Context

A Canadian research company General Fusion (2002-Present, Michel Laberge) is designing a reactor to yield sustainable fusion energy, which has never been achieved before. Success with this could hold immense promise for clean and sustainable energy sources in the future.

In this talk, we will study the reactor design by:

- Deriving a reactor model
- Studying the model numerically with finite volumes
- Qualitatively describing the operation with formal asymptotic approach
- Assessing the effects of instability via linearization
Thanks to...

Thanks to the following people for helpful discussions, collaboration, and support:

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- Randy LeVeque, University of Washington Applied Mathematics Department
- Russ Caflisch, John Garnett, James Ralston, Marcus Roper, UCLA Math Department
Fusion

- Fusion occurs naturally in sun: pressure $\sim 300$ G atm, temperature $\sim 10^7$ K
- Potential clean energy source
- Tritium-deuterium plasma reaction:
  $$^3_1\text{H} + ^2_1\text{H} \rightarrow ^4_2\text{He} + n + 17.6 \text{ MeV of energy}$$
- Complex: thermal distribution, radiative losses, etc.
- Lawson criterion $^1$ for energy yield:
  $$\text{density} \times \text{temperature} \times \text{time} \geq 4 \times 10^{15} \text{ cm}^{-3} \text{ KeV s}$$

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Modelling of a Magnetized Target Fusion Reactor
General Fusion design

How feasible is such a design?

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Lead-Lithium

- Density $\rho$, velocity $\nu$, and pressure $P$:
  \[
  \rho_t + \nabla \cdot (\rho \nu) = 0 \quad \text{(mass)}
  \]
  \[
  (\rho \nu)_t + \nabla \cdot (\rho \nu \otimes \nu) + \nabla P = 0 \quad \text{(momentum)}
  \]

- Quadratic fit to equation of state $2P = P(\rho)$

![Equation of State Fit for High Densities](image)
Assumptions and simplifications

- Spherical symmetry
- Piston pressure:
  \[ P(t) = P_{\text{atmospheric}} + (P_{\text{impact}} - P_{\text{atmospheric}})e^{-t^2/t_0^2} \theta(t) \]
- Adiabatic: plasma gas pressure \( \propto V^{-5/3} \)
- Plasma magnetic pressure \( \propto R_{\text{plasma}}^{-4} \)
- Initially \( P_{\text{gas}} \approx 0.1 P_{\text{magnetic}} \)
- Initial plasma temperature 100 eV
- System starts in equilibrium
- No mixing/escape yields free boundary problems:
  \[ \frac{d}{dt} r_{\text{boundary}}(t) = \nu(r_{\text{boundary}}(t), t) \]
- Neglect thermal and radiative energy losses, and “slow” rotation
Overall model

Lead-Lithium

Plasma

$v < 0$: particles move inward

$v_L$, $v_R$

$r_L$, $r_R$

Lead-Lithium

Plasma

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Coordinate transformation

- Transform from moving to fixed computational domain
- Let $\tau = t$, $\Delta(\tau) = r_R(\tau) - r_L(\tau)$, $\Gamma(\tau) = v_R(\tau) - v_L(\tau)$
- Set $y = \frac{r - r_L(\tau)}{\Delta(\tau)}$, $y \in [0, 1]$

\[\rho(r_R(t), t) = \rho(1, \tau), \text{ etc.}\]

- New conservation laws e.g. mass:

\[\rho_\tau + \left\{ \frac{1}{\Delta} \left[ -(v_L + \Gamma y)\rho + \rho v \right] \right\}_y = \frac{-2}{r_L + \Delta y} - \frac{\Gamma}{\Delta} \rho\]
Finite volume approach

- Split stepping: update via $\rho_\tau + \left\{ \frac{1}{\Delta} \left[ - (v_L + \Gamma y) \rho + \rho v \right] \right\}_y = 0$ then via
  \[
  \rho_\tau = \frac{-2}{r_L + \Delta y} - \frac{\Gamma}{\Delta} \rho
  \]
  - similarly for momentum

- Code summary 4:
  - constant extrapolation to ghost points
  - upwind homogeneous system with second-order correction: use local eigenvector bases
  - use approximate Riemann solver
  - update with source term
  - update time and boundary data
Pulse profiles

Velocity, Density, and Pressure Profiles at \( t = 0.098 \) [3.6 ms]

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Abridged sensitivity analysis

Table: Min radius $R_{\text{min}}$, Lawson triple product $\Pi_L$, impact pressure $P_{\text{impact}}$, initial plasma radius $R_{\text{plasma,0}}$, initial sphere radius $R_{\text{lead,0}}$.

