+2)(k+3)}, \quad \cdots \quad \frac{1}{(k+1)^{n-k-1}} > \frac{1}{(k+1)(k+2)\cdots n},$$

and

$$s_n - s_k < \frac{1}{(k+1)!} \sum_{\ell=0}^{n-k-1} \frac{1}{(k+1)^\ell} = \frac{1}{(k+1)!} \frac{1 - \frac{1}{(k+1)^{n-k}}}{1 - \frac{1}{(k+1)}},$$

$$< \frac{1}{k(k+1)!}.$$
Taking the limit of the left hand side as \( n \to \infty \), we get

\[
e - s_k \leq \frac{1}{k(k+1)!} < \frac{1}{k \cdot k!}.
\]

b) Now suppose that e is rational, i.e., \( e = p/q \) for some \( p, q \in \mathbb{N} \). Then we must have

\[
\frac{p}{q} - s_q \leq \frac{1}{q \cdot q!} \quad \text{and} \quad 0 < q! \left( \frac{p}{q} - s_q \right) < \frac{1}{q}.
\]

But now \( q!(p/q) \) is an integer and \( p!s_q \) is also because \( 2!, 3!, \cdots \) and \( q! \), appearing in the denominators of the summation of \( s_q \), all divide \( q! \), i.e.,

\[
q!s_q = q! \left( 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} \right) = q! + q! + (3 \cdot 4 \cdots q) + (4 \cdot 5 \cdots q) + \cdots + (k \cdot (k+1) \cdots q) + \cdots + q + 1.
\]

Thus the number \( q!(p/q - s_q) \) is a positive integer smaller than \( 1/q \), which is not possible. This contradiction comes from the assumption that \( e = p/q, p, q \in \mathbb{N} \). Therefore we conclude that there is no pair of natural numbers \( p, q \) such that \( e = p/q \), i.e., \( e \) is an irrational number. \( \heartsuit \)

Page 33, Problem 1. Compute enough terms of the following sequences to guess what their limits are:

a) 
\[
a_n = n \sin \frac{1}{n}.
\]

b) 
\[
a_n = \left( 1 + \frac{1}{n} \right)^n.
\]

c) 
\[
a_{n+1} = \frac{1}{2}a_n + 2, \quad a_1 = \frac{1}{2}.
\]

d) 
\[
a_{n+1} = \frac{5}{2}a_n(1 - a_n), \quad a_1 = 0.3.
\]
**Answer.** a) It is not easy to compute \( \sin \frac{1}{2}, \sin \frac{1}{3} \) and so on. So let us take a closer look at the function \( \sin x \) near \( x = 0 \):

Consider the circle of radius 1 with center 0, i.e., \( \overline{0A} = 1 \), and draw a line \( \overline{0B} \) with angle \( \angle A0B = x \). Let \( C \) be the intersection of the line \( \overline{0B} \) and the circle. Draw a line \( \overline{CD} \) through \( C \) and perpendicular to the line \( \overline{0A} \) with \( D \) the intersection of the new line and the line \( \overline{0A} \). So obtain the figure on the right. Now the length \( \overline{CD} = \sin x \).

We have the arc length \( \widehat{AC} = x \) and the line length \( \overline{AB} = \tan x \). To compare the sizes of \( x, \sin x \) and \( \tan x \), we consider the areas of the triangle \( \triangle A0C \), the pizza pie cut shape \( <0AC \) and \( \triangle 0AB \) which are respectively \( (\sin x)/2, x/2 \) and \( (\tan x)/2 \). As these three figures are in the inclusion relations:

\[
\triangle 0A0C \subset <0AC \subset \triangle 0AB,
\]

we have

\[
\frac{\sin x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2} = \frac{1}{2} \sin x \cos x
\]

Consequently we conclude that

\[
\cos x \leq \frac{\sin x}{x} \leq 1.
\]

Therefore we have

\[
\cos \left( \frac{1}{n} \right) \leq n \sin \left( \frac{1}{n} \right) \leq 1 \quad \text{and} \quad \lim_{n \to \infty} n \sin \left( \frac{1}{n} \right) = 1.
\]

b) We simply compute a few terms:

\[
a_1 = \left(1 + \frac{1}{1}\right)^1 = 2, a_2 = \left(1 + \frac{1}{2}\right)^2 = 1 + 1 + \frac{1}{4} = 2.25.
\]

\[
a_3 = \left(1 + \frac{1}{3}\right)^3 = 1 + 1 + \frac{3}{9} + \frac{1}{27} = 2\frac{10}{27},
\]

\[
a_4 = \left(1 + \frac{1}{4}\right)^4 = 1 + 1 + 6 \cdot \frac{1}{16} + 4 \cdot \frac{1}{64} + \frac{1}{256} = 2\frac{6 \cdot 16 + 4 \cdot 4 + 1}{256}
\]

\[
= 2\frac{96 + 16 + 1}{256} = 2\frac{103}{256},
\]

\[
a_5 = \left(1 + \frac{1}{5}\right)^5 = 1 + 1 + \left(\frac{5}{2}\right)\frac{1}{25} + \left(\frac{5}{3}\right)\frac{1}{125} + \left(\frac{5}{4}\right)\frac{1}{625} + \frac{1}{3125}
\]

\[
= 2\frac{10 \cdot 125 + 10 \cdot 25 + 5 \cdot 5 + 1}{3125} = 2\frac{1256}{3125}.
\]
It is still hard to make the guess of the limit of \((1 + 1/n)^n\). So let us try something else.

\[
a_n = \left(1 + \frac{1}{n}\right)^n = 1 + 1 + \left(\frac{n}{2}\right) \frac{1}{n^2} + \left(\frac{n}{3}\right) \frac{1}{n^3} + \cdots + \left(\frac{n}{k}\right) \frac{1}{n^k} + \cdots + \frac{1}{n^n}
\]

\[
= 2 + \frac{1}{2!} \frac{n(n-1)}{n^2} + \frac{1}{3!} \frac{n(n-1)(n-2)}{n^3} + \cdots + \frac{1}{k!} \frac{n(n-1)(n-2) \cdots (n-k+1)}{n^k}
\]

\[
= 2 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k}{n}\right)
\]

\[
< \sum_{k=0}^{n} \frac{1}{k!} = s_n.
\]

\[
a_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1} = 2 + \frac{1}{2!} \left(1 - \frac{1}{n+1}\right) + \frac{1}{3!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) + \cdots + \frac{1}{(n+1)!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{n}{n+1}\right)
\]

