Magnetized Target Fusion: Insights from Mathematical Modelling

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Context

A local Canadian research company is designing a reactor to produce fusion energy, which has never been achieved before. Success with this could hold immense promise for clean and sustainable energy sources in the future. In this talk, we will look into the mathematical analysis used to yield insights into the design work. Key components:

- Reactor model
- Numerical finite volume approach
- Formal asymptotic approach
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- George Bluman, Michael Ward, and Brian Wetton, UBC Math Department
- Randy LeVeque, University of Washington Applied Mathematics Department
- My colleagues and fellow graduate students
Fusion

- Fusing atomic nuclei yield new nuclei plus energy
- Occurs naturally in sun: pressure $\sim 300 \text{ G atm}$, temperature $\sim 10^7 \text{ K}$
- Potential clean energy source
- Tritium-deuterium plasma reaction:
  \[ ^3_1 \text{H} + ^2_1 \text{H} \rightarrow ^4_2 \text{He} + n + 17.6 \text{ MeV of energy} \]
- Depends on temperature, thermal distribution, etc.
Challenges

- Immense temperature and pressure must be sustained
- Thermal and radiative losses
- Lawson criterion \(^1\) for energy yield:
  \[
  \text{density} \times \text{temperature} \times \text{time} \geq 4 \times 10^{15} \text{ cm}^{-3} \text{ KeV s}
  \]
**General Fusion design**

- General Fusion founded by Michel Laberge in 2002
- Design magnetized target fusion reactor
- Isolate spheromak plasma with magnetic field
- Crush plasma in imploding metal cavity
General Fusion design

How feasible is such a design?
Lead-Lithium

- Euler equations for mass and momentum conservation

- With density $\rho$, velocity $v$, and pressure $P$:

  $$\rho_t + \nabla \cdot (\rho v) = 0 \quad \text{(mass)}$$
  $$\rho v_t + \nabla \cdot (\rho v \otimes v) + \nabla P = 0 \quad \text{(momentum)}$$
  $$P = P(\rho) \quad \text{(equation of state)}$$

- Nonlinear coupled system of PDEs
Lead-Lithium

- Quadratic fit to high-pressure Lead experiments $^2$

Equation of State Fit for High Densities

- Pressure vs. Density graph

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Piston and plasma pressure

- Gaussian piston pressure:

\[ P(t) = P_{\text{atmospheric}} + (P_{\text{impact}} - P_{\text{atmospheric}})e^{-t^2/t_0^2} \theta(t) \]

- Plasma has gas and magnetic pressure

- Ideal gas of volume \( V \):

\[ P_{\text{gas}} \propto V^{-\gamma}, \quad \gamma \geq 1 \]

- Magnetic pressure \(^3\):

\[ P_{\text{magnetic}} \propto R_{\text{major}}^{-4} \]
Assumptions and simplifications

- Spherical symmetry
- Isentropic (reversible) conditions so $\gamma = 5/3$
- Initially $P_{\text{gas}} \approx 0.1 P_{\text{magnetic}}$
- Initial plasma temperature 100 eV
- System starts in equilibrium
- No mixing implies free boundary problem at plasma-metal and metal-piston interfaces:
  \[ \frac{d}{dt} r_{\text{boundary}}(t) = v(r_{\text{boundary}}(t), t) \]
- Neglect thermal and radiative energy losses
Overall model

[Diagram showing a model of nuclear fusion with labels for Lead-Lithium, Plasma, and other components]
Overall model

In $r_L(t) < r < r_R(t)$, $t > 0$, dimensionless system has form:

\[ \rho_t + \frac{1}{r^2} (r^2 \rho v)_r = 0, \quad (\rho v)_t + p_r + \frac{1}{r^2} (r^2 \rho v^2)_r = 0 \]

\[ p = p(\rho), \quad \frac{dr_{L,R}}{dt} = v(r_{L,R}(t), t) \]

\[ p(r_L(t), t) = p_L(r_L(t)), \quad p(r_R(t), t) = f(t) \]

\[ v(r, -\infty) = 0, \quad p(r, -\infty) = p_L(r_L(-\infty)) \]

\[ r_L(-\infty) = \chi \epsilon^{1/2}, \quad r_R(-\infty) = 1 \]
First-order finite volumes