<table>
<thead>
<tr>
<th>System</th>
<th>$R_{\text{min}}$ (cm)</th>
<th>$\Pi_L \times 10^{15}$ keV s cm$^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>3.6</td>
<td>0.52</td>
</tr>
<tr>
<td>$R_{\text{plasma,0}} \times 1.1$</td>
<td>5.5</td>
<td>0.25</td>
</tr>
<tr>
<td>$P_{\text{impact}} \times 1.1$</td>
<td>3.0</td>
<td>0.64</td>
</tr>
<tr>
<td>$R_{\text{lead,0}} \times 1.1$</td>
<td>3.0</td>
<td>0.92</td>
</tr>
<tr>
<td>$P_{\text{impact}} \times 2$</td>
<td>1.2</td>
<td>16</td>
</tr>
<tr>
<td>$R_{\text{lead,0}} \times 2$</td>
<td>0.84</td>
<td>2.5</td>
</tr>
</tbody>
</table>
Big picture

- Modifications:
  - Gaussian impulse pressure without heaviside
  - linearized $P(\rho)$ equation of state
  - only magnetic pressure

- Relevant scales:
  - asymptotic parameter $\epsilon \approx 0.013 \ll 1$
  - big sound speed: $b\epsilon^{-1/2}$
  - impulse pressure: $O(1)$
  - small initial plasma radius: $\chi\epsilon^{1/2}$
  - very small impulse time: $O(\epsilon)$
  - very, very small initial pressure: $\mu\epsilon^{3/2}$

- Matched asymptotics 5
Asymptotic regimes

- **V max compression**: $O(\varepsilon) \times O(\varepsilon^{5/4})$
- **IV slow collapse**: $O(\varepsilon)$
- **III reflecting**: $O(\varepsilon^{1/2})$
- **II focusing**: $O(\varepsilon^{1/2})$
- **I formation**: $O(\varepsilon)$

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Pulse formation and focusing

- **Formation:**
  - \( \rho(y, \tau) \sim 1 + \epsilon \rho_1 + \epsilon^{3/2} \rho_2, \quad v(y, \tau) \sim \epsilon^{1/2} v_0 + \epsilon v_1 \)
  - Riemann Invariants: plane wave solutions in \((-\infty, 0] \times [0, \infty)\):
    \[
    \rho_1(y, \tau) = \frac{1}{b^2} e^{-(\tau+y/b)^2}, \quad v_0(y, \tau) = -\frac{1}{b} e^{-(\tau+y/b)^2}
    \]
  - need divergent \( \rho_2, v_1 \) for matching

- **Focusing:**
  - amplitude growth:
    \[
    \rho \sim 1 + \epsilon \rho_1 + \epsilon^{3/2} \rho_2, \quad v \sim \epsilon^{1/2} v_0 + \epsilon v_1 \quad \text{(outer)}
    \]
    \[
    \rho \sim 1 + \epsilon^{1/2} \rho_1 + \epsilon \rho_2, \quad v \sim v_0 + \epsilon^{1/2} v_1 \quad \text{(inner)}
    \]
  - linear acoustic limit:
    \[
    \rho_{1,t} + v_{0,\sigma} + \frac{2}{\sigma} v_0 = 0, \quad v_{0,t} + b^2 \rho_{1,\sigma} = 0
    \]

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Complete reflection of $\rho_1$ and $v_0$ but
\[ v_1,\dot{t} + b^2 \rho_{2,\sigma} = - (\rho_1 v_0)_{\dot{t}} - (v_0^2)_{\sigma} - \frac{2}{\sigma} v_0^2 \]

Find $v_1(\sigma, \infty) = \frac{-\sqrt{2\pi}}{b^2 \chi \sigma^2}$ so $v = O(\epsilon^{1/2})$

Rayleigh argument or more dynamics: take $v_L = v_L(r_L)$.