As each term of \(a_{n+1}\) is greater than the corresponding term of \(a_n\), we have \(a_n \leq a_{n+1}, n \in \mathbb{N}\), i.e., the sequence \(\{a_n\}\) is increasing and bounded by \(e\) as \(a_n \leq s_n \leq e\). Therefore we conclude that the sequence \(\{a_n\}\) converges and the limit is less than or equal to \(e\).\(^1\)

c) Skip.

d) Let us check a few terms:

\[
a_1 = 0.3, \quad a_2 = \frac{5}{2} a_1 (1 - a_1) = 0.525
\]

\[
a_3 = \frac{5}{2} a_2 (1 - a_2) = 0.6234375
\]

\[
a_4 = 0.58690795898, a_5 = 0.60611751666, a_6 = 0.59684768164
\]

\[
a_7 = 0.6015513164, a_8 = 0.59921832534, a_9 = 0.60038930979, a_{10} = 0.5998049662
\]

With

\[
f(x) = \frac{5}{2} x (1 - x) = \frac{5}{2} (x - x^2) = \frac{5}{2} \left(\frac{1}{4} - \left(\frac{1}{2} - x\right)^2\right) \leq \frac{5}{8} < 1
\]

\(^1\)In fact the limit of \(\{a_n\}\) is the natural logarithm number \(e\), which will be shown later.
we have
\[ 0 \leq f(x) \leq \frac{5}{8} = 0.625 \quad \text{for all} \quad x \in [0, 1], \]
and consequently
\[ 0 \leq a_{n+1} = f(a_n) \leq \frac{5}{8} = 0.625, \quad n \geq 3. \]

To compare \( a_n \) and \( a_{n+1} = f(a_n) \), we consider
\[
x - f(x) = x - \frac{5}{2}x(1 - x) \\
= \frac{2x - 5x^2}{2} = \frac{5x^2 - 3x}{2} = \frac{x(5x - 3)}{2}.
\]

This means that \( a_{n+1} \leq a_n \) if \( 0 \leq a_n \leq 0.6 \) and \( a_{n+1} \geq a_n \) if \( a_n < 0 \) or \( a_n > 0.6 \). But the case \( a_n < 0 \) has been excluded by the above arguments. From the computation of the first three terms we observe that the sequence \( \{a_n\} \) seems to oscillate. At any rate, if the sequence \( \{a_n\} \) converges, then we must have \( a = \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1} \), i.e., we must have \( a = f(a) \), which narrows the candidate of the limit down to either 0 or \( 3/5 = 0.6 \). Let us examine the candidate \( 3/5 \) first. So we compute the error
\[
\left| \frac{3}{5} - f(x) \right| = \left| \frac{3}{5} - \frac{5}{2}x(1 - x) \right| = \left| \frac{6 - 25x(1 - x)}{10} \right| \\
= \left| \frac{25x^2 - 25x + 6}{10} \right| = \left| \frac{(5x - 2)(5x - 3)}{10} \right| \\
= \frac{1}{2} |5x - 2| \left| x - \frac{3}{5} \right| = \frac{1}{2} \left| 5 \left( x - \frac{3}{5} \right) + 1 \right| \left| x - \frac{3}{5} \right|.
\]

Therefore, if \( |x - 3/5| = \delta \), then
\[
\left| \frac{3}{5} - f(x) \right| \leq \frac{1}{2} (5\delta + 1)\delta.
\]

Thus if \( \delta < 1/5 \), then with \( r = (5\delta + 1)/2 < 1 \) we have
\[
\left| \frac{3}{5} - f(x) \right| \leq r\delta,
\]
in other words
\[ r^k \left| \frac{3}{5} - a_n \right| \geq r^{k-1} \left| \frac{3}{5} - a_{n+1} \right| \geq r^{k-2} \left| \frac{3}{5} - a_{n+2} \right| \geq r^{k-3} \left| \frac{3}{5} - a_{n+3} \right| \geq \cdots \geq \left| \frac{3}{5} - a_{n+k} \right|. \]

Therefore, if we get \(|3/5 - a_n| < 1/5\) for some \(n \in \mathbb{N}\), then we have
\[
\lim_{k \to \infty} a_k = \frac{3}{5}.
\]

But we know
\[
\left| \frac{3}{5} - a_3 \right| = 0.6234375 - 0.6 = 0.0234375 < 0.2 = \frac{1}{5}.
\]

Therefore the limit of the sequence \(\{a_n\}\) is 0.6 as seen in the first computation.

**Page 33, Problem 2.** Prove directly that each of the following sequences converges by letting \(\varepsilon > 0\) be given and finding \(N(\varepsilon)\) so that
\[
|a - a_n| < \varepsilon \quad \text{for every } n \geq N(\varepsilon).
\]

\[ (1) \]

a) 
\[
a_n = 1 + \frac{10}{\sqrt{n}}.
\]

b) 
\[
a_n = 1 + \frac{1}{\sqrt{n}}.
\]

c) 
\[
a_n = 3 + 2^{-n}.
\]

d) 
\[
a_n = \sqrt{\frac{n}{n+1}}.
\]

**Solution.** a) Obviously our guess on the limit \(a\) is \(a = 1\). So let us try with \(a = 1\) to find \(N(\varepsilon)\) which satisfy the condition (1):

\[
|1 - a_n| = \frac{10}{\sqrt{n}} < \varepsilon \quad \text{for every } n \geq N(\varepsilon),
\]
which is equivalent to the inequality:

\[ \sqrt{n} > \frac{10}{\varepsilon} \iff n > \frac{100}{\varepsilon^2} \]

for every \( n \geq N(\varepsilon) \). Thus if we choose \( N(\varepsilon) \) to be

\[ N(\varepsilon) = \left\lfloor \frac{100}{\varepsilon^2} \right\rfloor + 1, \]

where \([x], x \in \mathbb{R}\), means the largest integer which is less than or equal to \( x \), i.e., the integer \( m \) such that \( m \leq x < m + 1 \), then for every \( n \geq N(\varepsilon) \), we have

\[ \frac{100}{\varepsilon^2} < N(\varepsilon) \leq n, \quad \text{hence} \quad \varepsilon^2 > \frac{100}{n} \quad \text{and} \quad \varepsilon > \frac{10}{\sqrt[3]{n}} = |1 - a_n|. \]

