SOLUTION SET FOR THE HOMEWORK PROBLEMS

Page 5. Problem 8. *Prove that if x and y are real numbers, then*

$$2xy \leq x^2 + y^2.$$

Proof. First we prove that if x is a real number, then $x^2 \geq 0$. The product of two positive numbers is always positive, i.e., if $x \geq 0$ and $y \geq 0$, then $xy \geq 0$. In particular if $x \geq 0$ then $x^2 = x \cdot x \geq 0$. If x is negative, then $-x$ is positive, hence $(-x)^2 \geq 0$. But we can conduct the following computation by the associativity and the commutativity of the product of real numbers:

$$\begin{aligned} 0 \geq (-x)^2 &= (-x)(-x) = ((-1)x)((-1)x) = (((-1)x))(-1))x \\ &= (((-1)(x(-1)))x = (((-1)(-1))x)x = (1x)x = xx = x^2. \end{aligned}$$

The above change in bracketting can be done in many ways. At any rate, this shows that the square of any real number is non-negaitive. Now if x and y are real numbers, then so is the difference, $x - y$ which is defined to be $x + (-y)$. Therefore we conclude that $0 \leq (x + (-y))^2$ and compute:

$$\begin{aligned} 0 &\leq (x + (-y))^2 = (x + (-y))(x + (-y)) = x(x + (-y)) + (-y)(x + (-y)) \\ &= x^2 + x(-y) + (-y)x + (-y)^2 = x^2 + y^2 + (-xy) + (-xy) \\ &= x^2 + y^2 + 2(-xy); \end{aligned}$$

adding $2xy$ to the both sides,

$$\begin{aligned} 2xy = 0 + 2xy &\leq (x^2 + y^2 + 2(-xy)) + 2xy = (x^2 + y^2) + (2(-xy) + 2xy) \\ &= (x^2 + y^2) + 0 = x^2 + y^2. \end{aligned}$$

Therefore, we conclude the inequality:

$$2xy \leq x^2 + y^2$$

for every pair of real numbers x and y. ♡

Page 5. Problem 11. *If a and b are real numbers with $a < b$, then there exists a pair of integers m and n such that*

$$a < \frac{m}{n} < b, \quad n \neq 0.$$

Proof. The assumption $a < b$ is equivalent to the inequality $0 < b - a$. By the Archimedian property of the real number field, $\mathbb{R}$, there exists a positive integer n such that

$$n(b - a) > 1.$$

Of course, $n \neq 0$. Observe that this n can be 1 if $b - a$ happen to be large enough, i.e., if $b-a > 1$. The inequality $n(b-a) > 1$ means that $nb-na > 1$, i.e., we can conclude that

$$na + 1 < nb.$$

Let m be the smallest integer such that $na < m$. Does there exists such an integer? To answer to the question, we consider the set $A = \{k \in \mathbb{Z} : k > na\}$ of integers. First $A \neq \emptyset$. Because if $na \geq 0$ then $1 \in A$ and if $na > 0$ then by the Archimedian property of $\mathbb{R}$, there exists $k \in \mathbb{Z}$ such that $k = k \cdot 1 > na$. Hence $A \neq \emptyset$. Choose $\ell \in A$ and consider the following chain:

$$\ell > \ell - 1 > \ell - 2 > \cdots > \ell - k, \quad k \in \mathbb{N}.$$

This sequence eventually goes down beyond na. So let k be the first natural number such that $\ell - k \leq na$, i.e., the natural number k such that $\ell - k \leq na < \ell - k + 1$. Set $m = \ell - k + 1$ and observe that

$$na < m = \ell - k \leq +1 \leq na + 1 < nb.$$

Therefore, we come to the inequality $na < m < nb$. Since n is a positive integer, we devide the inequlity by n withoug changing the direction of the inequality:

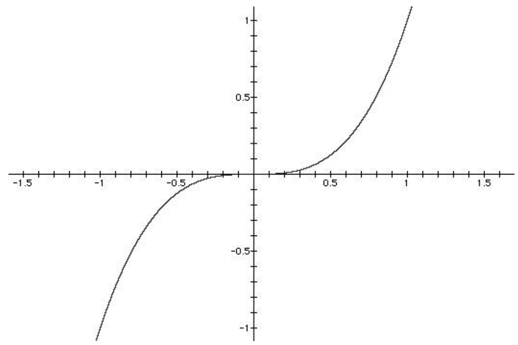
$$a = \frac{na}{n} < \frac{m}{n} < \frac{nb}{n} = b.$$

♡

Page 14, Problem 6. *Generate the graph of the following functions on* $\mathbb{R}$ *and use it to determine the range of the function and whether it is onto and one-to-one*:

a) $f(x) = x^3$.
b) $f(x) = \sin x$.
c) $f(x) = e^x$.
d) $f(x) = \frac{1}{1+x^4}$.

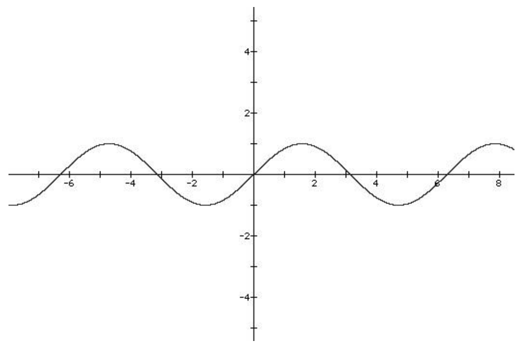
Solution. a) The function f is bijection since $f(x) < f(y)$ for any pair $x, y \in \mathbb{R}$ with the relation $x < y$ and for every real number $y \in \mathbb{R}$ there exists a real numbe $x \in \mathbb{R}$ such that $y = f(x)$.



b) The function f is neither injective nor surjective since

$$f(x + 2\pi) = f(x)$$

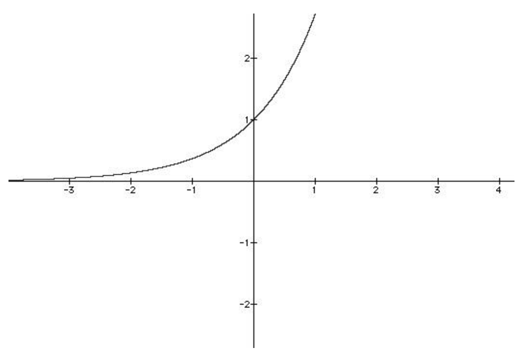
$x + \pi \neq x, x \in \mathbb{R}$, and if $y > 1$ then there is no $x \in \mathbb{R}$ such that $y = f(x)$.



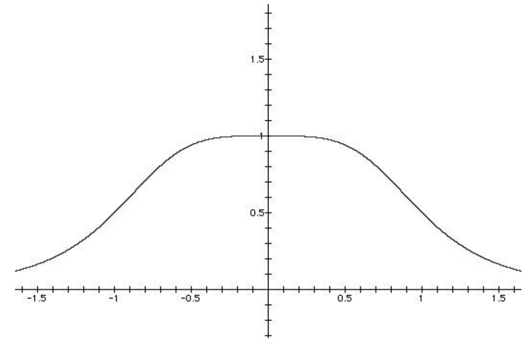
c) The function f is injective because

$$f(x) < f(y)$$

if $x < y, x, y \in \mathbb{R}$, but not surjective as a map from $\mathbb{R}$ to $\mathbb{R}$, because there exists no $x \in \mathbb{R}$ such that $f(x) = -1$.



d) The function f is not injective as $f(x) = f(-x)$ and $x \neq -x$ for $x \neq 0$, nor surjective as there is no $x \in \mathbb{R}$ such that $f(x) = -1$.



♡

Page 14, Problem 8. *Let $\mathcal{P}$ be the set of polynomials of one real variable. If $p(x)$ is such a polynomial, define $I(p)$ to be the function whose value at x is*

$$I(p)(x) \equiv \int_0^x p(t)\mathrm{d}t.$$

Explain why I is a function from $\mathcal{P}$ to $\mathcal{P}$ and determine whether it is one-to-one and onto.

Solution. Every element $p \in \mathcal{P}$ is of the form:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}, \quad x \in \mathbb{R},$$

with $a_0, a_1, \cdots, a_{n-1}$ real numbers. Then we have

$$\begin{aligned} I(p)(x) &= \int_0^x (a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1}t^{n-1})\mathrm{d}t \\ &= a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_{n-1}}{n}x^n. \end{aligned}$$

Thus $I(p)$ is another polynomial, i.e., an element of $\mathcal{P}$. Thus I is a function from $\mathcal{P}$ to $\mathcal{P}$.

We claim that I is injective: If

$$\begin{aligned} p(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1}x^{m-1}; \\ q(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1}x^{n-1} \end{aligned}$$

have $I(p)(x) = I(q)(x), x \in R$,i.e.,

$$a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_{m-1}}{m}x^m = b_0 x + \frac{b_1}{2}x^2 + \frac{b_2}{3}x^3 + \cdots + \frac{b_{n-1}}{n}x^n.$$

Let $P(x) = I(p)(x)$ and $Q(x) = I(q)(x)$. Then the above equality for all $x \in \mathbb{R}$ allows us to differentiate the both sides to obtain

$$P'(x) = Q'(x) \quad \text{for every } x \in \mathbb{R},$$

in particular $a_0 = P'(0) = Q'(0) = b_0$. The second differentiation gives

$$P''(x) = Q''(x) \quad \text{for every } x \in \mathbb{R},$$

in particular $a_1 = P''(0) = Q''(0) = b_1$.