- $\rho \sim 1 + \epsilon^2 \hat{\rho}$, $v \sim \epsilon^{1/2} \hat{v}$: plasma radius $\sigma_L$
- find $\hat{v}(\sigma_L) = \hat{v}_L = \frac{-\sqrt{2\pi}}{b^2 \chi^{3/2} \sigma_L^{3/2}}$

Finally

- $\rho \sim 1 + \epsilon^{1/2} \hat{\rho}$, $v \sim \epsilon^{-1/4} \hat{v}$: plasma radius $z_L$
- find $\hat{v}(z_L) = \hat{v}_L = \frac{-\sqrt{2Az_L-2\mu}}{z_L^2}$

Matching gives $A$: turnaround point when $\hat{v}_L = 0$
Minimum radius

- Minimum radius:
  \[ r_{\text{min}} \sim \frac{b^4 \chi^3 \mu}{\pi} \varepsilon \]

- Agreement with numerics as \( \varepsilon \downarrow 0 \) with \( b = 1.05, \chi = 0.937, \mu = \pi \):

| \( \varepsilon \) | \( |r_{\text{min, num}} - r_{\text{min, asy}}| \) |
|----------------|----------------|
| 0.02           | 0.00845        |
| 0.01           | 0.00400        |
| 0.005          | 0.00056        |
| 0.0025         | 0.00008        |
Key insights

- Dimensional minimum radius

\[ R_{\text{min}} \approx \frac{C_s^4 P_{\text{plasma},0} R_{\text{plasma},0}^7 \rho_0^3}{\pi P_{\text{impact}}^4 R_{\text{lead},0}^4 t_0^2} = 1.6 \text{ cm} \]

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_s )</td>
<td>lead sound speed</td>
</tr>
<tr>
<td>( R_{\text{plasma},0} )</td>
<td>initial plasma radius</td>
</tr>
<tr>
<td>( P_{\text{impact}} )</td>
<td>piston pressure</td>
</tr>
<tr>
<td>( t_0 )</td>
<td>impulse time scale</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_{\text{plasma},0} )</td>
<td>initial pressure</td>
</tr>
<tr>
<td>( \rho_0 )</td>
<td>lead density</td>
</tr>
<tr>
<td>( R_{\text{lead},0} )</td>
<td>initial lead radius</td>
</tr>
</tbody>
</table>

- Qualitative consistency with numerics for all parameters
Key insights

- consistent compression profile
- wave-like behaviour: sound speed dominates
- almost all input energy reflected:
  \[ E_{\text{input}} \sim \frac{\sqrt{8\pi^3}}{b} \epsilon^{3/2}, \quad E_{\text{compression}} \sim \frac{4\pi^2}{b^4 \chi^3} \epsilon^{5/2} \]
Asymmetric context

- Finite (∼ 100 pistons on sphere), time differences, etc.
- Perturbations to axially symmetric collapse: linearize!

\[
P(r_R, \theta, t) = P_{\text{atm}} + (P_{\text{impact}} - P_{\text{atm}}) \times e^{t^2/t_0^2} \left( F + \sum_{k=1}^{\infty} \frac{2}{\pi k} \sin(k\pi F) \text{Re}(e^{ikN_c \theta}) \right), \quad N_c \approx 15
\]

- Pistons cover fraction \( F \) of angle \( 2\pi/N_c \):
  \[
P(r_R, \theta, t) = P_{\text{atm}} + (P_{\text{impact}} - P_{\text{atm}}) \times e^{t^2/t_0^2} \left( F + \sum_{k=1}^{\infty} \frac{2}{\pi k} \sin(k\pi F) \text{Re}(e^{ikN_c \theta}) \right), \quad N_c \approx 15
\]

- Dimensionless: study boundary pressure
  \[
p(1, \theta, t) = (1 + \eta e^{im\theta})e^{-t^2/\tau^2} \quad \eta \ll 1 \text{ controllable}
\]
Consider $u_t + \left( \frac{1}{2} u^2 \right)_x = u^2$ with a perturbation parameter $\eta$ with $u^\eta(x, 0) = (1 - \eta)\Theta(-x) + \eta\Theta(x)$.