This shows that

\[ \lim_{n \to \infty} \left( 1 + \frac{10}{\sqrt[3]{n}} \right) = 1. \]

b) It is also easy to guess that the limit \( a \) of \( \{a_n\} \) is 1. So let \( \varepsilon > 0 \) and try to find \( N(\varepsilon) \) which satisfy the condition (1) above which is:

\[ \frac{1}{\sqrt[3]{n}} = |1 - a_n| < \varepsilon \quad \text{for every} \quad n \geq N(\varepsilon). \]

So we look for the smallest integer \( N \) which satisfy

\[ \frac{1}{\sqrt[3]{n}} < \varepsilon \quad \text{equivalently} \quad \frac{1}{\varepsilon} < \sqrt[3]{n}, \]

which is also equivalent to

\[ n > \frac{1}{\varepsilon^3}. \]

So with \( N(\varepsilon) = \lceil 1/\varepsilon^3 \rceil + 1 \), if \( n \geq N(\varepsilon) \), then

\[ \frac{1}{\varepsilon^3} < N(\varepsilon) \leq n \quad \text{consequently} \quad \frac{1}{\sqrt[3]{n}} < \varepsilon. \]

d) First, we make a small change in the form of \( a_n \):

\[ a_n = \sqrt[3]{\frac{n}{n+1}} = \sqrt[3]{\frac{1}{1 + \frac{1}{n}}}, \]
and guess that the limit \( a \) of \( \{a_n\} \) would be 1. So we compute:

\[
1 - \sqrt{\frac{n}{n+1}} = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1}} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1}(n+1)} \leq \frac{1}{n}.
\]

Hence if \( n \geq [1/\varepsilon] + 1 \), then \( 1/n \leq 1/([1/\varepsilon] + 1) < 1/(1/\varepsilon) = \varepsilon \), i.e.,

\[
|1 - \sqrt{\frac{n}{n+1}}| < \varepsilon \quad \text{for every} \quad n \geq N(\varepsilon).
\]

Page 33, Problem 3. Prove directly that each of the following sequences converges by letting \( \varepsilon > 0 \) be given and finding \( N(\varepsilon) \) so that

\[
|a - a_n| < \varepsilon \quad \text{for every} \quad n \geq N(\varepsilon). \tag{1}
\]

a) \[
a_n = 5 - \frac{2}{\ln n} \quad \text{for} \quad n \geq 2.
\]

b) \[
a_n = \frac{3n + 1}{n + 2}.
\]

c) \[
a_n = \frac{n^2 + 6}{2n^2 - 2} \quad \text{for} \quad n \geq 2.
\]

d) \[
a_n = \frac{2^n}{n!}.
\]

Solution. a) From the form of the sequence, we guess that the limit \( a \) would be 5. So we try 5 as \( a \):

\[
|5 - a_n| = \left|5 - \left(5 + \frac{2}{\ln n}\right)\right| = \frac{2}{\ln n},
\]

which we want to make smaller than a given \( \varepsilon > 0 \). So we want find how large \( n \) ought to be in order to satisfy the inequality:

\[
\varepsilon > \frac{2}{\ln 2}.
\]
This inequality is equivalent to $\ln n > 2/\varepsilon$. Taking the exponential of the both sides, we must have $n > \exp(2/\varepsilon)$. So if we take

$$N(\varepsilon) = \left\lceil \exp \left( \frac{2}{\varepsilon} \right) \right\rceil + 1,$$

then for every $n \geq N(\varepsilon)$ the inequality (1) holds.

b) First we change the form of each term slightly:

$$a_n = \frac{3n + 1}{n + 2} = \frac{3 + \frac{1}{n}}{1 + \frac{2}{n}},$$

to make a guess on $a$. This indicates that the limit $a$ would be 3. So we try to fulfil the requirement of (1) with $a = 3$:

$$\left| 3 - \frac{3n + 1}{n + 2} \right| = \left| \frac{3(n + 2) - (3n + 1)}{n + 2} \right| = \frac{5}{n + 2} < \frac{5}{n}.$$  

So if the inequality $5/n < \varepsilon$ holds, then $|3 - a_n| < \varepsilon$ holds. Thus $N(\varepsilon) = \left\lceil 5/\varepsilon \right\rceil + 1$ gives that $|5 - a_n| < \varepsilon$ for every $n \geq N(\varepsilon)$.

c) We alter the form of the sequence slightly:

$$a_n = \frac{n^2 + 6}{2n^2 - 2} = \frac{1 + \frac{6}{n^2}}{2 - \frac{2}{n^2}},$$

in order to make a good guess on the limit $a$, which looks like $1/2$. Let us try with this $a$:

$$\left| \frac{1}{2} - \frac{n^2 + 6}{2n^2 - 2} \right| = \left| \frac{(n^2 - 1) - (n^2 + 6)}{2n^2 - 2} \right| = \frac{7}{2n^2 - 2}, \quad \text{for} \quad n \geq 2.$$  

If $n \geq 2$, then $(2n^2 - 2) - 2(n - 1)^2 = 4n > 0$, so that $2(n - 1)^2 < 2n^2 - 2$ and therefore

$$\left| \frac{1}{2} - \frac{n^2 + 6}{2n^2 - 2} \right| < \frac{7}{2(n - 1)^2}.$$  