Suppose that with $k \in \mathbb{N}$ we have $P^{(k)}(x) = Q^{(k)}(x)$ for every $x \in \mathbb{R}$. Then the differentiation of the both sides gives

$$P^{(k+1)}(x) = Q^{(k+1)}(x) \quad forevery\ x \in \mathbb{R},$$

in particular $a_{k+1} = P^{(k+1)}(0) = Q^{(k+1)}(0) = b_{k+1}$. Therefore the mathematical induction gives

$$a_0 = b_0, a_1 = b_1, \cdots, a_{m-1} = b_{m-1} \quad \text{and } m = n,$$

i.e., $p = q$. Hence the function I is injective.

We claim that I is not surjective: As $I(p)(0) = 0$, the constant polynomial $q(x) = 1$ cannot be of the form $q(x) = I(p)(x)$ for any $p \in \mathcal{P}$, i.e., there is no $p \in \mathcal{P}$ such that $I(p)(x) = 1$. Hence the constant polynimial q is not in the image $I(\mathcal{P})$. ♡

Page 19, Problem 3. *Prove that*:

a) *The union of two finite sets is finite.*
b) *The union of a finite sent and a countable set is countable.*
c) *The union of two contable sets is countable.*

Proof. a) Let A and B be two finite sets. Set $C = A \cap B$ and $D = A \cup B$. First, let a, b and c be the total number of elements of A, B and C respectively. As $C \subset A$ and $C \subset B$, we know that $c \leq a$ and $c \leq b$. We then see that the union:

$$D = C \cup (A \backslash C) \cup (B \backslash C)$$

is a disjoint union, i.e., the sets C, $A\backslash C$ and $B\backslash C$ are mutually disjoint. Thus the total number d of elements of D is precisely $c + (a - c) + (b - c) = a + b - c$ which is a finite number, i.e., D is a finite set with the total number d of elements.

b) Let A be a finite set and B a countable set. Set $C = A \cap B$ and $D = A \cup B$. Since C is a subset of the finite set A, C is finite. Let m be the total number of elements of C and $\{c_1, c_2, \cdots, c_m\}$ be the list of elemtns of C. Let n be the total number of elements of A and let $\{a_1, a_2, \cdots, a_{n-m}\}$ be the leballing of the set $A\backslash C$. Arrange an enumeration of the elements of B in the following fashion:

$$B = \{c_1, c_2, \cdots, c_m, b_{m+1}, b_{m+2}, \cdots\}.$$

Arranging the set A in the following way:

$$A = \{a_1, a_2, \cdots, a_{n-m}, c_1, c_2, \cdots, c_m\},$$

we enumerate the elements of $D = A \cup B$ in the following way:

$$d_i = \begin{cases} a_i & \text{for } 1 \le i \le n - m; \\ c_{i-n-m} & \text{for } n - m < i \le n; \\ b_{i-n+m} & \text{for } i > n. \end{cases}$$

This gives an enumeration of the set D. Hence D is countable.

c) Let A and B be two countable sets. Let $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$ be enumerations of A and B respectively. Define a map f from the set $\mathbb{N}$ of natural numbers in the following way:

$$f(2n-1) = a_n, \quad n \in \mathbb{N};$$
$$f(2n) = b_n, \quad n \in \mathbb{N}.$$

Then f maps $\mathbb{N}$ onto $D = A \cup B$. The surjectivity of the map f guarantees that $f^{-1}(d) \neq \emptyset$ for every $d \in D$. For each $d \in \mathbb{N}$, let $g(d) \in \mathbb{N}$ be the first element of $f^{-1}(d)$. Since $f^{-1}(d) \cap f^{-1}(d') = \emptyset$ for every distinct pair $d, d' \in D$, $g(d) \neq g(d')$ for every distinct pair $d, d' \in D$. Hence the map g is injective. Now we enumerate the set D by making use of g. Let $d_1 \in D$ be the element of D such that $g(d_1)$ is the least element of $g(D)$. After $\{d_1, d_2, \cdots, d_n\}$ were chosen, we choose $d_{n+1} \in D$ as the element such that $g(d_{n+1})$ is the least element of $g(D \backslash \{d_1, d_2, \cdots, d_n\})$. By induction, we choose a sequence $\{d_n\}$ of elements of D. Observe that $1 \le g(d_1) < g(d_2) < \cdots < g(d_n) < \cdots$ in $\mathbb{N}$. Hence we have $n \le g(d_n)$. This means that every $d \in \mathbb{N}$ appears in the list $\{d_1, d_2, \cdots\}$. Hence D is countable. ♡

Page 24, Problem 1. *Give five examples which show that P implies Q does not necessarily mean that Q implies P.*

Examples. 1) P is the statement that $x = 1$ and Q is the statement that $x^2 = 1$.

2) P is the statement that $x \le 1$ and Q is the statement that $x \le 2$.

3) Let A be a subset of a set B with $A \neq B$. P is the statement that x is an element of A and Q is the statement that x is an element of B.

4) P is the statement that x is a positive real number and Q is the statement that x^2 is a positive real number.

5) P is the statement that $x = 0$ or $x = 1$ and Q is the statement that $x(x-1)(x-2) = 0$. ♡

Page 24, Problem 3. *Suppose that a, b, c, and d are positive real numbers such that $a/b < c/d$. Prove that*

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

Proof. The inequality $a/b < c/d$ is equivalent to the inequality $bc - ad > 0$. We compare two numbers by subtracting one from the other. So we compare the first two of the above three fractions first and then the second pair of the fractions:

$$\frac{a+c}{b+d} - \frac{a}{b} = \frac{b(a+c) - a(b+d)}{b(b+d)} = \frac{bc - ad}{b(b+d)} > 0;$$
$$\frac{c}{d} - \frac{a+c}{b+d} = \frac{c(b+d) - (a+c)d}{c(b+d)} = \frac{bc - ad}{c(b+d)} > 0.$$

Therefore the desired inequalities follows. ♡

Page 24, Problem 4. *Suppose that* $0 < a < b$. *Prove that*

a) $a < \sqrt{ab} < b$.
b) $\sqrt{ab} < \frac{a+b}{2}$.

Proof. a) We compute, based on the fact that the inequality $a < b$ implies the inequality $\sqrt{a} < \sqrt{b}$,

$$b = \sqrt{b}\sqrt{b} > \sqrt{a}\sqrt{b} = \sqrt{ab} > \sqrt{a}\sqrt{a} = a.$$

b) We simply compute:

$$\frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{a}\sqrt{b}}{2}$$
$$= \frac{(\sqrt{a} - \sqrt{b})^2}{2} > 0,$$

where we used the fact that $(x+y)^2 = x^2 - 2xy + y^2$ which follows from the distributive law and the commutativity law in the field of real numbers as seen below:

$$(x-y)^2 = (x-y)x - (x-y)y \quad \text{by the distributive law}$$
$$= x^2 - yx - xy + y^2 = x^2 - 2xy + y^2 \quad \text{by the commutativity law.}$$

♡

Remark. In the last computation, is

$$(x-y)^2 = x^2 - 2xy + y^2$$

obvious?
If so, take

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and } y = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

and compute

$$x - y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad (x-y)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$
$$x^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ y^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ xy = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},\ yx = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix};$$
$$x^2 - 2xy + y^2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (x-y)^2.$$

Therefore, the formula

$$(x-y)^2 = x^2 - 2xy + y^2$$

is **not universally true**. This is a **consequence of the distributive law and the commutative law** which governs the field $\mathbb{R}$ of real numbers as discussed in the very early class.

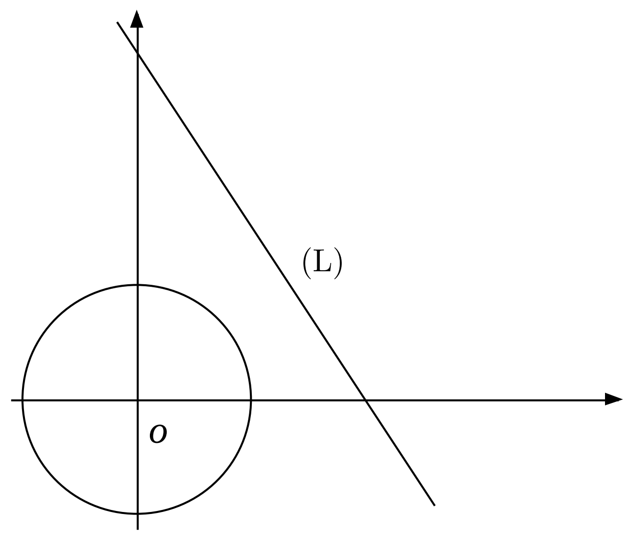
Page 24, Problem 5. *Suppose that x and y satisfy $\frac{x}{2} + \frac{y}{3} = 1$. Prove that $x^2 + y^2 > 1$.*

Proof. The point (x, y) lies on the line:

$$\frac{x}{2} + \frac{y}{3} = 1 \tag{L}$$

which cuts through x-axis at $(2, 0)$ and y-axis at $(0, 3)$. The line L is also discribed by parameter $(2t, 3(1-t)), t \in \mathbb{R}$. So we compute

$$\begin{aligned} x^2 + y^2 &= (2t)^2 + (3(1-t))^2 = 4t^2 + 9(1-t)^2 \\ &= 4t^2 + 9t^2 - 18t + 9 \\ &= 13t^2 - 18t + 9 = 13\left(t^2 - \frac{18}{13}t + \frac{9}{13}\right) \\ &= 13\left\{\left(t - \frac{9}{13}\right)^2 + \frac{9}{13} - \left(\frac{9}{13}\right)^2\right\} > 0 \end{aligned}$$



for all $t \in \mathbb{R}$ as $\frac{9}{13} < 1$. ♡

Page 25, Problem 8. *Prove that for all positive integers n,*

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

Proof. Suppose $n = 1$. Then the both sides of the above identity is one. So the formula hold for $n = 1$.