- **Given** $u_t + (f(u))_x = 0$ denote $\bar{u}(x_i, t) = \frac{1}{h} \int_{x_{i-h/2}}^{x_{i+h/2}} u(x, t) dx$

- $\frac{d\bar{u}(x_i,t)}{dt} = \frac{1}{h} \int_{x_{i-h/2}}^{x_{i+h/2}} u_t(x, t) dx = - \frac{f(u(x_{i+1/2}, t)) - f(u(x_{i-1/2}, t))}{h}$

- **Choose** $\mathcal{F}^L(u_{i-1}, u_i) = f(u_{i-1})$ or $f(u_i)$ depending on $f'$

- **Stable time-step** $k < h \sup |f'|$,
  
  $u_{i+1}^j = u_i^j - \frac{k}{h} (\mathcal{F}^L(u_i^j, u_{i+1}^j) - \mathcal{F}^L(u_{i-1}^j, u_i^j))$
Second-order flux-limited finite volume idea

- Smoothness: \( u_t + f'(u)u_x = 0 \)
- Use \( u_{tt} = -f''(u)u_tu_x - f'(u)u_{tx} = (f'(u)^2)_xu_x + f'(u)^2u_{xx} \)
- Lax-Wendroff:
  \[
  u(x, t + k) = u(x, t) + ku_t(x, t) + \frac{k^2}{2} u_{tt}(x, t) + O(k^3)
  \]
- Corresponding high-resolution flux \( \mathcal{F}^\mathcal{H} \)
- Flux limited: take flux \( \mathcal{F} = \phi \mathcal{F}^\mathcal{L} + (1 - \phi) \mathcal{F}^\mathcal{H} \)
- Limiter \( \phi \) largest where derivative \( |u'(x)| \) largest
Convergence

- Numerical solution $u_{\text{num}}(x, t)$ and exact solution $u_{\text{ex}}(x, t)$:
  \[ E = \int |u_{\text{num}}(x, t) - u_{\text{ex}}(x, t)| \, dx \to 0, \quad \text{as } N \to \infty \]

- Near shocks, $E = O(h)$

- Near smooth regions $E = O(h^n)$ for $n^{\text{th}}$-order method
Coordinate transformation

- Transform from moving to fixed computational domain

- Let $\tau = t$ and define $\Delta(\tau) = r_R(\tau) - r_L(\tau)$,
  $\Gamma(\tau) = v_R(\tau) - v_L(\tau)$

- Set $y = \frac{r - r_L(\tau)}{\Delta(\tau)}$, $y \in [0, 1]$

- $p(r_R(t), t) = p(1, \tau)$, etc.

- New conservation laws e.g. mass:

  \[
  \rho_\tau + \left\{ \frac{1}{\Delta} \left[ -(v_L + \Gamma y)\rho + \rho v \right] \right\}_y = \frac{-2}{r_L + \Delta y} - \frac{\Gamma}{\Delta} \rho
  \]
Finite volume overview

- Pseudocode:\n  - eigen analysis for system
  - constant extrapolation to ghost-points
  - use projections for limiters
  - homogeneous Riemann problem then use source
  - update time and boundary data

- Verify $L^1$-convergence

Table: Convergence of Numerical Scheme at fixed time

<table>
<thead>
<tr>
<th>N</th>
<th>Velocity Error</th>
<th>Density Error</th>
<th>Mass Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>$5.68 \times 10^{-4}$</td>
<td>$1.11 \times 10^{-3}$</td>
<td>$4.79 \times 10^{-4}$</td>
</tr>
<tr>
<td>4000</td>
<td>$1.39 \times 10^{-4}$</td>
<td>$2.77 \times 10^{-4}$</td>
<td>$1.20 \times 10^{-4}$</td>
</tr>
<tr>
<td>16000</td>
<td>$3.51 \times 10^{-5}$</td>
<td>$6.93 \times 10^{-5}$</td>
<td>$2.99 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Rate: 1.01  1.00  1.00
Pulse profiles

![Velocity, Density, and Pressure Profiles at t=0.098 [3.6 ms]](image)

- **Pressure**
- **Density**
- **Velocity**
**Abridged sensitivity analysis**

**Table:** Min radius $R_{\text{min}}$, Lawson triple product $\Pi_L$, impact pressure $P_{\text{impact}}$, initial plasma radius $R_{\text{plasma,0}}$, initial sphere radius $R_{\text{lead,0}}$, time scale $t_0$.