Thus if $N(\varepsilon) = \left\lceil \sqrt{7/(2\varepsilon)} \right\rceil + 2$, then for every $n \geq N(\varepsilon)$ we have

$$\left| \frac{1}{2} - \frac{n^2 + 6}{2n^2 - 2} \right| < \frac{7}{2(n - 1)^2} \leq \frac{7}{2(N(\varepsilon) - 1)^2} = \frac{7}{2\left( \left\lceil \sqrt{\frac{7}{2\varepsilon}} \right\rceil + 1 \right)^2} < \frac{7}{2 \cdot \frac{\sqrt{7}}{2\varepsilon}} = \varepsilon.$$
Therefore
\[
\lim_{n \to \infty} a_n = \frac{1}{2}.
\]

d) With
\[
a_n = \frac{2^n}{n!}
\]
we look at the ratio \(a_n/a_{n+1}^+\):
\[
\frac{a_n}{a_{n+1}^+} = \frac{2^n}{n!} \cdot \frac{(n+1)!}{2^{n+1}} = \frac{n+1}{2} \geq 2 \quad \text{for} \quad n \geq 3.
\]

Therefore, we have for every \(k \geq 2\)
\[
a_3 \geq 2^k a_{3+k} \quad \text{equivalently} \quad a_{k+3} \leq \frac{a_3}{2^k} = \frac{8}{6} \cdot \frac{1}{2} \leq \frac{1}{2^{k-1}} \cdot \frac{1}{k-1}.
\]

So for any \(\varepsilon > 0\) if \(n \geq N(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil + 5\), then we have \(0 < a_n < \varepsilon\). 

\textbf{Page 24, Problem 6.} Suppose that \(a_n \to a\) and let \(b\) be any number strictly less than \(a\). Prove that \(a_n > b\) for all but finitely many \(n\).

\textit{Proof.} The assumption \(a > b\) yields \(b - a > 0\) so that there exists \(N \in \mathbb{N}\) such that \(|a - a_n| < b - a\) for every \(n \geq N\), equivalently

\[b - a < a - a_n < a - b\quad \text{for every} \quad n \geq N, \quad \text{hence} \quad b < a_n \quad \text{for every} \quad n \geq N.
\]

Thus the total number of \(n\) with \(b \geq a_n\) is at most \(N - 1\) which is of course finite. Thus \(a_n > b\) for all but finitely many \(n\). 

\textbf{Page 34, Problem 9.} a) Find a sequence \(\{a_n\}\) and a real number \(a\) so that
\[|a_{n+1} - a| < |a_n - a| \quad \text{for each} \quad n,
\]
but \(\{a_n\}\) does not converge to \(a\).

b) Find a sequence \(\{a_n\}\) and a real number \(a\) so that \(a_n \to a\) but so that the above inequality is violated for infinitely many \(n\).

\textit{Answer.} a) Take \(a_n = \frac{1}{n}\) and \(a = -1\). Then
\[|a_{n+1} - a| = \frac{1}{n+1} + 1 < \frac{1}{n} + 1 = |a_n - a| \quad \text{but} \quad a_n \nrightarrow a.
\]

b) Set
\[a_n = \frac{1}{n} \left(1 + \frac{(-1)^n}{2}\right) \quad \text{and} \quad a = 0.
\]
Then we have

\[ a_n = \begin{cases} \frac{1}{2n} & \text{for odd } n; \\ \frac{3}{2n} & \text{for even } n \end{cases} \]

and \( a_n \to 0 \).

If \( n \) is odd, then

\[ a_{n+1} = \frac{3}{2(n+1)} > \frac{1}{2n} = a_n. \]

This occurs infinitely many times, i.e., at every odd \( n \).

Page 39, Problem 1. Prove that each of the following limits exists:

a) \[ a_n = 5 \left(1 + \frac{1}{\sqrt[3]{n}}\right)^2. \]

b) \[ a_n = \frac{3n + 1}{n + 2}. \]

c) \[ a_n = \frac{n^2 + 6}{3n^2 - 2}. \]

d) \[ a_n = \frac{5 + \left(\frac{2}{3n}\right)^2}{2 + \frac{2n+5}{3n-2}}. \]

Proof. a) First \( \lim_{n \to \infty} 1/\sqrt[3]{n} = 0 \) because for any given \( \varepsilon > 0 \) if \( n \geq N(\varepsilon) = [1/\varepsilon^3] + 1 \) then

\[ \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N(\varepsilon)}} = \frac{1}{\sqrt[3]{[1/\varepsilon]^3}} + 1 < \frac{1}{\sqrt[3]{[1/\varepsilon^3}}} = \frac{1}{(\varepsilon)^{\frac{1}{3}}} = \varepsilon. \]

Hence we get

\[ \lim_{n \to \infty} 5 \left(1 + \frac{1}{\sqrt[3]{n}}\right)^2 = 5 \]

by the combination of Theorem 2.2.3, Theorem 2.2.4 and Theorem 2.2.5 as seen below:

\[ 1 + \frac{1}{\sqrt[3]{n}} \to 1 \quad \text{by Theorem 2.2.3} \Rightarrow \left(1 + \frac{1}{\sqrt[3]{n}}\right)^2 \to 1 \quad \text{by Theorem 2.2.5} \]

\[ \downarrow \]

\[ 5\left(1 + \frac{1}{\sqrt[3]{n}}\right)^2 \to 5 \quad \text{by Theorem 2.2.4}. \]
b) We change the form of each term $a_n$ slightly:

$$a_n = a_n = \frac{3n + 1}{n + 2} = 3 + \frac{1}{n}. $$

We know that $1/n \to 0$ and $2/n \to 0$ as $n \to \infty$. Thus we get the following chain of deduction:

$$3 + \frac{1}{n} \to 3 \quad \text{and} \quad 2 + \frac{2}{n} \to 2 \quad \text{by Theorem 2.2.3}$$

$$\downarrow$$

$$\frac{3 + \frac{1}{n}}{2 + \frac{2}{n}} \to \frac{3}{2} \quad \text{by Theorem 2.26.}$$

c) We change the form of each term $a_n$ in the following way:

$$a_n = \frac{n^2 + 6}{3n^2 - 2} = \frac{1 + \frac{6}{n}}{3 - \frac{2}{n^2}}.$$ 

As $1/n \to 0$, Theorem 2.2.5 yields that $1/n^2 \to 0$ and therefore

$$\frac{1 + \frac{6}{n}}{3 - \frac{2}{n^2}} \to \frac{1 + 6 \cdot 0}{3 - 2 \cdot 0} = \frac{1}{3} \quad \text{by Theorem 2.2.4 and Theorem 2.2.6.}$$

d) As seen before, we have

$$\frac{1}{3^n} \leq \frac{1}{n} \to 0 \quad \text{and} \quad \frac{2n + 5}{3n - 2} = \frac{2 + \frac{5}{2}}{3 - \frac{2}{n}} \to \frac{2}{3}.$$ 