Suppose that the formula hold for n, i.e.,

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

Adding $(n+1)^3$ to the both sides, we get

$$\begin{aligned} 1 + 2 + \cdots + n^3 + (n+1)^3 &= (1 + 2 + \cdots + n)^2 + (n+1)^3 \\ &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad \text{by Proposition 1.4.3} \\ &= (n+1)^2\left(\frac{n^2 + 4n + 4}{4}\right) = \frac{(n+1)^2(n+2)^2}{4} \\ &= (1 + 2 + \cdots + n + n + 1)^2 \quad \text{by Proposition 1.4.3.} \end{aligned}$$

Thus the formula holds for $n+1$. Therefore mathematical induction assures that the formula holds for every $n \in \mathbb{N}$. ♡

Page 25, Problem 9. *Let $x > -1$ and n be a positive integer. Prove Bernoulli's inequality:*

$$(1+x)^n \geq 1 + nx.$$

Proof. If $x \geq 0$, then the binary expansion theorem for real numbers gives

$$\begin{aligned}(1+x)^n &= 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n-1}x^{n-1} + x^n \\ &\geq 1 + nx\end{aligned}$$

as $x^2, x^3, \cdots, x^{n-1}$ and x^n are all non-negative. If $-1 < x < 0$, then the inequality is more delicate. But we can proceed in the following way:

$$\begin{aligned}&(1+x)^n - 1 = x\Big((1+x)^{n-1} + (1+x)^{n-1} + \cdots + (1+x) + 1\Big); \\ &n \geq (1+x)^{n-1} + (1+x)^{n-1} + \cdots + (1+x) + 1 \quad \text{as } 1 + x < 1.\end{aligned}$$

As $x < 0$, we get

$$nx \leq x\Big((1+x)^{n-1} + (1+x)^{n-1} + \cdots + (1+x) + 1\Big),$$

consequently the desired inequality:

$$(1+x)^n - 1 = x\Big((1+x)^{n-1} + (1+x)^{n-1} + \cdots + (1+x) + 1\Big) \geq nx.$$

♡

Page 25, Problem 11. *Suppose that $c < d$.*

a) *Prove that there is a $q \in \mathbb{Q}$ so that $|q - \sqrt{2}| < d - c$.*
b) *Prove that $q - \sqrt{2}$ is irrational.*
c) *Prove that there is an irrational number between c and d.*

Proof. a) Choose $a_1 = 1$ and $b_1 = 2$ and observe that

$$a_1^2 = 1 < 2 < 4 = b_1^2 \quad \text{consequently} \quad a_1 < \sqrt{2} < b_1.$$

Consider $(a_1 + b_1)/2$ and square it to get

$$\left(\frac{a_1 + b_1}{2}\right)^2 = \frac{9}{4} > 2$$

and put $a_2 = a_1$ and $b_2 = (a_1 + b_1)/2$. Suppose that $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ were chosen in such a way that

i) if

$$\left(\frac{a_{k-1} + b_{k-1}}{2}\right)^2 < 2,$$

then $a_k = (a_{k-1} + b_{k-1})/2$ and $b_k = b_{k-1}$;

ii) if

$$\left(\frac{a_{k-1} + b_{k-1}}{2}\right)^2 > 2,$$

then $a_k = a_{k-1}$ and $b_k = (a_{k-1} + b_{k-1})/2$.

Thus we obtain sequences $\{a_n\}$ and $\{b_k\}$ of rational numbers such that

$$a_1 \leq a_2 \leq a_3 \leq \cdots a_n \leq \cdots\cdots b_n \leq b_{n-1} \leq \cdots b_2 \leq b_1 \quad \text{and} \quad b_k - a_k = \frac{b_{k-1} - a_{k-1}}{2};$$
$$a_n^2 < 2 < b_n^2, \quad n \in \mathbb{N}, \quad \text{hence} \quad a_n < \sqrt{2} < b_n.$$

As $b_1 - a_1 = 1$, we have $b_n - a_n = 1/2^n$. For a large enough $n \in \mathbb{N}$ we have $1/2^n < d - c$. Now we conclude

$$0 < \sqrt{2} - a_n \leq b_n - a_n = \frac{1}{2^n} < d - c \quad \text{and} \quad a_n \in \mathbb{Q}.$$

Thus $q = a_n$ has the required property.

b) Set $p = \sqrt{2} - q$. If $p \in \mathbb{Q}$, then $\sqrt{2} = p + q \in \mathbb{Q}$ which is impossible. Therefore, p cannot be rational.

c) From (b), $p = \sqrt{2} - q$ is an irrational number and $0 < p < d - c$ from (a). Thus we get $c < c + p < d$. As seen in (b), $c + p$ cannot be rational. Because if $c + p = a$ is rational, then $p = a - c$ has to be rational which was just proven not to be the case. ♡

Problems.

1) *Negate the following statement on a function f on an interval $[a, b], a < b$. The function f has the property:*

$$f(x) \geq 0 \quad \textit{for every } x \in [a, b].$$

2) *Let*

$$f(x) = x^2 + bx + c, \quad x \in \mathbb{R}.$$

What can you say about the relation on the constants b and c in each of the following cases?

a)

$$f(x) \geq 0 \quad \textit{for every } x \in \mathbb{R}.$$

b)

$$f(x) \geq 0 \quad \textit{for som } x \in \mathbb{R}.$$

3) *Let*

$$f(x) = x^2 + 4x + 3, \quad x \in \mathbb{R}.$$

Which of the following statements on f is true? State the proof of your conclusion.

a)

$$f(x) < 0 \quad \textit{for some } x \in \mathbb{R};$$

b)

$$f(x) > 0 \quad \textit{for some } x \in \mathbb{R};$$

c)

$$f(x) \geq 0 \quad \textit{for every } x \in \mathbb{R}.$$

d)

$$f(x) < 0 \quad \textit{for every } x \in \mathbb{R}.$$

Solution. 1) There exists an $x_0 \in [a, b]$ such that

$$f(x_0) < 0.$$

2-a) First, we look at the function f closely:

$$\begin{aligned} f(x) &= x^2 + bx + c = x^2 + bx + \frac{b^2}{4} + c - \frac{b^2}{4} \\ &= \left(x + \frac{b}{2}\right)^2 + \frac{4c - b^2}{4} \geq \frac{4c - b^2}{4} \quad \text{for every } x \in \mathbb{R}. \end{aligned}$$

The function f assumes its smallest value

$$f\left(-\frac{b}{2}\right) = \frac{4c - b^2}{4}$$

at $x = -\frac{b}{2}$. Thus $f(x) \geq 0$ for every $x \in \mathbb{R}$ if and only if

$$f\left(-\frac{b}{2}\right) = \frac{4c - b^2}{4} \geq 0 \quad \text{if and only if} \quad 4c \geq b^2.$$

2-b) If $4c \geq b^2$, then $f(x) \geq 0$ for every $x \in \mathbb{R}$, in particular $f(0) \geq 0$. If $4c < b^2$, then we have

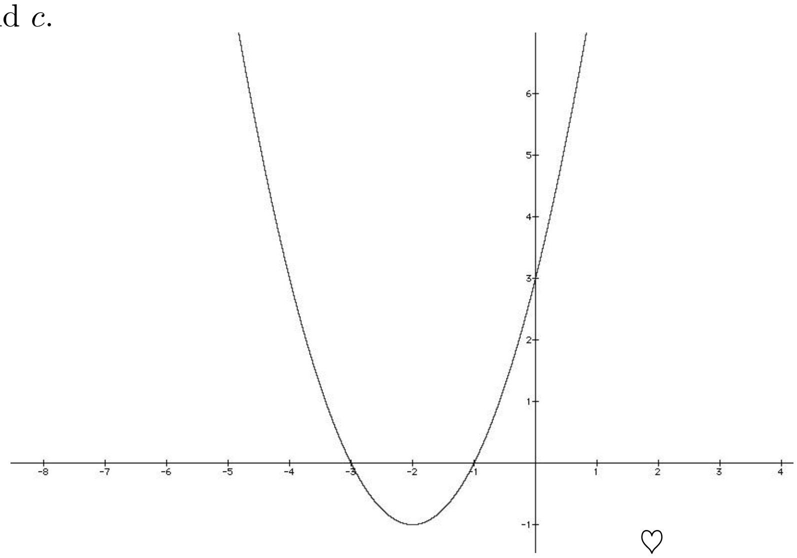
$$f\left(\frac{-b + \sqrt{b^2 - 4c}}{2}\right) = 0.$$

Therefore the condition that $f(x) \geq 0$ for some $x \in \mathbb{R}$ holds regardless of the values of b and c. So we have no relation between b and c.

3) First, we factor the polynomial f and draw the graph:

$$f(x) = x^2 + 4x + 3 = (x + 3)(x + 1).$$

We conclude that $f(x) \leq 0$ is equivalent to the condition that $-3 \leq x \leq -1$. Therefore we conclude that (a) and (b) are both true and that (c) and (d) are both false.



♡

Page 25, Problem 12. *Prove that the constant*

$$\mathrm{e} = \sum_{k=0}^{\infty} \frac{1}{k!}$$

is an irrational number.