Gateaux derivative $\frac{u^\eta - u^0}{\eta}$ as $\eta \to 0$:
\[
\begin{cases} 
-1/(1 - t)^2, & x < x_0^0(t) \\
1, & x > x_0^0(t)
\end{cases} + \frac{t^2}{2(1-t)^2} \delta(x - x_0^0(t))
\]
Perturbed inviscid Burgers’ equation

- If \( u = u^0 + \eta u^1 \) then \( u^1_t + (u^1 u^0)_x = 2u^0 u^1 \)

- Let \( u^1 = \tilde{u}^1 + M\delta(x - x_s^0), \ a < x_s^0 < b \) then
  \[
  \frac{d}{dt} \int_a^b u^1 dx = \frac{dM}{dt} + \int_a^{x_s^0} \tilde{u}_t^1 dx + \int_{x_s^0}^b \tilde{v}_t^1 dx + \tilde{v}_1(x_s^0^-)(x_s^0'(t)) = \tilde{u}^1(x_s^0^+)(x_s^0'(t))
  \]

- \[ \frac{d}{dt} \int_a^b u^1 dx = (u^0 u^1)^- - (u^0 u^1)^+ + \int_a^b 2u^0 u^1 dx. \]

- Obtain \( \frac{dM}{dt} = x_s^0'(t)[u^1] - [u^0 u^1] + \frac{1}{2}(2u^0^- + 2u^0^+)M. \)

- Burger: \( M(t) = \Delta_{\eta=0}[u^0] \implies \text{gap } \Delta = \eta M(t)/[u^0] \)
Perturbed equations

- Density $\rho = \rho_0(r, t) + \eta \bar{\rho}(r, t)e^{im\theta}$, momentum density in $\hat{r}$- and $\hat{\theta}$-directions $\mu = \mu_0(r, t) + \eta s(r, t)e^{im\theta}$, $\eta \psi(r, t)e^{im\theta}$.

- Obtain $3 \times 3$ perturbed system
Abridged sensitivities

<table>
<thead>
<tr>
<th>Dimensionless time</th>
<th>m</th>
<th>$\Delta_{\eta}^{(\bar{\rho})}$</th>
<th>$\Delta_{\eta}^{(s)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0</td>
<td>0.2800</td>
<td>0.2836</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.2378</td>
<td>0.2404</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.0111</td>
<td>0.0101</td>
</tr>
<tr>
<td>0.15</td>
<td>0</td>
<td>0.5738</td>
<td>0.5809</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.0279</td>
<td>0.0280</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

$\Delta_{\eta}$: sensitivity of deviation amplitude with respect to perturbation size

- Suponitsky et al. suggest low mode numbers interacting with plasma pose few problems for operation
- Work here suggests high mode numbers damped out
Pulse formation as $m \to \infty$

- Near outer boundary with $m \to \infty$ acoustic model:
  \[
  p_{TT} = p_{xx} - p, \quad p(x, 0) = p_t(x, 0) = 0, \quad p(0, t) = 1 \quad \text{for } x, T \geq 0
  \]
  \[
  p \sim e^{-x} + \sqrt{\frac{2}{\pi}} \left( \frac{1 - \hat{\sigma}/T^2}{\sqrt{2\hat{\sigma} + \hat{\sigma}^2/T^2}} \sin(\sqrt{2\hat{\sigma} + \hat{\sigma}^2/T^2} + \pi/4) \right)
  \]
  \[
  - \frac{1}{\sqrt{2\hat{\sigma}}} \sin(\sqrt{2\hat{\sigma} + \pi/4})) - \frac{1}{\pi} F(\hat{\sigma}), \quad x < T \quad (T \to \infty)
  \]
  \[
  \bar{\rho} = 1/2, \quad x = T
  \]
  \[
  \bar{\rho} = 0, \quad x > T
  \]
- $F(z) = \int_0^1 \frac{1}{\Omega} (\sin(z\Omega + \frac{1}{2\Omega}) + \sin(\frac{z}{\Omega} + \frac{\Omega}{2}))d\Omega$. 

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Results and future work

- Results:
  - numerics and asymptotics consistent
  - sensitivity to parameters, energy yield may be within reach
  - larger outer sphere radius and impact pressure appear important
  - much energy is reflected
  - asymmetries may be dampened

- Future directions:
  - incorporate more plasma physics
  - more precise assessment of design
References