Thus we get

$$a_n = \frac{5 + \left(\frac{2}{3^n}\right)^2}{2 + \frac{2n + 5}{3n - 2}} \to \frac{5 + 4 \cdot 0 \cdot 0}{2 + \frac{2}{3}} = \frac{15}{8}$$

by a combination of Theorem 2.2.3, Theorem 2.2.4 and Theorem 2.2.6. ♦

Page 39, Problem 6. Let $p(x)$ be any polynomial and suppose that $a_n \to a$. Prove that

$$\lim_{n \to \infty} p(a_n) = p(a).$$

Proof. Suppose that the polynomial $p(x)$ has the form:

$$p(x) = p_k x^k + p_{k-1} x^{k-1} + \cdots + p_1 x + p_0.$$ 

We claim that $a_n^\ell \to a^\ell$ for each $\ell \in \mathbb{N}$. If $\ell = 1$, then certainly we have the convergence: $a_1^n = a_n \to a = a^1$. Suppose $a_n^{\ell-1} \to a^{\ell-1}$. Then by Theorem 2.2.5 we have $a_n^\ell = a_n^{\ell-1} a_n \to a^{\ell-1} a = a^\ell$. By mathematical induction we have $a_n^\ell \to a^\ell$ for each $\ell \in \mathbb{N}$. Therefore, each term $p_\ell a_n^\ell$ converges to $p_\ell a^\ell$ for $\ell = 1, 2, \cdots, k$. A repeated use of Theorem 2.2.3 yields that

$$p(a_n) = p_k a_n^k + p_{k-1} a_n^{k-1} + \cdots + p_1 a_n + p_0 \to p_k a^k + p_{k-1} a^{k-1} + \cdots + p_1 a + p_0 = p(a).$$

♥
Page 39, Problem 7. Let \( \{a_n\} \) and \( \{b_n\} \) be sequences and suppose that \( a_n \leq b_n \) for all \( n \) and that \( a_n \to \infty \). Prove that \( b_n \to \infty \).

Proof. The divergence \( a_n \to \infty \) means that for every \( M \) there exists \( N \in \mathbb{N} \) such that \( a_n > M \) for every \( n \geq N \). The assumption that \( a_n \leq b_n \) gives \( M < a_n \leq b_n \) for every \( n \geq N \). Hence \( b_n \to \infty \). \( \diamondsuit \)

Page 39, Problem 9. a) Let \( \{a_n\} \) be the sequence given by

\[
a_{n+1} = \frac{1}{2} a_n + 2, \quad a_1 = 0.5
\]

Prove that \( a_n \to 4 \).

b) Consider the sequence defined by

\[
a_{n+1} = \alpha a_n + 2.
\]

Show that if \( |\alpha| < 1 \), then the sequence has a limit independent of \( a_1 \).

Proof. a) Based on the hint, we compute

\[
a_{n+1} - 4 = \frac{1}{2} a_n + 2 - 4 = \frac{1}{2} a_n - 2 = \frac{1}{2} (a_n - 4).
\]

Hence we get

\[
|a_n - 4| = \frac{1}{2} |a_{n-1} - 4| = \frac{1}{2^2} |a_{n-2} - 4| = \cdots = \frac{1}{2^{n-1}} |a_1 - 4| = \frac{3.5}{2^{n-1}} \to 0.
\]

b) We just compute

\[
a_{n+1} - \frac{2}{1 - \alpha} = \alpha a_n + 2 - \frac{2}{1 - \alpha} = \alpha a_n + \frac{2(1 - \alpha) - 2}{1 - \alpha}
\]

\[
= \alpha a_n - \frac{2\alpha}{1 - \alpha} = \alpha \left( a_n - \frac{2}{1 - \alpha} \right);\]

\[
a_n - \frac{2}{1 - \alpha} = \alpha \left( a_{n-1} - \frac{2}{1 - \alpha} \right) = \alpha^2 \left( a_{n-2} - \frac{2}{1 - \alpha} \right) = \cdots
\]

\[
= \alpha^{n-1} \left( a_1 - \frac{2}{1 - \alpha} \right) \to 0 \quad \text{as} \quad |\alpha| < 1.
\]

Hence \( \{a_n\} \) converges and

\[
\lim_{n \to \infty} a_n = \frac{2}{1 - \alpha}
\]

which is independent of \( a_1 \). \( \diamondsuit \)
Page 40, Problem 10. For a pair \((x, y)\) of real numbers, define
\[
\|(x, y)\| = \sqrt{x^2 + y^2}.
\]

a) Let \(x_1, x_2, y_1, y_2\) be real numbers. Prove that
\[
|x_1x_2 + y_1y_2| \leq \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}.
\]

b) Prove that for any two dimensional vectors \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\)
\[
\|(x_1, y_1) + (x_2, y_2)\| \leq \|(x_1, y_1)\| + \|(x_2, y_2)\|.
\]

c) Let \(p_n = (x_n, y_n)\) be a sequence of points in the plane \(\mathbb{R}^2\) and let \(p = (x, y)\). We say that \(p_n \to p\) if \(\|p_n - p\| \to 0\). Prove that \(p_n \to p\) if and only if \(x_n \to x\) and \(y_n \to y\).