Proof. Set

$$s_k = \sum_{j=0}^{k} \frac{1}{j!}, \quad k \in \mathbb{N}.$$

Clearly we have $2 = s_1 \le s_2 \le s_3 \le \cdots \le s_k \le \cdots$, i.e., the sequence $\{s_k\}$ is an increasing sequence. If $k \ge 3$, we have

$$\begin{aligned} s_k &= 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots + \frac{1}{1\cdot 2\cdot 3\cdot \cdots \cdot k} \\ &\le 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 2} + \cdots + \frac{1}{2^{k-1}} = 1 + \frac{1 - \frac{1}{2^k}}{1 - \frac{1}{2}} \\ &< 1 + 2 = 3. \end{aligned}$$

Thus the sequence $\{s_k\}$ is bounded. The Bounded Monotonce Convergence Axiom (MC) guarantees the convergence of $\{s_k\}$. Thus the limit e of $\{s_k\}$ exists and $\mathrm{e} \le 3$.

a) If $n > k$, then we have

$$\begin{aligned} s_n - s_k &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \frac{1}{(k+1)!} + \cdots + \frac{1}{n!} \\ &\quad - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!}\right) \\ &= \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \cdots + \frac{1}{n!} \\ &= \frac{1}{(k+1)!} \\ &\quad \times \left(1 + \frac{1}{k+2} + \frac{1}{(k+2)(k+3)} + \frac{1}{(k+2)(k+3)(k+4)} + \cdots + \frac{1}{(k+1)\cdots n}\right). \end{aligned}$$

Since

$$k+1 < k+2 < k+3 < \cdots < n, \quad \text{and} \quad \frac{1}{k+1} > \frac{1}{k+2}\frac{1}{k+3} > \cdots > \frac{1}{n},$$

we have

$$\frac{1}{(k+1)^2} > \frac{1}{(k+1)(k+2)}, \ \frac{1}{(k+1)^3} > \frac{1}{(k+1)(k+2)(k+3)}, \cdots$$
$$\frac{1}{(k+1)^{n-k-1}} > \frac{1}{(k+1)(k+2)\cdots n}$$

and

$$\begin{aligned} s_n - s_k &< \frac{1}{(k+1)!}\sum_{\ell=0}^{n-k-1}\frac{1}{(k+1)^\ell} = \frac{1}{(k+1)!}\frac{1 - \frac{1}{(k+1)^{n-k}}}{1 - \frac{1}{(k+1)}} \\ &< \frac{1}{k(k+1)!}. \end{aligned}$$

Taking the limit of the left hand side as $n \to \infty$, we get

$$\mathrm{e} - s_k \le \frac{1}{k(k+1)!} < \frac{1}{k \cdot k!}.$$

b) Now suppose that e is rational, i.e., $\mathrm{e} = p/q$ for some $p, q \in \mathbb{N}$. Then we must have

$$\frac{p}{q} - s_q \le \frac{1}{q \cdot q!} \quad \text{and} \quad 0 < q!\left(\frac{p}{q} - s_q\right) < \frac{1}{q}.$$

But now $q!(p/q)$ is an integer and $p!s_q$ is also because $2!$, $3!, \cdots$ and $q!$, appearing in the denominators of the summation of s_q, all divide $q!$, i.e.,

$$\begin{aligned} q!s_q &= q!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right) \\ &= q! + q! + (3 \cdot 4 \cdots q) + (4 \cdot 5 \cdots q) + \cdots + (k \cdot (k+1) \cdots q) + \cdots + q + 1. \end{aligned}$$

Thus the number $q!(p/q - s_q)$ is a positive integer smaller than $1/q$, which is not possible. This contradiction comes from the assumption that $\mathrm{e} = p/q, p, q \in \mathbb{N}$. Therefore we conclude that there is no pair of natural numbers p, q such that $\mathrm{e} = p/q$, i.e., e is an irrational number. ♡

Page 33, Problem 1. *Compute enough terms of the following sequences to guess what their limits are*:

a)

$$a_n = n \sin \frac{1}{n}.$$

b)

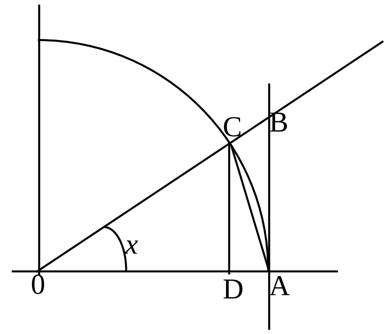
$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

c)

$$a_{n+1} = \frac{1}{2}a_n + 2, \quad a_1 = \frac{1}{2}.$$

d)

$$a_{n+1} = \frac{5}{2}a_n(1 - a_n), \quad a_1 = 0.3.$$



Answer. a) It is not easy to compute $\sin 1/2, \sin 1/3$ and so on. So let us take a closer look at the function $\sin x$ near $x = 0$:
Consider the circle of radius 1 with center 0, i.e., $\overline{0A} = 1$, and draw a line $\overline{0B}$ with angle $\angle A0B = x$. Let C be the intersection of the line $\overline{0B}$ and the circle. Draw a line $\overline{CD}$ through C and perpendicular to the line $\overline{0A}$ with D the intersection of the new line and the line $\overline{0A}$. So obtain the figure on the right. Now the length $\overline{CD} = \sin x$. We have the arc length $\widehat{AC} = x$ and the line length $\overline{AB} = \tan x$. To compare the sizes of $x, \sin x$ and $\tan x$, we consider the areas of the triangle $\triangle 0AC$, the piza pie cut shape $\triangleleft 0AC$ and $\triangle 0AB$ which are respectively $(\sin x)/2, x/2$ and $(\tan x)/2$. As these three figures are in the inclusion relations:

$$\triangle 0AC \subset \triangleleft 0AC \subset \triangle 0AB,$$

we have

$$\frac{\sin x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2} = \frac{1}{2}\frac{\sin x}{\cos x}$$

Consequently we conclude that

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

Therefore we have

$$\cos\left(\frac{1}{n}\right) \leq n \sin\left(\frac{1}{n}\right) \leq 1 \quad \text{and} \quad \lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1.$$

b) We simply compute a few terms:

$$a_1 = \left(1 + \frac{1}{1}\right)^1 = 2, a_2 = \left(1 + \frac{1}{2}\right)^2 = 1 + 1 + \frac{1}{4} = 2.25.$$

$$a_3 = \left(1 + \frac{1}{3}\right)^3 = 1 + 1 + \frac{3}{9} + \frac{1}{27} = 2\frac{10}{27},$$

$$a_4 = \left(1 + \frac{1}{4}\right)^4 = 1 + 1 + 6 \cdot \frac{1}{16} + 4 \cdot \frac{1}{64} + \frac{1}{256} = 2\frac{6 \cdot 16 + 4 \cdot 4 + 1}{256}$$

$$= 2\frac{96 + 16 + 1}{256} = 2\frac{103}{256},$$

$$a_5 = \left(1 + \frac{1}{5}\right)^5 = 1 + 1 + \binom{5}{2}\frac{1}{25} + \binom{5}{3}\frac{1}{125} + \binom{5}{4}\frac{1}{625} + \frac{1}{3125}$$

$$= 2\frac{10 \cdot 125 + 10 \cdot 25 + 5 \cdot 5 + 1}{3125} = 2\frac{1256}{3125}.$$

It is still hard to make the guess of the limit of $(1+1/n)^n$. So let us try something else.

$$\begin{aligned}
a_n = \left(1+\frac{1}{n}\right)^n &= 1+1+\binom{n}{2}\frac{1}{n^2}+\binom{n}{3}\frac{1}{n^3}+\cdots+\binom{n}{k}\frac{1}{n^k}+\cdots+\frac{1}{n^n}\\
&= 2+\frac{1}{2!}\frac{n(n-1)}{n^2}+\frac{1}{3!}\frac{n(n-1)(n-2)}{n^3}+\cdots+\frac{1}{k!}\frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k}\\
&\qquad+\cdots+\frac{1}{n!}\frac{n(n-1)(n-2)\cdots 2\cdot 1}{n^n}\\
&= 2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k}{n}\right)\\
&\qquad+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)\\
&< \sum_{k=0}^{n}\frac{1}{k!} = s_n.
\end{aligned}$$

$$\begin{aligned}
a_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} &= 2+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\\
&\qquad++\frac{1}{k!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k}{n+1}\right)+\cdots\\
&\qquad+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+}\right)\cdots\left(1-\frac{n}{n+1}\right)
\end{aligned}$$

As each term of a_{n+1} is greater than the corresponding term of a_n, we have $a_n \le a_{n+1}, n \in \mathbb{N}$, i.e., the sequence $\{a_n\}$ is increasing and bounded by e as $a_n \le s_n \le$ e. Therefore we conclude that the sequence $\{a_n\}$ converges and the limit is less than or equal to e.[1]

c) Skip.

d) Let us check a few terms:

$$\begin{aligned}
a_1 &= 0.3, \quad a_2 = \frac{5}{2}a_1(1-a_1) = 0.525\\
a_3 &= \frac{5}{2}a_2(1-a_2) = 0.6234375\\
a_4 &= 0.58690795898, a_5 = 0.60611751666, a_6 = 0.59684768164\\
a_7 &= 0.6015513164, a_8 = 0.59921832534, a_9 = 0.60038930979, a_{10} = 0.5998049662
\end{aligned}$$

With

$$f(x) = \frac{5}{2}x(1-x) = \frac{5}{2}(x-x^2) = \frac{5}{2}\left(\frac{1}{4}-\left(\frac{1}{2}-x\right)^2\right) \le \frac{5}{8} < 1$$

[1] In fact the limit of $\{a_n\}$ is the natural logarithm number e, which will be shown later.

we have

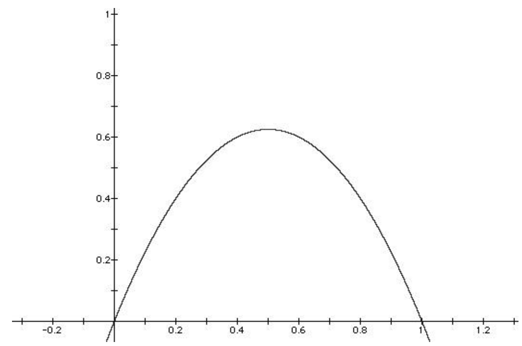
$$0 \le f(x) \le \frac{5}{8} = 0.625 \quad \text{for all} \quad x \in [0, 1],$$

and consequently

$$0 \le a_{n+1} = f(a_n) \le \frac{5}{8} = 0.625, \quad n \ge 3.$$

To compare a_n and $a_{n+1} = f(a_n)$, we consider

$$\begin{aligned} x - f(x) &= x - \frac{5}{2}x(1-x) \\ &= \frac{2x - 5(x - x^2)}{2} \\ &= \frac{5x^2 - 3x}{2} = \frac{x(5x-3)}{2} \\ &\begin{cases} \le 0 & \text{for all} \quad x \in [0, 0.6] \\ \ge 0 & \text{for all} \quad x \notin [0, 0.6] \end{cases}. \end{aligned}$$



