Mathematical Modelling of a Magnetized Target Fusion Reactor

Michael Lindstrom
PIC Assistant Adjunct Professor, Mathematics
UCLA

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Thanks to...

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Fusion

- Fusion occurs naturally in sun: pressure $\sim 300 \text{ G atm}$, temperature $\sim 10^7 \text{ K}$
- Potential clean energy source
- Tritium-deuterium plasma reaction:
  $$^3_1\text{H} + ^2_1\text{H} \rightarrow ^4_2\text{He} + n + 17.6 \text{ MeV of energy}$$
- Complex: thermal distribution, radiative losses, etc.
- Lawson criterion for energy yield:
  $$\text{density} \times \text{temperature} \times \text{time} \geq 4 \times 10^{15} \text{ cm}^{-3} \text{ KeV s}$$
How feasible is such a design?
Pistons, plasma, and general simplifications

- Spherical symmetry
- Pressure: Piston (Gaussian), plasma (gas and magnetic)
- Empirical quadratic equation of state for lead lithium
- No mixing of plasma and lead-lithium:

\[ \frac{d}{dt} r_{\text{boundary}}(t) = v(r_{\text{boundary}}(t), t) \]
Overall model

In \( r_L(t) < r < r_R(t), \ t > 0 \), dimensionless system has form:

\[
\begin{align*}
\rho_t + \frac{1}{r^2}(r^2 \rho v)_r &= 0, \\
(\rho v)_t + p_r + \frac{1}{r^2}(r^2 \rho v^2)_r &= 0 \\
p &= p(\rho), \\
\frac{dr_L, R}{dt} &= v(r_L, R(t), t) \\
p(r_L(t), t) &= p_L(r_L(t)), \\
p(r_R(t), t) &= f(t) \\
v(r, 0) &= 0, \\
p(r, 0) &= \text{constant} \\
r_L(0) &= \text{given}, \\
r_R(0) &= 1
\end{align*}
\]
Finite volume methodology

- Conservation $u_t + (f(u))_x = 0$:
  $$\bar{u}_i^{j+1} = \bar{u}_i^j - k \frac{\mathcal{F}_i^{j+1/2} - \mathcal{F}_i^{j-1/2}}{h}$$
- $\mathcal{F}$ combination of low/high resolution via limiters
- $L^1$ convergence: $\int |u_{num}(x, t) - u_{ex}(x, t)| dx = O(h)$
- Fixed space domain via coordinate change
- Local linearized systems, approximate Riemann solvers
- Split stepping for geometric sources
Pulse profiles

Velocity, Density, and Pressure Profiles at t = 0.098 [3.6 ms]

- **Pressure**
- **Density**
- **Velocity**

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**Modelling of a Magnetized Target Fusion Reactor**
Abridged sensitivity analysis

Table: Min radius $R_{\text{min}}$, Lawson triple product $\Pi_L$, impact pressure $P_{\text{impact}}$, initial plasma radius $R_{\text{plasma},0}$, initial sphere radius $R_{\text{lead},0}$.

<table>
<thead>
<tr>
<th>System</th>
<th>$R_{\text{min}}$ (cm)</th>
<th>$\Pi_L$ ($10^{15}$ keV s cm$^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>3.6</td>
<td>0.52</td>
</tr>
<tr>
<td>$R_{\text{plasma},0} \times 1.1$</td>
<td>5.5</td>
<td>0.25</td>
</tr>
<tr>
<td>$P_{\text{impact}} \times 1.1$</td>
<td>3.0</td>
<td>0.64</td>
</tr>
<tr>
<td>$R_{\text{lead},0} \times 1.1$</td>
<td>3.0</td>
<td>0.92</td>
</tr>
<tr>
<td>$P_{\text{impact}} \times 2$</td>
<td>1.2</td>
<td>16</td>
</tr>
<tr>
<td>$R_{\text{lead},0} \times 2$</td>
<td>0.84</td>
<td>2.5</td>
</tr>
</tbody>
</table>
Big picture

- **Modifications:**
  - Gaussian impulse pressure without heaviside
  - linearized $P(\rho)$ equation of state
  - only magnetic pressure

- **Relevant scales:**
  - asymptotic parameter $\epsilon \approx 0.013 \ll 1$
  - big sound speed: $b\epsilon^{-1/2}$
  - impulse pressure: $O(1)$
  - small initial plasma radius: $\chi\epsilon^{1/2}$
  - very small impulse time: $O(\epsilon)$
  - very, very small initial pressure: $\mu\epsilon^{3/2}$

- **Matched asymptotics**
Asymptotic regimes

- **I** Formation: $O(\epsilon)$
- **II** Focusing: $O(\epsilon^{1/2})$
- **III** Reflecting: $O(\epsilon)$
- **IV** Slow collapse: $O(\epsilon^{1/2}) 	imes O(\epsilon^{5/4})$
- **V** Max compression: $O(\epsilon) 	imes O(\epsilon^{5/4})$
- Negligible movement: $O(\epsilon^{3/2})$
Qualitative story and techniques