Proof. a) Let us compute:
\[
(x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1y_1 + x_2y_2)^2
= x_1^2y_1^2 + x_2^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2 - (x_1y_1^2 + 2x_1y_1x_2y_2 + x_2y_2^2)
= x_1^2y_1^2 + x_2^2y_2^2 - 2x_1y_1x_2y_2 = (x_1y_2 - x_2y_1)^2 \geq 0.
\]

b) We also compute directly:
\[
\|(x_1, y_1) + (x_2, y_2)\|^2 = \|(x_1 + x_2, y_1 + y_2)\|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2
\]
\[
= x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2 = x_1^2 + x_2^2 + 2(x_1x_2 + y_1y_2) + y_1^2 + y_2^2
\]
\[
\leq x_1^2 + y_1^2 + 2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} + x_2^2 + y_2^2
\]
\[
= \left(\sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}\right)^2 = \left(\|(x_1, y_1)\| + \|(x_2, y_2)\|\right)^2.
\]

This shows the inequality:
\[
\|(x_1, y_1) + (x_2, y_2)\| \leq \|(x_1, y_1)\| + \|(x_2, y_2)\|.
\]

c) Since we have the inequalities:
\[
\max\{|x_n - x|, |y_n - y|\} \leq \sqrt{(x_n - x)^2 + (y_n - y)^2} \leq 2\max\{|x_n - x|, |y_n - y|\},
\]
show that \(\|p_n - p\| \to 0\) if and only if \(\max\{|x_n - x|, |y_n - y|\} \to 0\) if and only if \(|x_n - x| \to 0\) and \(|y_n - y| \to 0\).
Page 50, Problem 1. Prove directly that $a_n = 1 + \frac{1}{\sqrt{n}}$ is a Cauchy sequence.

Proof. We just compute for $m < n$:

$$|a_m - a_n| = \left| \left(1 + \frac{1}{\sqrt{m}}\right) - \left(1 + \frac{1}{\sqrt{n}}\right) \right| = \left| \frac{1}{\sqrt{m}} - \frac{1}{\sqrt{n}} \right|$$

$$\leq \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}} \leq \frac{2}{\sqrt{m}}$$

since $m < n$.

So if $\varepsilon > 0$ is given, then we take $N$ to be $\lceil (2/\varepsilon)^2 \rceil$ so that for every $n > m \geq N$ we have

$$|a_m - a_n| \leq \frac{2}{\sqrt{m}} \leq \frac{2}{\sqrt{\lceil (2/\varepsilon)^2 \rceil}} < \frac{2}{\sqrt{\frac{2}{\varepsilon}^2}} = \varepsilon.$$

Page 50, Problem 2. Prove that the rational numbers are dense in the real numbers.

Proof. We have to prove that for every $\varepsilon > 0$ $(a - \varepsilon, a + \varepsilon) \cap \mathbb{Q} \neq \emptyset$. Choose $m = \lceil 1/\varepsilon \rceil + 1$ so that $1/m < \varepsilon$. If $a - \varepsilon > 0$, then the Archimedian property of $\mathbb{R}$ yields the existence of $k \in \mathbb{N}$ such that

$$\frac{k}{m} = k \cdot \frac{1}{m} > a - \varepsilon.$$

Let $n$ be the first such a number. Then we have

$$\frac{n-1}{m} \leq a - \varepsilon < \frac{n}{m} = \frac{n-1}{m} + \frac{1}{m} \leq a - \varepsilon + \frac{1}{m} < a - \varepsilon + \varepsilon = a.$$

Therefore, we conclude that $a - \varepsilon < \frac{n}{m} < a$ therefore $\frac{n}{m} \in (a - \varepsilon, a + \varepsilon) \cap \mathbb{Q}$.

If $a - \varepsilon < 0$, then we apply the Archimedian property of $\mathbb{R}$ to the pair $1/m$ and $\varepsilon - a > 0$ to find a natural number $k \in \mathbb{N}$ such that

$$\varepsilon - a < k \cdot \frac{1}{m} = \frac{k}{m}.$$

Let $n \in \mathbb{N}$ be the smallest natural number such that $n/m \geq \varepsilon - a$, so that

$$\frac{n-1}{m} < \varepsilon - a \leq \frac{n}{m}, \quad \text{equivalently} \quad -\frac{n}{m} \leq a - \varepsilon < \frac{1-n}{m} = -\frac{n}{m} + \frac{1}{m}.$$

As we have chosen $m \in \mathbb{N}$ so large that $1/m < \varepsilon$, the above inequality yields

$$\frac{1-n}{m} = \frac{1}{m} - \frac{n}{m} < \varepsilon - \frac{n}{m} \leq \varepsilon + (a - \varepsilon) = a, \quad \text{i.e.,} \quad a - \varepsilon < \frac{1-n}{m} < a.$$

Therefore we have

$$\frac{1-n}{m} \in (a - \varepsilon, a + \varepsilon) \cap \mathbb{Q} \quad \text{consequently} \quad (a - \varepsilon, a + \varepsilon) \cap \mathbb{Q} \neq \emptyset$$

for arbitrary $\varepsilon > 0$. Hence $a$ is a limit point of $\mathbb{Q}$. 
SOLUTION SET FOR THE HOMEWORK PROBLEMS

Page 59, Problem 3. Suppose that the sequence \( \{a_n\} \) converges to \( a \) and \( d \) is a limit point of the sequence \( \{b_n\} \). Prove that \( ad \) is a limit point of the sequence \( \{a_nb_n\} \).

Proof. By the assumption on the sequence \( \{b_n\} \), there exists a subsequence \( \{b_{n_k}\} \) of the sequence \( \{b_n\} \) such that
\[
\lim_{k \to \infty} b_{n_k} = d.
\]
The subsequence \( \{a_{n_k}b_{n_k}\} \) of the sequence \( \{a_nb_n\} \) converges to \( ad \) because the subsequence \( \{a_{n_k}\} \) of \( \{a_n\} \) converges to the same limit \( a \). Hence \( ad \) is a limit point of \( \{a_nb_n\} \). \(

Page 59, Problem 6. Consider the following sequence: \( a_1 = \frac{1}{2} \); the next three terms are \( \frac{1}{4}, \frac{1}{3}, \frac{3}{4} \); the next seven terms are \( \frac{1}{8}, \frac{1}{5}, \frac{3}{8}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, \cdots \) and so forth. What are the limit points.