This means that $a_{n+1} \le a_n$ if $0 \le a_n \le 0.6$ and $a_{n+1} \ge a_n$ if $a_n < 0$ or $a_n > 0.6$. But the case $a_n < 0$ has been excluded by the above arguments. From the computation of the first three terms we observe that the sequence $\{a_n\}$ seems to oscillate. At any rate, if the sequence $\{a_n\}$ converges, then we must have $a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, i.e., we must have $a = f(a)$, which narrows the candidate of the limit down to either 0 or $3/5 = 0.6$. Let us examine the candidate $3/5$ first. So we compute the error

$$\begin{aligned} \left|\frac{3}{5} - f(x)\right| &= \left|\frac{3}{5} - \frac{5}{2}x(1-x)\right| = \frac{|6 - 25x(1-x)|}{10} \\ &= \frac{|25x^2 - 25x + 6|}{10} = \frac{|(5x-2)(5x-3)|}{10} \\ &= \frac{1}{2}|5x - 2|\left|x - \frac{3}{5}\right| = \frac{1}{2}\left|5\left(x - \frac{3}{5}\right) + 1\right|\left|x - \frac{3}{5}\right| \end{aligned}$$

Therefore, if $|x - 3/5| = \delta$, then

$$\left|\frac{3}{5} - f(x)\right| \le \frac{1}{2}(5\delta + 1)\delta.$$

Thus if $\delta < 1/5$, then with $r = (5\delta + 1)/2 < 1$ we have

$$\left|\frac{3}{5} - f(x)\right| \le r\delta,$$

in other words

$$r^k \left|\frac{3}{5} - a_n\right| \geq r^{k-1}\left|\frac{3}{5} - a_{n+1}\right| \geq r^{k-2}\left|\frac{3}{5} - a_{n+2}\right|$$
$$\geq r^{k-3}\left|\frac{3}{5} - a_{n+3}\right| \geq \cdots \geq \left|\frac{3}{5} - a_{n+k}\right|.$$

Therefore, if we get $|3/5 - a_n| < 1/5$ for some $n \in \mathbb{N}$, then we have

$$\lim_{k\to\infty} a_k = \frac{3}{5}.$$

But we know

$$\left|\frac{3}{5} - a_3\right| = 0.6234375 - 0.6 = 0.0234375 < 0.2 = \frac{1}{5}.$$

Therefore the limit of the sequence $\{a_n\}$ is 0.6 as seen in the first computation. ♡

Page 33, Problem 2. *Prove directly that each of the following sequences converges by letting $\varepsilon > 0$ be given and finding $N(\varepsilon)$ so that*

$$|a - a_n| < \varepsilon \qquad \textit{for every } n \geq N(\varepsilon). \tag{1}$$

a)

$$a_n = 1 + \frac{10}{\sqrt[2]{n}}.$$

b)

$$a_n = 1 + \frac{1}{\sqrt[3]{n}}.$$

c)

$$a_n = 3 + 2^{-n}.$$

d)

$$a_n = \sqrt{\frac{n}{n+1}}.$$

Solution. a) Obviously our guess on the limit a is $a = 1$. So let us try with $a = 1$ to find $N(\varepsilon)$ which satisfy the condition (1):

$$|1 - a_n| = \frac{10}{\sqrt[2]{n}} < \varepsilon \quad \text{for every } n \geq N(\varepsilon),$$

which is equivalent to the inequality:

$$\sqrt{n} > \frac{10}{\varepsilon} \quad \Leftrightarrow n > \frac{100}{\varepsilon^2}$$

for every $n \geq N(\varepsilon)$. Thus if we choose $N(\varepsilon)$ to be

$$N(\varepsilon) = \left[\frac{100}{\varepsilon^2}\right] + 1,$$

where $[x], x \in \mathbb{R}$, means the largest integer which is less than or equal to x, i.e., the integer m such that $m \leq x < m + 1$, then for every $n \geq N(\varepsilon)$, we have

$$\frac{100}{\varepsilon^2} < N(\varepsilon) \leq n, \quad \text{hence} \quad \varepsilon^2 > \frac{100}{n} \quad \text{and} \quad \varepsilon > \frac{10}{\sqrt[2]{n}} = |1 - a_n|.$$

This shows that

$$\lim_{n\to\infty}\left(1 + \frac{10}{\sqrt[2]{n}}\right) = 1.$$

b) It is also easy to guess that the limit a of $\{a_n\}$ is 1. So let $\varepsilon > 0$ and try to find $N(\varepsilon)$ which satisfy the condition (1) above which is:

$$\frac{1}{\sqrt[3]{n}} = |1 - a_n| < \varepsilon \quad \text{for every} \quad n \geq N(\varepsilon).$$

So we look for the smallest integer N which satisfy

$$\frac{1}{\sqrt[3]{n}} < \varepsilon \quad \text{equivalently} \quad \frac{1}{\varepsilon} < \sqrt[3]{n},$$

which is also equivalent to

$$n > \frac{1}{\varepsilon^3}.$$

So with $N(\varepsilon) = \left[1/\varepsilon^3\right] + 1$, if $n \geq N(\varepsilon)$, then

$$\frac{1}{\varepsilon^3} < N(\varepsilon) \leq n \quad \text{consequently} \quad \frac{1}{\sqrt[3]{n}} < \varepsilon.$$

d) First, we make a small change in the form of a_n:

$$a_n = \sqrt{\frac{n}{n+1}} = \sqrt{\frac{1}{1 + \frac{1}{n}}},$$

and guess that the limit a of $\{a_n\}$ would be 1. So we compute:

$$\left|1 - \sqrt{\frac{n}{n+1}}\right| = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1}} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1}(\sqrt{n+1} + \sqrt{n})}$$
$$= \frac{n+1-n}{n+1+\sqrt{(n+1)n}} \leq \frac{1}{n}.$$

Hence if $n \geq [1/\varepsilon] + 1$, then $1/n \leq 1/([1/\varepsilon] + 1) < 1/(1/\varepsilon) = \varepsilon$, i.e.,

$$\left|1 - \sqrt{\frac{n}{n+1}}\right| < \varepsilon \quad \text{for every} \quad n \geq N(\varepsilon).$$

♡

Page 33, Problem 3. *Prove directly that each of the following sequences converges by letting $\varepsilon > 0$ be given and finding $N(\varepsilon)$ so that*

$$|a - a_n| < \varepsilon \qquad \text{for every } n \geq N(\varepsilon). \tag{1}$$

a)
$$a_n = 5 - \frac{2}{\ln n} \quad \text{for} \quad n \geq 2.$$

b)
$$a_n = \frac{3n+1}{n+2}.$$

c)
$$a_n = \frac{n^2+6}{2n^2-2} \quad \text{for} \quad n \geq 2.$$

d)
$$a_n = \frac{2^n}{n!}.$$

Solution. a) From the form of the sequence, we guess that the limit a would be 5. So we try 5 as a:

$$|5 - a_n| = \left|5 - \left(5 + \frac{2}{\ln n}\right)\right| = \frac{2}{\ln n},$$

which we want to make smaller than a given $\varepsilon > 0$. So we want find how large n ought to be in order to satisfy the inequlity:

$$\varepsilon > \frac{2}{\ln 2}.$$

This inequality is equivalent to $\ln n > 2/\varepsilon$. Taking the exponential of the both sides, we must have $n > \exp(2/\varepsilon)$. So if we take

$$N(\varepsilon) = \left[\exp\left(\frac{2}{\varepsilon}\right)\right] + 1,$$

then for every $n \geq N(\varepsilon)$ the inequality (1) holds.

b) First we change the form of each term slightly:

$$a_n = \frac{3n+1}{n+2} = \frac{3+\frac{1}{n}}{1+\frac{2}{n}},$$

to make a guess on a. This indicates that the limit a would be 3. So we try to fulfil the requirement of (1) with $a = 3$:

$$\left|3 - \frac{3n+1}{n+2}\right| = \left|\frac{3(n+2)-(3n+1)}{n+2}\right| = \frac{5}{n+2} < \frac{5}{n}.$$

So if the inequality $5/n < \varepsilon$ holds, then $|3 - a_n| < \varepsilon$ holds. Thus $N(\varepsilon) = [5/\varepsilon] + 1$ gives that $|5 - a_n| < \varepsilon$ for every $n \geq N(\varepsilon)$.

c) We alter the form of the sequence slightly:

$$a_n = \frac{n^2+6}{2n^2-2} = \frac{1+\frac{6}{n^2}}{2-\frac{2}{n^2}},$$

in order to make a good guess on the limit a, which looks like $1/2$. Let us try with this a:

$$\left|\frac{1}{2} - \frac{n^2+6}{2n^2-2}\right| = \left|\frac{(n^2-1)-(n^2+6)}{2n^2-2}\right| = \frac{7}{2n^2-2}, \quad \text{for} \quad n \geq 2.$$

If $n \geq 2$, then $(2n^2-2) - 2(n-1)^2 = 4n > 0$, so that $2(n-1)^2 < 2n^2 - 2$ and therefore

$$\left|\frac{1}{2} - \frac{n^2+6}{2n^2-2}\right| < \frac{7}{2(n-1)^2}.$$

Thus if $N(\varepsilon) = \left[\sqrt{7/(2\varepsilon)}\right] + 2$, then for every $n \geq N(\varepsilon)$ we have

$$\left|\frac{1}{2} - \frac{n^2+6}{2n^2-2}\right| < \frac{7}{2(n-1)^2} \leq \frac{7}{2(N(\varepsilon)-1)^2} = \frac{7}{2\left(\left[\sqrt{\frac{7}{2\varepsilon}}\right]+1\right)^2}$$
$$< \frac{7}{2\cdot\frac{7}{2\varepsilon}} = \varepsilon.$$

