Math 31B
Integration and Infinite Series

Midterm 1 Practice

Instructions: You will have 50 minutes to complete the exam.

There will be four questions, worth a total of 50 points.

There are more questions here so that you can practice.

This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: ─────────────────────────────

Student ID number: ─────────────────────

Discussion: ───────────────────────────

Problem 1.

    (a) [4pts.] Calculate $\frac{d}{dx}\left[e^{e^{(x^2+2)}}\right]$

    (b) [6pts.] Calculate $\int \frac{1}{x(\ln x)^2}\, dx$.

Problem 2.
    Calculate the following integrals.

    (a) [4pts.] $\int e^{x^2+x} + 2xe^{x^2+x}\, dx$.

    (b) [6pts.] $\int_0^1 \frac{x+1}{x^2+2x+2}\, dx$.

Problem 3.

    (a) [6pts.] Calculate $\lim_{x\to 0}\left[\frac{1}{x} - \frac{1}{\sin x}\right]$.

    (b) [6pts.] Calculate $\lim_{x\to 1}(1 + \ln x)^{\frac{1}{x-1}}$.

    (c) [5pts.] Calculate $\lim_{x\to 1+}(\ln x)^2 \ln(\ln x)$.

Problem 4.
    Use L'Hôpital's rule and associated tricks to calculate the following limits.

    (a) $\lim_{x\to 0+} x(\ln x)^2$

    (b) $\lim_{x\to 0+} x^{\sin x}$.

Problem 5.
    Calculate the limits of the following sequences or say that they don't exist.

    (a) (a_n) where $a_n = \frac{2^{2n}}{n!}$.

    (b) (b_n) where $b_n = \frac{e^n + (-3)^n}{5^n}$.

    (c) (c_n) where $c_1 = 1$ and, for $n \geq 1$, $c_{n+1} = \sqrt{2c_n}$. You can assume (c_n) converges.

Problem 6.

    (a) [5pts.] The following series converges. Calculate its limit.

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n}$$

    (b) [7pts.] Prove the convergence or divergence of the following series using either the direct or limit comparison test.

$$\sum_{n=7}^{\infty} \frac{1}{\sqrt[6]{n^5 - 4n^3 + 2n - 1}}$$

Problem 7.

Prove the convergence or divergence of the following series using any of the tests which work. You should verify the hypotheses of any test you use carefully.

(a) [5pts.] $\sum_{n=1}^{\infty} \frac{n}{n+1}$

(b) [5pts.] $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+3n+2}$.

Problem 8.

Prove the convergence or divergence of the following series using any test that works. You should verify the hypotheses of any test you use carefully.

(a) $\sum_{n=1}^{\infty} \frac{\ln n}{n^{\frac{21}{20}}}$ (probably too difficult for the actual exam).

(b) $\sum_{n=1}^{\infty} \frac{1}{5^n-3^n}$.

Problem 9.

For each part of this question:

- *either* give an example of series satisfying the given conditions;
- *or* say that such series do not exist, i.e. the statement is always false.

(a) [3pts.] $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n$ diverges.

(b) [3pts.] $\sum_{n=1}^{\infty} a_n$ diverges, and $\sum_{n=10^{10}}^{\infty} a_n$ converges.

(c) [3pts.] $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ converges, and $a_n \le b_n$.
Notice that we have *not* said that $a_n, b_n \ge 0$.

(d) [3pts.] $\sum_{n=1}^{\infty} a_n$ diverges and its sequence of partial sums $(s_N)_{N=1}^{\infty}$ is given by

$$s_N = 1 - \frac{1}{2^N}.$$

Problem 10.

For each part of this question:

- *either* give an example of series satisfying the given conditions;
- *or* say that such series do not exist, i.e. the statement is always false.

(a) The n-th term test says that $\sum_{n=1}^{\infty} a_n$ converges.

(b) $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} b_n$ diverges, and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

(c) $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} a_n^2$ converges.

(d) $\sum_{n=1}^{\infty} a_n$ has as its sequence of partial sums $(s_N)_{N=1}^{\infty}$ where $s_N = 2^{N+1} - 2$.

(e) $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ diverges, $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

(f) $\sum_{n=1}^{\infty} a_n$ diverges, and every partial sum s_N is a number between -2 and 2.

(g) Each $a_n > 0$, $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} \sin(a_n)$ diverges.