Answer. The sequence \( \{a_n\} \) consists of the numbers \( \{k/2^n : k = 1, 2, \cdots, 2^n - 1, n \in \mathbb{N}\} \). Fix \( x \in [0, 1] \). We are going to construct a subsequence \( \{b_n\} \) of the sequence \( \{a_n\} \) by induction. For \( n = 1 \), choose
\[
k_1 = \begin{cases} 
0 & \text{if } x \leq \frac{1}{2}; \\
1 & \text{if } \frac{1}{2} < x \leq 1.
\end{cases}
\]
For each \( n > 1 \), let \( k_n \) be the natural number such that
\[
\frac{k_n}{2^n} \leq x < \frac{k_n + 1}{2^n}.
\]
Then the ratio \( b_n = k_n/2^n \) is in the sequence \( \{a_n\} \) and
\[
|b_n - x| < \frac{1}{2^n} \to 0 \quad \text{as } n \to \infty.
\]
Hence \( \lim_{n \to \infty} b_n = x \). Therefore, every \( x \in [0, 1] \) is a limit point of \( \{a_n\} \). Thus the sequence \( \{a_n\} \) is dense in the closed unit interval \([0, 1]\). \(

Page 59, Problem 8. Let \( \{I_k : k \in \mathbb{N}\} \) be a nested family of closed, finite intervals; that is, \( I_1 \supset I_2 \supset \cdots \). Prove that there is a point \( p \) contained in all the intervals, that is \( p \in \cap_{k=1}^{\infty} I_k \).

Proof. The assumption means that if \( I_k = [a_k, b_k], k \in \mathbb{N} \), then
\[
a_1 \leq a_2 \leq \cdots \leq a_k \leq \cdots \leq b_k \leq b_{k-1} \leq \cdots \leq b_2 \leq b_1.
\]
The sequence \( \{a_k\} \) is increasing and bounded by any of \( \{b_k\} \). Fix \( k \in \mathbb{N} \). Then we have
\[
a = \lim_{n \to \infty} a_n \leq b_k \quad \text{for } k \in \mathbb{N},
\]
where the convergence of \( \{a_n\} \) is guaranteed by the boundedness of the sequence. Now look at the sequence \( \{b_k\} \) which is decreasing and bounded below by \( a \). Hence it converges to \( b \in \mathbb{R} \) and \( a \leq b \). Thus the situation is like the following:

\[
a_1 \leq a_2 \leq \cdots \leq a_k \leq \cdots \leq a \leq b \leq \cdots \leq b_{k-1} \leq \cdots \leq b_2 \leq b_1.
\]

Hence the interval \([a, b]\) is contained in the intersection \( \bigcap_{k=1}^{\infty} I_k \). Any point \( p \) in the interval \([a, b]\) is a point of \( \bigcap_{k=1}^{\infty} I_k \); in fact \([a, b] = \bigcap_{k=1}^{\infty} I_k \).♥

**Page 59, Problem 9.** Suppose that \( \{x_n\} \) is a monotone increasing sequence of points in \( \mathbb{R} \) and suppose that a subsequence of \( \{x_n\} \) converges to a finite limit. Prove that \( \{x_n\} \) converges to a finite limit.

**Proof.** Let \( \{x_{n_k}\} \) be the subsequence converging to the finite limit \( x_0 \). As \( n_1 < n_2 < \cdots < n_k < \cdots \), we have \( k \leq n_k \) for every \( k \in \mathbb{N} \). If \( \varepsilon > 0 \) is given, then choose \( K \) so large that \( |x_{n_k} - x_0| < \varepsilon \) for every \( k \geq K \), i.e., \( x_0 - \varepsilon < x_{n_k} \leq x_0 \) for every \( k \geq K \). Set \( N = n_K \). Then if \( m \geq N \), then we have

\[
x_0 - \varepsilon < x_{n_K} = x_N \leq x_m \leq x_{n_m} \leq x_0.
\]

Hence we have \( 0 \leq x_0 - x_m < \varepsilon \) for every \( m \geq N \). Hence \( \{x_n\} \) converges to the same limit \( x_0 \).♥

**Page 79, Problem 3.** Let \( f(x) \) be a continuous function. Prove that \( |f(x)| \) is a continuous function.

**Proof.** Let \( x \in [a, b] \) be a point in the domain \([a, b]\) of the function \( f \). If \( \varepsilon > 0 \) is given, then choose a \( \delta > 0 \) so small that \( |f(x) - f(y)| < \varepsilon \) whenever \( |x - y| < \delta \). If \( |x - y| < \delta \), then

\[
||f(x)| - |f(y)|| \leq |f(x) - f(y)| < \varepsilon.
\]

Hence \( |f| \) is continuous at \( x \).♥

**Page 79, Problem 5.** Suppose that \( f \) is a continuous function on \( \mathbb{R} \) such that \( f(q) = 0 \) for every \( q \in \mathbb{Q} \). Prove that \( f(x) = 0 \) for every \( x \in \mathbb{R} \).

**Proof.** Choose \( x \in \mathbb{R} \) and \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that \( |f(x) - f(y)| < \varepsilon \) whenever \( |x - y| < \delta \). Take \( q \in \mathbb{Q} \cap (x - \delta, x + \delta) \), then

\[
|f(x)| = |f(x) - f(q)| < \varepsilon.
\]

Thus \( |f(x)| \) is less than any \( \varepsilon > 0 \) which is possible only when \( |f(x)| = 0 \).♥
Page 79, Problem 7. Let $f(x) = 3x - 1$ and let $\varepsilon > 0$ be given. How small $\delta$ be chosen so that $|x - 1| \leq \varepsilon$ implies $|f(x) - 2| < \varepsilon$?

**Answer.** To determine the magnitude of $\delta$, assume that $|x - 1| < \delta$ and see how the error becomes:

$$|f(x) - 2| = |3x - 1 - 2| = |3x - 3| = 3|x - 1| < 3\delta.$$  
Thus if $3\delta \leq \varepsilon$, i.e., if $\delta \leq \varepsilon/3$, then $|x - 1| < \delta$ implies $|f(x) - 2| < \varepsilon$.