<table>
<thead>
<tr>
<th>System</th>
<th>$R_{\text{min}}$ (cm)</th>
<th>$\Pi_L$ ($10^{15}$ keV s cm$^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>3.6</td>
<td>0.52</td>
</tr>
<tr>
<td>$R_{\text{plasma,0}} \times 1.1$</td>
<td>5.5</td>
<td>0.25</td>
</tr>
<tr>
<td>$P_{\text{impact}} \times 1.1$</td>
<td>3.0</td>
<td>0.64</td>
</tr>
<tr>
<td>$R_{\text{lead,0}} \times 1.1$</td>
<td>3.0</td>
<td>0.92</td>
</tr>
<tr>
<td>$t_0 \times 1.1$</td>
<td>3.1</td>
<td>0.61</td>
</tr>
<tr>
<td>$P_{\text{impact}} \times 2$</td>
<td>1.2</td>
<td>16</td>
</tr>
<tr>
<td>$R_{\text{lead,0}} \times 2$</td>
<td>0.84</td>
<td>2.5</td>
</tr>
<tr>
<td>$t_0 \times 2$</td>
<td>1.3</td>
<td>1.9</td>
</tr>
</tbody>
</table>
Big picture

- **Modifications:**
  - Gaussian impulse pressure without heaviside
  - linearized $P(\rho)$ equation of state
  - only magnetic pressure

- **Relevant scales:**
  - asymptotic parameter $\epsilon \ll 1$
  - big sound speed: $b\epsilon^{-1/2}$
  - impulse pressure: $O(1)$
  - small initial plasma radius: $\chi\epsilon^{1/2}$
  - very small impulse time: $O(\epsilon)$
  - very, very small initial pressure: $\mu\epsilon^{3/2}$

- **Matched asymptotics**  

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Asymptotic regimes

- **Max compression**: \( O(\epsilon) \times O(\epsilon^{5/4}) \)
- **Slow collapse**: \( O(\epsilon^{1/2}) \)
- **Reflecting**: \( O(\epsilon) \)
- **Focusing**: \( O(\epsilon^{1/2}) \)
- **Formation**: \( O(\epsilon) \)
Pulse formation and focusing

- **Formation:**
  - \( \rho(y, \tau) \sim 1 + \epsilon \rho_1 + \epsilon^{3/2} \rho_2, \quad v(y, \tau) \sim \epsilon^{1/2} v_0 + \epsilon v_1 \)
  - Riemann Invariants \( ^6 \) - plane wave solutions in \((-\infty, 0] \times [0, \infty)\):
    \[
    \rho_1(y, \tau) = \frac{1}{b^2} e^{-(\tau + y/b)^2}, \quad v_0(y, \tau) = \frac{-1}{b} e^{-(\tau + y/b)^2}
    \]
  - need divergent \( \rho_2, v_1 \) for matching

- **Focusing:**
  - amplitude growth:
    \[
    \rho \sim 1 + \epsilon \rho_1 + \epsilon^{3/2} \rho_2, \quad v \sim \epsilon^{1/2} v_0 + \epsilon v_1 \quad \text{(outer)}
    \]
    \[
    \rho \sim 1 + \epsilon^{1/2} \rho_1 + \epsilon \rho_2, \quad v \sim v_0 + \epsilon^{1/2} v_1 \quad \text{(inner)}
    \]
  - linear acoustic limit \( ^7 \):
    \[
    \rho_{1,\hat{t}} + v_{0,\sigma} + \frac{2}{\sigma} v_0 = 0, \quad v_{0,\hat{t}} + b^2 \rho_{1,\sigma} = 0
    \]
Pulse reflection

