Lecture 9 - LU Decomposition & Basis II.

LU Decomp & another method for systems

Def A matrix A is
- lower triangular if $a_{ij} = 0$ $i<j$
- upper triangular if $a_{ij} = 0$ $i>j$

If the coefficient matrix of a system is triangular, it is easy to solve systems: substitute.

L lower triangular: $L\bar{x} = \bar{b} \longleftrightarrow$

$$\begin{aligned} ax \quad\quad\quad &= d_1 \\ a_2x + b_2y \quad &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

$\Rightarrow$ solve for x, plug in, etc

U upper triangular: $U\bar{x} = \bar{b} \longleftrightarrow$

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2y + c_2z &= d_2 \\ c_3z &= d_3 \end{aligned}$$

$\Rightarrow$ solve for z & plug in.

Def The LU Decomposition of a matrix A is a product $A = LU$ s.t.
- L is (square) lower triangular w/ 1 on diagonal
- U is upper triangular.

Not all A have these. How do we know? Row reduce.

If Gaussian elimination requires now row swapping, then there is an LU decomp.

No swapping means the elementary row operations to kill off everything below the diagonal correspond to lower triangular matrices, the product of which is lower triangular.

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 25 \end{bmatrix} \begin{matrix} \\ -2 \\ -3 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 4 & 16 \end{bmatrix} \begin{matrix} \\ \\ -4 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$\underbrace{\begin{bmatrix} 1 & & \\ 0 & 1 & \\ -3 & 0 & 1 \end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -4 & 1 \end{bmatrix}}_{E_3}$$

$\Rightarrow U = E_3 \cdot E_2 \cdot E_1 \cdot A$

$\Rightarrow A = (E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1})\, U$

$$\overset{E_2^{-1}}{\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 3 & 0 & 1 \end{bmatrix}} \overset{E_1^{-1}}{\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 0 & 0 & 1 \end{bmatrix}} \overset{E_3^{-1}}{\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 4 & 1 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & 4 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 2 & 3 \\ & 1 & 4 \\ & & 0 \end{bmatrix}}$$

Something to note: l_{ij} is the multiple of row j we <u>subtract</u> from row i in row reduction.

Why is this easier to solve?

$$A\bar{x} = \bar{b} \longleftrightarrow L(U\bar{x}) = \bar{b}$$

1) Solve $L(\bar{y}) = \bar{b}$ (always has a unique sol: L is invertible)

2) Solve $U\bar{x} = \bar{y}$ (could have lots of sol.)

$\Rightarrow L(U\bar{x}) = L(\bar{y}) = \bar{b}$ ✓.

Ex $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 25 \end{bmatrix} \bar{x} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$

1) $L\bar{y} = \bar{b} \implies \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \longleftrightarrow \begin{matrix} x & = 1 \\ 2x + y & = 3 \\ 3x + 4y + z & = 7 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

2) $U\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{-2}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\rightsquigarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$; these are all the solutions.