Therefore
$$\lim_{n\to\infty} a_n = \frac{1}{2}.$$

d) With
$$a_n = 2^n/n!$$
we look at the ratio a_n/a_{n+1}:
$$\frac{a_n}{a_{n+1}} = \frac{2^n}{n!}\cdot\frac{(n+1)!}{2^{n+1}} = \frac{n+1}{2} \geq 2 \quad \text{for} \quad n \geq 3.$$
Therefore, we have for every $k \geq 2$
$$a_3 \geq 2^k a_{3+k} \quad \text{equivalently} \quad a_{k+3} \leq \frac{a_3}{2^k} = \frac{8}{6\cdot 2^k} = \frac{1}{3\cdot 2^{k-2}} < \frac{1}{2^{k-1}} < \frac{1}{k-1}.$$
So for any $\varepsilon > 0$ if $n \geq N(\varepsilon) = \left[\frac{1}{\varepsilon}\right] + 5$, then we have $0 < a_n < \varepsilon$. ♡

Page 24, Problem 6. *Suppose that $a_n \to a$ and let b be any number strictly less than a. Prove that $a_n > b$ for all but finitely many n.*

Proof. The assumption $a > b$ yields $b - a > 0$ so that there exists $N \in \mathbb{N}$ such that $|a - a_n| < b - a$ for every $n \geq N$, equivalently
$$b - a < a - a_n < a - b \quad \text{for every} \quad n \geq N, \qquad \text{hence} \quad b < a_n \quad \text{for every} \quad n \geq N.$$
Thus the total number of n with $b \geq a_n$ is at most $N - 1$ which is of course finite. Thus $a_n > b$ for all but finitely many n. ♡

Page 34, Problem 9. a) *Find a sequence $\{a_n\}$ and a real number a so that*
$$|a_{n+1} - a| < |a_n - a| \quad \textit{for each} \quad n,$$
but $\{a_n\}$ does not converge to a.

b) *Find a sequence $\{a_n\}$ and a real number a so that $a_n \to a$ but so that the above inequality is violated for infinitely many n.*

Answer. a) Take $a_n = 1/n$ and $a = -1$. Then
$$|a_{n+1} - a| = \frac{1}{n+1} + 1 < \frac{1}{n} + 1 = |a_n - a| \quad \text{but} \quad a_n \not\to a.$$

b) Set
$$a_n = \frac{1}{n}\left(1 + \frac{(-1)^n}{2}\right) \quad \text{and} \quad a = 0.$$

Then we have

$$a_n = \begin{cases} \frac{1}{2n} & \text{for odd } n; \\ \frac{3}{2n} & \text{for even } n \end{cases} \quad \text{and} \quad a_n \to 0.$$

If n is odd, then

$$a_{n+1} = \frac{3}{2(n+1)} > \frac{1}{2n} = a_n.$$

This occurs infinitely many times, i.e., at every odd n. ♡

Page 39, Problem 1. *Prove that each of the following limits exists:*

a)
$$a_n = 5\left(1 + \frac{1}{\sqrt[3]{n}}\right)^2.$$

b)
$$a_n = \frac{3n+1}{n+2}.$$

c)
$$a_n = \frac{n^2+6}{3n^2-2}.$$

d)
$$a_n = \frac{5 + \left(\frac{2}{3^n}\right)^2}{2 + \frac{2n+5}{3n-2}}$$

Proof. a) First $\lim_{n\to\infty} 1/\sqrt[3]{n} = 0$ because for any given $\varepsilon > 0$ if $n \geq N(\varepsilon) = [1/\varepsilon^3] + 1$ then

$$\frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N(\varepsilon)}} = \frac{1}{\sqrt[3]{\left[\frac{1}{\varepsilon^3}\right] + 1}} < \frac{1}{\sqrt[3]{\frac{1}{\varepsilon^3}}} = \frac{1}{\left(\frac{1}{\varepsilon}\right)} = \varepsilon.$$

Hence we get

$$\lim_{n\to\infty} 5\left(1 + \frac{1}{\sqrt[3]{n}}\right)^2 = 5$$

by the combination of Theorem 2.2.3, Theorem 2.2.4 and Theorem 2.2.5 as seen below:

$$1 + \frac{1}{\sqrt[3]{n}} \to 1 \quad \text{by Theorem 2.2.3} \Rightarrow \left(1 + \frac{1}{\sqrt[3]{n}}\right)^2 \to 1 \quad \text{by Theorem 2.2.5}$$
$$\Downarrow$$
$$5\left(1 + \frac{1}{\sqrt[3]{n}}\right)^2 \to 5 \quad \text{by Theorem 2.2.4.}$$

b) We change the form of each term a_n slightly:

$$a_n = a_n = \frac{3n+1}{n+2} = \frac{3+\frac{1}{n}}{2+\frac{2}{n}}.$$

We know that $1/n \to 0$ and $2/n \to 0$ as $n \to \infty$. Thus we get the following chain of deduction:

$$3+\frac{1}{n} \to 3 \quad \text{and} \quad 2+\frac{2}{n} \to 2 \quad \text{by Theorem 2.2.3}$$
$$\Downarrow$$
$$\frac{3+\frac{1}{n}}{2+\frac{2}{n}} \to \frac{3}{2} \quad \text{by Theorem 2.26.}$$

c) We change the form of each term a_n in the following way:

$$a_n = \frac{n^2+6}{3n^2-2} = \frac{1+\frac{6}{n^2}}{3-\frac{2}{n^2}}.$$

As $1/n \to 0$, Theorem 2.2.5 yields that $1/n^2 \to 0$ and therefore

$$\frac{1+\frac{6}{n^2}}{3-\frac{2}{n^2}} \to \frac{1+6\cdot 0}{3-2\cdot 0} = \frac{1}{3} \quad \text{by Theorem 2.2.4 and Theorem 2.2.6.}$$

d) As seen before, we have

$$\frac{1}{3^n} \leq \frac{1}{n} \to 0 \quad \text{and} \quad \frac{2n+5}{3n-2} = \frac{2+\frac{5}{n}}{3-\frac{2}{n}} \to \frac{2}{3}.$$

Thus we get

$$a_n = \frac{5+\left(\frac{2}{3^n}\right)^2}{2+\frac{2n+5}{3n-2}} \to \frac{5+4\cdot 0\cdot 0}{2+\frac{2}{3}} = \frac{15}{8}$$

by a combination of Theorem 2.2.3, Theorem 2.2.4 and Theorem 2.2.6. ♡

Page 39, Problem 6. *Let $p(x)$ be any polynomial and suppose that $a_n \to a$. Prove that*

$$\lim_{n\to\infty} p(a_n) = p(a).$$

Proof. Suppose that the polynomial $p(x)$ has the form:

$$p(x) = p_k x^k + p_{k-1}x^{k-1} + \cdots + p_1 x + p_0.$$

We claim that $a_n^\ell \to a^\ell$ for each $\ell \in \mathbb{N}$. If $\ell = 1$, then certainly we have the convergence: $a_n^1 = a_n \to a = a^1$. Suppose $a_n^{\ell-1} \to a^{\ell-1}$. Then by Theorem 2.2.5 we have $a_n^\ell = a_n^{\ell-1}a_n \to a^{\ell-1}a = a^\ell$. By mathematical induction we have $a_n^\ell \to a^\ell$ for each $\ell \in \mathbb{N}$. Therefore, each term $p_\ell a_n^\ell$ converges to $p_\ell a^\ell$ for $\ell = 1, 2, \cdots, k$. A repeated use of Theorem 2.2.3 yields that

$$p(a_n) = p_k a_n^k + p_{k-1}a_n^{k-1} + \cdots + p_1 a_n + p_0 \to p_k a^k + p_{k-1}a^{k-1} + \cdots + p_1 a + p_0 = p(a).$$

♡

Page 39, Problem 7. *Let $\{a_n\}$ and $\{b_n\}$ be sequences and suppose that $a_n \le b_n$ for all n and that $a_n \to \infty$. Prove that $b_n \to \infty$.*

Proof. The divergence $a_n \to \infty$ means that for every M there exists $N \in \mathbb{N}$ such that $a_n > M$ for every $n \ge N$. The assumption that $a_n \le b_n$ gives $M < a_n \le b_n$ for every $n \ge N$. Hence $b_n \to \infty$. ♡

Page 39, Problem 9. a) *Let $\{a_n\}$ be the sequence given by*

$$a_{n+1} = \frac{1}{2}a_n + 2, \quad a_1 = 0.5$$

Prove that $a_n \to 4$.

b) *Consider the sequence defined by*

$$a_{n+1} = \alpha a_n + 2.$$

Show that if $|\alpha| < 1$, then the sequence has a limit independent of a_1.