Page 79, Problem 8. Let $f(x) = x^2$ and let $\varepsilon > 0$ be given.

a) Find a $\delta$ so that $|x - 1| \leq \delta$ implies $|f(x) - 1| \leq \varepsilon$.

b) Find a $\delta$ so that $|x - 2| \leq \delta$ implies $|f(x) - 2| \leq \varepsilon$.

c) If $n > 2$ and you had to find a $\delta$ so that $|x - n| \leq \delta$ implies $|f(x) - n^2| \leq \varepsilon$, would the $\delta$ be larger or smaller than the $\delta$ for parts (a) and (b)? Why?

**Answer.** a) Choose $\delta > 0$ and see how the error grows from $|x - 1| \leq \delta$:

$$|f(x) - 1| = |x^2 - 1| = |(x + 1)(x - 1)| = |x + 1||x - 1|$$
$$\leq |x + 1|\delta = |x - 1 + 1 + 1\delta| \leq (|x - 1| + 2)\delta$$
$$\leq (\delta + 2)\delta.$$  
So we want to make $(\delta + 2)\delta \leq \varepsilon$. Let us solve this inequality:

$$0 \geq \delta^2 + 2\delta - \varepsilon = (\delta + 1)^2 - \varepsilon - 1 \Leftrightarrow \varepsilon + 1 \geq (\delta + 1)^2$$  
$$\Leftrightarrow \quad -\sqrt{\varepsilon + 1} - 1 \leq \delta \leq \sqrt{\varepsilon + 1} - 1.$$  
But we know that $\delta$ must be positive. Hence $0 < \delta \leq \sqrt{\varepsilon + 1} - 1 = \varepsilon/\sqrt{\varepsilon + 1}$. If $\delta$ is chosen in the interval $(0, \sqrt{\varepsilon + 1} - 1)$, then the above calculation shows that $|x - 1| \leq \delta \Rightarrow |f(x) - 1| \leq \varepsilon$.

b) Now we continue to examine the case $|x - 2| \leq \delta$:

$$|f(x) - 4| = |x^2 - 4| = |(x + 2)(x - 2)| = |x + 2||x - 2|$$
$$\leq \delta|x + 2| = \delta|x - 2 + 4| \leq \delta(|x - 2| + 4)$$
$$\leq \delta(\delta + 4).$$  
So we want to make $(\delta + 4)\delta \leq \varepsilon$, equivalently:

$$\varepsilon \geq \delta(\delta + 4) = \delta^2 + 4\delta \Leftrightarrow \delta^2 + 4\delta - \varepsilon \leq 0$$  
$$\Leftrightarrow \quad -\sqrt{\varepsilon + 4} - 2 \leq \delta \leq \sqrt{\varepsilon + 4} - 2 = \frac{(\sqrt{\varepsilon + 4} - 2)(\sqrt{\varepsilon + 4} + 2)}{\sqrt{\varepsilon + 4} + 2} = \frac{\varepsilon}{\sqrt{\varepsilon + 4} + 2}. $$
Hence if we take \(0 < \delta \leq \varepsilon/(\sqrt{\varepsilon + 4} + 2)\), then
\[
|x - 2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \varepsilon.
\]

c) Similarly, we examine the case \(|x - n| \leq \delta\):
\[
|f(x) - n^2| = |x^2 - n^2| = |x + n||x - n| \leq \delta|x + n|
\leq \delta|x - n + 2n| \leq \delta(|x - n| + 2n) \leq \delta(\delta + 2n).
\]

So we want to make \((\delta + n)\delta \leq \varepsilon\), equivalently:
\[
\varepsilon \geq \delta(\delta + 4) = \delta^2 + 2n\delta \quad \Leftrightarrow \quad \delta^2 + 2n\delta - \varepsilon \leq 0
\]

\[
-\sqrt{\varepsilon + n^2} - n \leq \delta \leq \sqrt{\varepsilon + n^2} - n = \frac{(\sqrt{\varepsilon + n^2} - n)(\sqrt{\varepsilon + n^2} + n)}{\sqrt{\varepsilon + n^2} + n} = \frac{\varepsilon}{\sqrt{\varepsilon + n^2} + n}.
\]

Hence if we take \(\delta > 0\) so small that 
\(0 < \delta \leq \varepsilon/(\sqrt{\varepsilon + n^2} + n)\), then
\[
|x - n| \leq \delta \quad \Rightarrow \quad |f(x) - n| \leq \varepsilon.
\]

The largest possible \(\delta = \varepsilon/(\sqrt{\varepsilon + n^2} + n)\) is squeezed to zero when \(n\) glows indefinitely.

**Page 79, Problem 11.** Let \(f(x) = \sqrt{x}\) with domain \(\{x : x \geq 0\}\).

a) Let \(\varepsilon > 0\) be given. For each \(c > 0\), show how to choose \(\delta > 0\) so that \(|x - c| \leq \delta\) implies \(|\sqrt{x} - \sqrt{c}| \leq \varepsilon\).

b) Give a separate argument to show that \(f\) is continuous at zero.

**Solution.** a) Once again we examine the growth of error by letting \(|x - c| \leq \delta\) and compute:
\[
|\sqrt{x} - \sqrt{c}| = |\sqrt{x - c + c} - \sqrt{c}| = \frac{|x - c|}{\sqrt{x - c + c} + \sqrt{c}} \leq \frac{\delta}{\sqrt{x - c + c} + \sqrt{c}}
\]
\[
\leq \frac{\delta}{\sqrt{\frac{x}{2} + \sqrt{c}}}
\]
(under the assumption \(\delta \leq \frac{c}{2}\))
\[
\leq \frac{2\delta}{\sqrt{c}}.
\]

Thus if \(0 < \delta \leq \min\{c/2, \varepsilon\sqrt{c}/2\}\), then
\[
|x - c| \leq \delta \quad \Rightarrow \quad |\sqrt{x} - \sqrt{c}| \leq \varepsilon.
\]

Hence \(f\) is continuous at \(c > 0\).

b) If \(0 \leq x \leq \delta\), then
\[
|\sqrt{x} - \sqrt{0}| = \sqrt{x} \leq \sqrt{\delta}.
\]

Hence if \(0 < \delta \leq \varepsilon^2\), then
\[
0 \leq x \leq \delta \quad \Rightarrow \quad 0 \leq \sqrt{x} \leq \varepsilon.
\]

Therefore \(f\) is continuous at \(0\).