The Admissibility of PSL$(2, 7)$ **and** SL$(2, 7)$

by

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Def: Let G be a group and K a field. We say G is K-admissible if there is a finite-dim central division algebra D over K with G the Galois group of a subfield of D over K.

If K is a number field, then G so occurs if and only if G is the Galois group of a maximal subfield of some D.

Such occurences are solutions of a *local-global principle*: Let L/K be a G-Galois extension of K. Then L fits inside a div ring central over K if and only if: every Sylow subgroup P of G is contained in the decomposition group $G_q =$

Gal(L_q/K_q) for at least two q-adic completions K_q of K. The two q's depend on P.

Example: Let $f(x) = x^4+1$. Then any central division ring over $\mathbf{Q}$ containing a root of $f(x)$ is inf dimensional.

Theorem

1. Any finite group G is admissible over some number field.

2. If G is $\mathbf{Q}$ admissible, then every Sylow subgroup of G is meta-cyclic.

The converse of 2. is open; it is true for solvable G (Sonn), and all the alternating and symmetric groups satisfying the Sylow meta-cyclic condition.

The A_n and their double covers

The alternating groups A_n have a unique double cover, which we refer to as $\tilde{A}_n$.

Suppose $\mathrm{Gal}(L/K) = A_n$; we can realize L as the splitting field of a poly $f(x) \in K[x]$ with square disc. When can L be extended to $\tilde{L}$ with $\mathrm{Gal}(\tilde{L}/K) = \tilde{A}_n$?

Serre's Condition

Let $M = K[x]/(f)$. Diagonalize the quadratic form $\mathrm{tr}(x^2)$ on M, resulting in $D = (a_1, a_2, \ldots, a_n)$. Set $W = W(f) = \prod_{i<j}(a_i, a_j)$; $W \in \mathrm{Br}_2(K)$.

Serre: $\tilde{L}$ exists if and only if $W = 0$.

Mestre: Let $f(x) \in K[x]$ be a poly of degree n with square disc. Then there is a poly $q(x) \in$

$K[x]$ of degree $n-1$ so that $p(x) = f - tq$ has Gal group A_n over $K(t)$. Also $W(f) = W(p)$.

Consequence: (Feit, Sonn): A_n and $\tilde{A}_n$ are $\mathbf{Q}$-admissible infinitely often for $n = 5, 6, 7$.

The simple group of order 168

It is PSL$(2, 7)$, GL$(3, 2)$, or the automorphism group of the Fano plane. Call it G.

G subset A_7, and the restriction of the double cover $\tilde{A}_7$ produces the double cover SL$(2, 7)$ of G.

Mestre1: G has a generic Galois construction over any number field.

Mestre2: Let $f(x)$ be a poly of degree 7 with square disc and Galois group contained in G.

Then there is a poly $q(x)$ of degree 6 so that $p(x) = f - tq$ has Galois group G over $K(t)$.

Theorem (Allman-Schacher): G is admissible over a number field K if and only if either $\sqrt{-1} \notin K$ or K has two primes over 2.

So what about $\mathrm{SL}(2,7)$? Bad news: The passage from f to p no longer preserves trivial Witt invariant!

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Question: Is $\mathrm{SL}(2,7)$ admiss over $\mathbf{Q}$ inf often???

Let $0 \neq W = W(t) \in \mathrm{Br}_2(\mathbf{Q}(\mathbf{t}))$.

Theorem: (Fein-Saltman-Schacher) The set of $t_0 \in \mathbf{Q}$ with $0 \neq W(t_0)$ is infinite.

Suppose $W(s) = 0$ for some $s \in \mathbf{Q}$. Serre wonders: must the set of such s be infinite ???

If yes, $\mathrm{SL}(2,7)$ is $\mathbf{Q}$-admiss infinitely often.

Special Case:

(Using $W = (-1, f(t))$ as a model)

Let $f(t)$ be an irred poly in $\mathbf{Q(t)}$ of odd degree. Then f represents a sum of two squares infinitely often ????

Warning: For this it is necessary to allow *rational* values, even if $f(t)$ is monic with integer coeffs.

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