Proof. a) Based on the hint, we compute

$$a_{n+1} - 4 = \frac{1}{2}a_n + 2 - 4 = \frac{1}{2}a_n - 2 = \frac{1}{2}(a_n - 4).$$

Hence we get

$$|a_n - 4| = \frac{1}{2}|a_{n-1} - 4| = \frac{1}{2^2}|a_{n-2} - 4| = \cdots = \frac{1}{2^{n-1}}|a_1 - 4| = \frac{3.5}{2^{n-1}} \to 0.$$

b) We just compute

$$\begin{aligned} a_{n+1} - \frac{2}{1-\alpha} &= \alpha a_n + 2 - \frac{2}{1-\alpha} = \alpha a_n + \frac{2(1-\alpha) - 2}{1-\alpha} \\ &= \alpha a_n - \frac{2\alpha}{1-\alpha} = \alpha\left(a_n - \frac{2}{1-\alpha}\right); \\ a_n - \frac{2}{1-\alpha} &= \alpha\left(a_{n-1} - \frac{2}{1-\alpha}\right) = \alpha^2\left(a_{n-2} - \frac{2}{1-\alpha}\right) = \cdots \\ &= \alpha^{n-1}\left(a_1 - \frac{2}{1-\alpha}\right) \longrightarrow 0 \quad \text{as } |\alpha| < 1. \end{aligned}$$

Hence $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = \frac{2}{1-\alpha}$$

which is independent of a_1. ♡

Page 40, Problem 10. *For a pair (x, y) of real numbers, define*

$$\|(x,y)\| = \sqrt{x^2+y^2}.$$

a) *Let x_1, x_2, y_1, y_2 be real numbers. Prove that*

$$|x_1x_2 + y_1y_2| \le \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}.$$

b) *Prove that for any two dimensional vectors $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$*

$$\|(x_1,y_1)+(x_2,y_2)\| \le \|(x_1,y_1)\| + \|(x_2,y_2)\|.$$

c) *Let $p_n = (x_n, y_n)$ be a sequence of points in the plane $\mathbb{R}^2$ and let $p = (x, y)$. We say that $p_n \to p$ if $\|p_n - p\| \to 0$. Prove that $p_n \to p$ if and only if $x_n \to x$ and $y_n \to y$.*

Proof. a) Let us compute:

$$\begin{aligned}
&(x_1^2+x_2^2)(y_1^2+y_2^2) - (x_1y_1+x_2y_2)^2 \\
&\qquad = x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2 - (x_1^2y_1^2 + 2x_1y_1x_2y_2 + x_2^2y_2^2) \\
&\qquad = x_1^2y_2^2 + x_2^2y_1^2 - 2x_1y_1x_2y_2 = (x_1y_2 - x_2y_1)^2 \ge 0.
\end{aligned}$$

b) We also compute directly:

$$\begin{aligned}
\|(x_1,y_1)+(x_2,y_2)\|^2 &= \|(x_1+x_2, y_1+y_2)\|^2 = (x_1+x_2)^2 + (y_1+y_2)^2 \\
&= x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2 = x_1^2 + x_2^2 + 2(x_1x_2+y_1y_2) + y_1^2 + y_2^2 \\
&\le x_1^2 + y_1^2 + 2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2} + x_2^2 + y_2^2 \\
&= \left(\sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}\right)^2 = \Big(\|(x_1,y_1)\| + \|(x_2,y_2)\|\Big)^2.
\end{aligned}$$

This shows the inequality:

$$\|(x_1,y_1)+(x_2,y_2)\| \le \|(x_1,y_1)\| + \|(x_2,y_2)\|.$$

c) Since we have the inequalities:

$$\max\{|x_n - x|, |y_n - y|\} \le \sqrt{(x_n-x)^2 + (y_n-y)^2} \le 2\max\{|x_n-x|, |y_n-y|\},$$

show that $\|p_n - p\| \to 0$ if and only if $\max\{|x_n - x|, |y_n - y|\} \to 0$ if and only if $|x_n - x| \to 0$ and $|y_n - y| \to 0$. ♡

Page 50, Problem 1. *Prove directly that $a_n = 1 + \frac{1}{\sqrt{n}}$ is a Cauchy sequence.*

Proof. We just compute for $m < n$:

$$|a_m - a_n| = \left|\left(1 + \frac{1}{\sqrt{m}}\right) - \left(1 + \frac{1}{\sqrt{n}}\right)\right| = \left|\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{n}}\right|$$
$$\leq \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}} \leq \frac{2}{\sqrt{m}} \quad \text{since} \quad m < n.$$

So if $\varepsilon > 0$ is given, then we take N to be $[(2/\varepsilon)^2] + 1$ so that for every $n > m \geq N$ we have

$$|a_m - a_n| < \frac{2}{\sqrt{m}} \leq \frac{2}{\sqrt{\left[\left(\frac{2}{\varepsilon}\right)^2\right] + 1}} < \frac{2}{\sqrt{\left(\frac{2}{\varepsilon}\right)^2}} = \varepsilon.$$

♡

Page 50, Problem 2. *Prove that the rational numbers are dense in the real numbers.*

Proof. We have to prove that for every $\varepsilon > 0$ $(a - \varepsilon, a + \varepsilon) \cap \mathbb{Q} \neq \emptyset$. Choose $m = [1/\varepsilon] + 1$ so that $1/m < \varepsilon$. If $a - \varepsilon > 0$, then the Archimedian property of $\mathbb{R}$ yields the existence of $k \in \mathbb{N}$ such that

$$\frac{k}{m} = k \cdot \frac{1}{m} > a - \varepsilon.$$

Let n be the first such a number. Then we have

$$\frac{n-1}{m} \leq a - \varepsilon < \frac{n}{m} = \frac{n-1}{m} + \frac{1}{m} \leq a - \varepsilon + \frac{1}{m} < a - \varepsilon + \varepsilon = a.$$

Therefore, we conclude that

$$a - \varepsilon < \frac{n}{m} < a \quad \text{therefore} \quad \frac{n}{m} \in (a - \varepsilon, a + \varepsilon) \cap \mathbb{Q}.$$

If $a - \varepsilon < 0$, then we apply the Archimedian property of $\mathbb{R}$ to the pair $1/m$ and $\varepsilon - a > 0$ to find a natural number $k \in \mathbb{N}$ such that

$$\varepsilon - a < k \cdot \frac{1}{m} = \frac{k}{m}.$$

Let $n \in \mathbb{N}$ be the smallest natural number such that $n/m \geq \varepsilon - a$, so that

$$\frac{n-1}{m} < \varepsilon - a \leq \frac{n}{m}, \quad \text{equivalently} \quad -\frac{n}{m} \leq a - \varepsilon < \frac{1-n}{m} = -\frac{n}{m} + \frac{1}{m}.$$

As we have chosen $m \in \mathbb{N}$ so large that $1/m < \varepsilon$, the above inequality yields

$$\frac{1-n}{m} = \frac{1}{m} - \frac{n}{m} < \varepsilon - \frac{n}{m} \leq \varepsilon + (a - \varepsilon) = a, \quad \text{i.e.,} \quad a - \varepsilon < \frac{1-n}{m} < a.$$

Therefore we have

$$\frac{1-n}{m} \in (a - \varepsilon, a + \varepsilon) \cap \mathbb{Q} \quad \text{consequently} \quad (a - \varepsilon, a + \varepsilon) \cap \mathbb{Q} \neq \emptyset$$

for arbitrary $\varepsilon > 0$. Hence a is a limit point of $\mathbb{Q}$.

Page 59, Problem 3. *Suppose that the sequence $\{a_n\}$ converges to a and d is a limit point of the sequence $\{b_n\}$. Prove that ad is a limit point of the sequence $\{a_n b_n\}$.*

Proof. By the assumption on the sequence $\{b_n\}$, there exists a subsequence $\{b_{n_k}\}$ of the sequence $\{b_n\}$ such that

$$\lim_{k\to\infty} b_{n_k} = d.$$

The subsequence $\{a_{n_k} b_{n_k}\}$ of the sequence $\{a_n b_n\}$ converges to ad because the subsequence $\{a_{n_k}\}$ of $\{a_n\}$ converges to the same limit a. Hence ad is a limit point of $\{a_n b_n\}$. ♡

Page 59, Problem 6. *Consider the following sequence: $a_1 = \frac{1}{2}$; the next three terms are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$; the next seven terms are $\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}; \cdots$ and so forth. What are the limit points.*

Answer. The sequence $\{a_n\}$ consists of the numbers $\{k/2^n : k = 1, 2, \cdots, 2^n - 1, n \in \mathbb{N}\}$. Fix $x \in [0, 1]$. We are going to construct a subsequence $\{b_n\}$ of the sequence $\{a_n\}$ by induction. For $n = 1$, choose

$$k_1 = \begin{cases} 0 & \text{if} \quad x \le \frac{1}{2}; \\ 1 & \text{if} \quad \frac{1}{2} < x \le 1. \end{cases}$$

For each $n > 1$, let k_n be the natural number such that

$$\frac{k_n}{2^n} \le x < \frac{k_n + 1}{2^n}.$$

Then the ratio $b_n = k_n/2^n$ is in the sequence $\{a_n\}$ and

$$|b_n - x| < \frac{1}{2^n} \to 0 \quad \text{as } n \to \infty.$$

Hence $\lim_{n\to\infty} b_n = x$. Therefore, every $x \in [0, 1]$ is a limit point of $\{a_n\}$. Thus the sequence $\{a_n\}$ is dense in the closed unit interval $[0, 1]$. ♡

Page 59, Problem 8. *Let $\{I_k : k \in \mathbb{N}\}$ be a nested family of closed, finite intervals; that is, $I_1 \supset I_2 \supset \cdots$. Prove that there is a point p contained in all the intervals, that is $p \in \cap_{k=1}^{\infty} I_k$.*

Proof. The assumption means that if $I_k = [a_k, b_k], k \in \mathbb{N}$, then

$$a_1 \le a_2 \le \cdots \le a_k \le \cdots\cdots \le b_k \le b_{k-1} \le \cdots \le b_2 \le b_1.$$