- Complete reflection of $\rho_1$ and $v_0$, negligible compression
- Next order:

$$v_{1,\hat{t}} + b^2 \rho_{2,\sigma} = -(\rho_1 v_0)_{\hat{t}} - (v_0^2)_{\sigma} - \frac{2}{\sigma} v_0^2$$

- Careful integration and boundary condition trickery gives

$$v_1(\sigma, \infty) = \frac{-\sqrt{2\pi}}{b^2 \chi \sigma^2}$$

- Residual velocity field $v \sim \epsilon^{1/2} v_1$ compresses plasma!
- Rayleigh energy argument 8 or more dynamics
Plasma compression

- Integration of PDEs with reformulation gives free boundary ODE

- Initially
  - $\rho \sim 1 + \epsilon^2 \hat{\rho}$, $\nu \sim \epsilon^{1/2} \hat{\nu}$
  - $O(\epsilon^{1/2})$ space scale, $O(1)$ time scale
  - find $\hat{\nu}(\sigma_L) = \hat{\nu}_L = -\frac{\sqrt{2\pi}}{b^2 \chi^{3/2} \sigma_L^{3/2}}$; inner wall at $\sigma_L$

- Finally
  - $\rho \sim 1 + \epsilon^{1/2} \hat{\rho}$, $\nu \sim \epsilon^{-1/4} \hat{\nu}$
  - $O(\epsilon)$ space scale, $O(\epsilon^{5/4})$ time scale
  - find $\hat{\nu}(z_L) = \hat{\nu}_L = -\frac{\sqrt{2A\mu - 2\mu}}{z_L^2}$; inner wall at $z_L$

- Matching gives $A$

- Turnaround point when $\hat{\nu}_L = 0$
Minimum radius

Minimum radius:

\[ r_{\text{min}} \sim \frac{b^4 \chi^3 \mu}{\pi} \epsilon \]

Agreement with numerics as \( \epsilon \downarrow 0 \) with

\( b = 1.05, \chi = 0.937, \mu = \pi \):

| \( \epsilon \) | \( |r_{\text{min,num}} - r_{\text{min,asy}}| \) |
|--------------|------------------|
| 0.02         | 0.00845          |
| 0.01         | 0.00400          |
| 0.005        | 0.00056          |
| 0.0025       | 0.00008          |
### Key insights

- **Dimensional minimum radius**

  \[ R_{\text{min}} \approx \frac{C_s^4 P_{\text{plasma},0} R_{\text{plasma},0}^7 \varrho_0^3}{\pi P_{\text{impact}}^4 R_{\text{lead},0}^4 t_0^2} = 1.6 \text{ cm} \]

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_s )</td>
<td>lead sound speed</td>
<td>( P_{\text{plasma},0} )</td>
<td>initial pressure</td>
</tr>
<tr>
<td>( R_{\text{plasma},0} )</td>
<td>initial plasma radius</td>
<td>( \varrho_0 )</td>
<td>lead density</td>
</tr>
<tr>
<td>( P_{\text{impact}} )</td>
<td>piston pressure</td>
<td>( R_{\text{lead},0} )</td>
<td>initial lead radius</td>
</tr>
<tr>
<td>( t_0 )</td>
<td>impulse time scale</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Qualitative consistency with numerics**
Key insights

- consistent compression profile
- wave-like behaviour: sound speed dominates
- almost all input energy reflected:

\[ E_{\text{input}} \sim \frac{\sqrt{8\pi^3}}{b} \epsilon^{3/2}, \quad E_{\text{compression}} \sim \frac{4\pi^2}{b^4 \chi^3} \epsilon^{5/2} \]
Results and future work

- Results:
  - numerics and asymptotics consistent
  - sensitivity to parameters, energy yield may be within reach
  - larger outer sphere radius and impact pressure

- Future directions:
  - incorporate more plasma physics
  - consider angular instabilities
  - more precise assessment of design