- I formation:
  \[ \rho \sim 1 + \epsilon \rho_1 + \epsilon^{3/2} \rho_2, \quad v \sim \epsilon^{1/2} v_0 + \epsilon v_1 \]
  Use Riemann invariants, two terms

- II focusing:
  \[ \rho \sim 1 + \epsilon^{1/2} \rho_1 + \epsilon \rho_2, \quad v \sim v_0 + \epsilon^{1/2} v_1 \]
  Use velocity potential

- III reflection:
  \[ \rho \sim 1 + \epsilon^2 \hat{\rho}, \quad v \sim \epsilon^{1/2} \hat{v} \]
  Boundary conditions imply long-term velocity

- IV/V compression:
  \[ \rho \sim 1 + \epsilon^{1/2} \hat{\rho}, \quad v \sim \epsilon^{-1/4} \hat{v} \]
  Plasma velocity depends on radius; matched asymptotics
Minimum radius

- Minimum radius:
  \[ r_{\text{min}} \sim \frac{b^4 \chi^3 \mu}{\pi \epsilon} \]

- Agreement with numerics as \( \epsilon \downarrow 0 \) with \( b = 1.05, \chi = 0.937, \mu = \pi \):

| \( \epsilon \)   | \( |r_{\text{min, num}} - r_{\text{min, asy}}| \) |
|-----------------|-------------------|
| 0.02            | 0.00845           |
| 0.01            | 0.00400           |
| 0.005           | 0.00056           |
| 0.0025          | 0.00008           |
Key insights

- **Dimensional minimum radius**

  \[ R_{\text{min}} \approx \frac{C_s^4 P_{\text{plasma},0} R_{\text{plasma},0}^7 \varrho_0^3}{\pi P_{\text{impact}}^4 R_{\text{lead},0}^4 t_0^2} = 1.6 \text{ cm} \]

- Qualitative consistency with numerics for all parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_s )</td>
<td>lead sound speed</td>
<td>( P_{\text{plasma},0} )</td>
<td>initial pressure</td>
</tr>
<tr>
<td>( R_{\text{plasma},0} )</td>
<td>initial plasma radius</td>
<td>( \varrho_0 )</td>
<td>lead density</td>
</tr>
<tr>
<td>( P_{\text{impact}} )</td>
<td>piston pressure</td>
<td>( R_{\text{lead},0} )</td>
<td>initial lead radius</td>
</tr>
<tr>
<td>( t_0 )</td>
<td>impulse time scale</td>
<td></td>
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</tr>
</tbody>
</table>
Key insights

- consistent compression profile
- wave-like behaviour: sound speed dominates
- almost all input energy reflected:
  \[ E_{\text{input}} \sim \frac{\sqrt{8\pi^3}}{b} \epsilon^{3/2} , \quad E_{\text{compression}} \sim \frac{4\pi^2}{b^4 \chi^3} \epsilon^{5/2} \]
Asymmetric context

- Finite (∼ 100 pistons on sphere), time differences, etc.
- Perturbations to axially symmetric collapse: linearize!

\[
p(1, \theta, t) = (1 + \eta e^{im\theta})e^{-t^2/\tau^2} \quad \eta \ll 1
\]

- Study boundary pressure \( p(1, \theta, t) = (1 + \eta e^{im\theta})e^{-t^2/\tau^2} \eta \ll 1 \) depends on width of pistons, decreases with \( m \).
Consider $u_t + \left( \frac{1}{2} u^2 \right)_x = u^2$ with two solutions $u^0$ and $u^\eta$ (a perturbed solution).

Gateaux derivative $(u^\eta - u^0)/\eta$ as $\eta \to 0$:

$$\begin{cases} -1/(1 - t)^2, & x < x_s^0(t) \\ 1, & x > x_s^0(t) \end{cases} - \frac{t^2}{2(1-t)^2} \delta(x - x_s^0(t))$$
Perturbed equations

- Density $\rho = \rho_0(r, t) + \eta \bar{\rho}(r, t)e^{im\theta}$, momentum density in $\hat{r}$- and $\hat{\theta}$-directions $\mu = \mu_0(r, t) + \eta s(r, t)e^{im\theta}$, $\eta \psi(r, t)e^{im\theta}$.
Abridged sensitivities

<table>
<thead>
<tr>
<th>Dimensionless time</th>
<th>m</th>
<th>$\Delta^{(\bar{\rho})}_\eta$</th>
<th>$\Delta^{(s)}_\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0</td>
<td>0.2800</td>
<td>0.2836</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.2378</td>
<td>0.2404</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.0111</td>
<td>0.0101</td>
</tr>
<tr>
<td>0.15</td>
<td>0</td>
<td>0.5738</td>
<td>0.5809</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.0279</td>
<td>0.0280</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

$\Delta_\eta$: sensitivity of deviation amplitude with respect to perturbation size

- Suponitsky et al. suggest low mode numbers interacting with plasma pose few problems for operation
- Work here suggests high mode numbers damped out
Results and future work

- Results:
  - numerics and asymptotics consistent
  - sensitivity to parameters, energy yield may be within reach
  - larger outer sphere radius and impact pressure appear important
  - much energy is reflected
  - asymmetries may be dampened

- Future directions:
  - incorporate more plasma physics
  - more precise assessment of design