The sequence $\{a_k\}$ is increasing and bounded by any of $\{b_\ell\}$. Fix $k \in \mathbb{N}$. Then we have

$$a = \lim_{n\to\infty} a_n \le b_k \quad \text{for} \quad k \in \mathbb{N},$$

where the convergence of $\{a_n\}$ is guaranteed by the boundedness of the sequence. Now look at the sequence $\{b_k\}$ whis is decreasing and bounded below by a. Hence it converges to $b \in \mathbb{R}$ and $a \le b$. Thus the situation is like the following:

$$a_1 \le a_2 \le \cdots \le a_k \le \cdots a \le b \le \cdots \le b_k \le b_{k-1} \le \cdots \le b_2 \le b_1.$$

Hence the interval $[a,b]$ is contained in the intersection $\cap_{k=1}^{\infty} I_k$. Any point p in the interval $[a,b]$ is a point of $\cap_{k=1}^{\infty} I_k$; in fact $[a,b] = \cap_{k=1}^{\infty} I_k$. ♡

Page 59, Problem 9. *Suppose that $\{x_n\}$ is a monotone increasing sequence of points in $\mathbb{R}$ and suppose that a subsequence of $\{x_n\}$ converges to a finite limit. Prove that $\{x_n\}$ converges to a finite limit.*

Proof. Let $\{x_{n_k}\}$ be the subsequence converging to the finite limit x_0. As $n_1 < n_2 < \cdots < n_k < \cdots$, we have $k \le n_k$ for every $k \in \mathbb{N}$. If $\varepsilon > 0$ is given, then choose K so large that $|x_{n_k} - x_0| < \varepsilon$ for every $k \ge K$, i.e., $x_0 - \varepsilon < x_{n_k} \le x_0$ for every $k \ge K$. Set $N = n_K$. Then if $m \ge N$, then we have

$$x_0 - \varepsilon < x_{n_K} = x_N \le x_m \le x_{n_m} \le x_0.$$

Hence we have $0 \le x_0 - x_m < \varepsilon$ for every $m \ge N$. Hence $\{x_n\}$ converges to the same limit x_0. ♡

Page 79, Problem 3. *Let $f(x)$ be a continuous function. Prove that $|f(x)|$ is a continuous function.*

Proof. Let $x \in [a,b]$ be a point in the domain $[a,b]$ of the function f. If $\varepsilon > 0$ is given, then choose a $\delta > 0$ so small that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. If $|x - y| < \delta$, then

$$\big||f(x)| - |f(y)|\big| \le |f(x) - f(y)| < \varepsilon.$$

Hence $|f|$ is continuous at x. ♡

Page 79, Problem 5. *Suppose that f is a continuous function on $\mathbb{R}$ such that $f(q) = 0$ for every $q \in \mathbb{Q}$. Prove that $f(x) = 0$ for every $x \in \mathbb{R}$.*

Proof. Choose $x \in \mathbb{R}$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Take $q \in \mathbb{Q} \cap (x - \delta, x + \delta)$, then

$$|f(x)| = |f(x) - f(q)| < \varepsilon.$$

Thus $|f(x)|$ is less than any $\varepsilon > 0$ which is possible only when $|f(x)| = 0$. ♡

Page 79, Problem 7. *Let $f(x) = 3x - 1$ and let $\varepsilon > 0$ be given. How small δ be chosen so that $|x - 1| \le \varepsilon$ implies $|f(x) - 2| < \varepsilon$?*

Answer. To determine the magnitude of δ, assume that $|x - 1| < \delta$ and see how the error becomes:

$$|f(x) - 2| = |3x - 1 - 2| = |3x - 3| = 3|x - 1| < 3\delta.$$

Thus if $3\delta \le \varepsilon$, i.e., if $\delta \le \varepsilon/3$, then $|x - 1| < \delta$ implies $|f(x) - 2| < \varepsilon$. ♡

Page 79, Problem 8. *Let $f(x) = x^2$ and let $\varepsilon > 0$ be given.*

a) *Find a δ so that $|x - 1| \le \delta$ implies $|f(x) - 1| \le \varepsilon$.*
b) *Find a δ so that $|x - 2| \le \delta$ implies $|f(x) - 2| \le \varepsilon$.*
c) *If $n > 2$ and you had to find a δ so that $|x - n| \le \delta$ implies $|f(x) - n^2| \le \varepsilon$, would the δ be larger or smaller than the δ for parts* (a) *and* (b)*? Why?*

Answer. a) Choose $\delta > 0$ and see how the error grows from $|x - 1| \le \delta$:

$$\begin{aligned}|f(x) - 1| &= |x^2 - 1| = |(x + 1)(x - 1)| = |x + 1||x - 1|\\ &\le |x + 1|\delta = |x - 1 + 1 + 1|\delta \le (|x - 1| + 2)\delta\\ &\le (\delta + 2)\delta.\end{aligned}$$

So we want to make $(\delta + 2)\delta \le \varepsilon$. Let us solve this inequality:

$$0 \ge \delta^2 + 2\delta - \varepsilon = (\delta + 1)^2 - \varepsilon - 1 \quad \Leftrightarrow \quad \varepsilon + 1 \ge (\delta + 1)^2$$
$$\Leftrightarrow$$
$$-\sqrt{\varepsilon + 1} - 1 \le \delta \le \sqrt{\varepsilon + 1} - 1.$$

But we know that δ must be positive. Hence $0 < \delta \le \sqrt{\varepsilon + 1} - 1 = \varepsilon/(\sqrt{\varepsilon + 1} + 1)$. If δ is chosen in the interval $(0, \sqrt{\varepsilon + 1} - 1)$, then the above calculation shows that

$$|x - 1| \le \delta \quad \Rightarrow \quad |f(x) - 1| \le \varepsilon.$$

b) Now we continue to examine the case $|x - 2| \le \delta$:

$$\begin{aligned}|f(x) - 4| &= |x^2 - 4| = |(x + 2)(x - 2)| = |x + 2||x - 2|\\ &\le \delta|x + 2| = \delta|x - 2 + 4| \le \delta(|x - 2| + 4)\\ &\le \delta(\delta + 4).\end{aligned}$$

So we want to make $(\delta + 4)\delta \le \varepsilon$, equivalently:

$$\varepsilon \ge \delta(\delta + 4) = \delta^2 + 4\delta \quad \Leftrightarrow \quad \delta^2 + 4\delta - \varepsilon \le 0$$
$$\Leftrightarrow$$
$$-\sqrt{\varepsilon + 4} - 2 \le \delta \le \sqrt{\varepsilon + 4} - 2 = \frac{(\sqrt{\varepsilon + 4} - 2)(\sqrt{\varepsilon + 4} + 2)}{\sqrt{\varepsilon + 4} + 2} = \frac{\varepsilon}{\sqrt{\varepsilon + 4} + 2}.$$

Hence if we take $0 < \delta \le \varepsilon/(\sqrt{\varepsilon+4}+2)$, then

$$|x-2| < \delta \quad \Rightarrow \quad |f(x)-4| < \varepsilon.$$

c) Similarly, we examine the case $|x-n| \le \delta$:

$$\begin{aligned}|f(x)-n^2| &= |x^2-n^2| = |x+n||x-n| \le \delta|x+n| \\ &\le \delta|x-n+2n| \le \delta(|x-n|+2n) \le \delta(\delta+2n).\end{aligned}$$

So we want to make $(\delta+n)\delta \le \varepsilon$, equivalently:

$$\varepsilon \ge \delta(\delta+4) = \delta^2 + 2n\delta \quad \Leftrightarrow \quad \delta^2+2n\delta-\varepsilon \le 0$$
$$\Leftrightarrow$$
$$-\sqrt{\varepsilon+n^2}-n \le \delta \le \sqrt{\varepsilon+n^2}-n = \frac{(\sqrt{\varepsilon+n^2}-n)(\sqrt{\varepsilon+n^2}+n)}{\sqrt{\varepsilon+n^2}+n} = \frac{\varepsilon}{\sqrt{\varepsilon+n^2}+n}.$$

Hence if we take $\delta > 0$ so small that $0 < \delta \le \varepsilon/(\sqrt{\varepsilon+n^2}+n)$, then

$$|x-n| \le \delta \quad \Rightarrow \quad |f(x)-n| \le \varepsilon.$$

The largest possible $\delta = \varepsilon/(\sqrt{\varepsilon+n^2}+n)$ is squeezed to zero when n glows indefinitely.

Page 79, Problem 11. *Let $f(x) = \sqrt{x}$ with domain $\{x : x \ge 0\}$.*

a) *Let $\varepsilon > 0$ be given. For each $c > 0$, show how to choose $\delta > 0$ so that $|x-c| \le \delta$ implies $|\sqrt{x}-\sqrt{c}| \le \varepsilon$.*
b) *Give a separate argument to show that f is continuous at zero.*

Solution. a) Once again we examine the growth of error by letting $|x-c| \le \delta$ and compute:

$$\begin{aligned}|\sqrt{x}-\sqrt{c}| = |\sqrt{x-c+c}-\sqrt{c} &= \frac{|x-c|}{\sqrt{x-c+c}+\sqrt{c}} \le \frac{\delta}{\sqrt{c-\delta}+\sqrt{c}} \\ &\le \frac{\delta}{\sqrt{\frac{c}{2}}+\sqrt{c}} \quad \left(\text{under the assumption} \quad \delta \le \frac{c}{2}\right) \\ &\le \frac{2\delta}{\sqrt{c}}.\end{aligned}$$

Thus if $0 < \delta \le \min\{c/2, \varepsilon\sqrt{c}/2\}$, then

$$|x-c| \le \delta \quad \Rightarrow \quad |\sqrt{x}-\sqrt{c}| \le \varepsilon.$$

Hence f is continuous at $c > 0$.

b) If $0 \le x \le \delta$, then

$$|\sqrt{x}-\sqrt{0}| = \sqrt{x} \le \sqrt{\delta}.$$

Hence if $0 < \delta \le \varepsilon^2$, then

$$0 \le x \le \delta \quad \Rightarrow 0 \le \sqrt{x} \le \varepsilon.$$

Therefore f is continuous at 0. ♡