Lecture 6 - Matrix Operations

Note Title 2/4/2008

Today we're starting a slightly different look at systems et al.

Terms form matrices:

Size = # rows × # columns

Elements are indexed by a pair of numbers: $a_{i,j}$ is the entry in row i, column j

main diagonal = entries with $i=j$: $a_{1,1}, a_{2,2}, \dots$

A & B are **equal** if they are the same size and $a_{i,j} = b_{i,j}$ $\forall i,j$

If A & B are the same size, then the sum $C = A+B$ is the matrix with

$$c_{i,j} = a_{i,j} + b_{i,j}.$$

Can also scale a matrix by scaling every entry:

$$B = cA \leftrightarrow b_{i,j} = c\,a_{i,j}$$

Ex: $A = \begin{bmatrix} \underline{0} & 1 \\ 1 & \underline{2} \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} \underline{1} & 1 \\ 2 & \underline{3} \\ 5 & 8 \end{bmatrix}$, $C = \begin{bmatrix} \underline{13} & 21 \\ 34 & \underline{55} \end{bmatrix}$ $\Rightarrow$ A 3×2, B is 3×2, C is 2×2

_ = main diagonal.

Then $a_{3,2} = 5 = b_{3,1}$ $\quad A+B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 8 & 13 \end{bmatrix}$, $A+C$ not defined.

$$2A = \begin{bmatrix} 0 & 2 \\ 2 & 4 \\ 6 & 10 \end{bmatrix} \qquad 2C = \begin{bmatrix} 26 & 42 \\ 68 & 110 \end{bmatrix}, \qquad 0 \cdot C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For every size, there is a **zero matrix** $0_{m,n}$ in which all entries are zero.

Properties:

1) $(A+B)+C = A+(B+C)$
2) $A+0 = 0+A = A$
3) $A+(-1)A = (-1)A+A = 0$
4) $A+B = B+A$
5) $1 \cdot A = A$
6) $a\cdot(b\cdot C) = (a\cdot b)\cdot C$
7) $a\cdot(B+C) = a\cdot B + a\cdot C$
8) $(a+b)\cdot C = a\cdot C + b\cdot C$

This shows that $m\times n$ matrices forms a vector space (basically $\mathbb{R}^{mn}$).

Real power comes from matrix multiplication.

If A is $m\times n$ and B is $n\times k$, then AB is defined and is $m\times k$.

Fast def: The rows of A are vectors in $\mathbb{R}^n$

$$A = \begin{bmatrix} \bar{a}^1 \\ \vdots \\ \bar{a}^m \end{bmatrix} \qquad \bar{a}^i \text{ are row vectors}$$

The columns of B are also vectors in $\mathbb{R}^n$

$$B = \begin{bmatrix} \bar{b}_1 & \cdots & \bar{b}_k \end{bmatrix}$$

The $C = AB$ has $c_{ij} = \bar{a}^i \cdot \bar{b}_j$.

Ex: 1) $\underset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}}, \underset{B}{\begin{bmatrix} 1 & 1 & 3 \\ 5 & 7 & 9 \end{bmatrix}}$ $\quad AB = \begin{bmatrix} 1\cdot 1 + 2\cdot 5 & 1\cdot 1 + 2\cdot 7 & 1\cdot 3 + 2\cdot 9 \\ 0\cdot 1 + 3\cdot 5 & 0\cdot 1 + 3\cdot 7 & 0\cdot 3 + 3\cdot 9 \end{bmatrix}$

2) $\overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}, \overset{B}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}$ $\quad AB = \begin{bmatrix} a + 2b + 3c \\ 4a + 5b + 6c \\ 7a + 8b + 9c \end{bmatrix}$

Goal: rewrite systems as $A\bar{x} = \bar{b}$, where $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\bar{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$, $A_{m\times n}$.

$\longleftrightarrow$ multiply A $m\times n$ by $\bar{x}$ $n\times 1$, getting $\bar{b}$ $m\times 1$ and

$A = [a_{ij}]$, then $A\bar{x} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

$$\longleftrightarrow \quad \left.\begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ \underbrace{a_{m1}x_1 + \cdots + a_{mn}x_n}_{\substack{\#\text{ of var } = \\ \#\text{ of col of } A}} = b_m \end{array}\right\} \text{number of eqns} = \#\text{ of rows in } A$$

Def A diagonal matrix is a square matrix such that the non-diagonal elements are 0.

An identity matrix is a diagonal matrix with ones on the diagonal.

Properties of Matrix mult:

1) $A(BC) = (AB)C$ — associativity

2) A $m \times n$ $\Rightarrow I_m \cdot A = A \cdot I_n = A$

3) $A(B+C) = AB + AC$

4) $(A+B)C = AC + BC$

$\Big\}$ matrix mult distributes over $+$

5) $A \cdot cB = cA \cdot B = c(A \cdot B)$ — matrix and scalar mults commute

Application: The solutions to a homogeneous system form a subspace.

Why? Let $\bar{x}$ and $\bar{y}$ be solutions to $A\bar{x} = \bar{0}$. Then we need to show $\bar{x} + \bar{y}$ and $k\bar{x}$ are also solutions:

$$A \cdot (\bar{x} + \bar{y}) = A\bar{x} + A\bar{y} = \bar{0} + \bar{0} = \bar{0} \quad \checkmark$$

$$A \cdot (k\bar{x}) = k(A\bar{x}) = k \cdot \bar{0} = \bar{0} \quad \checkmark$$

Could have done this directly. This was a lot easier.

One last topic: Block matrices.

Def A block partition of a matrix is a grouping of rows and columns to form submatrices.

Ex:

$$\left[\begin{array}{c|cc} 1 & 2 & 3 \\ 3 & 9 & 14 \\ \hline 0 & 1 & 6 \end{array}\right] \rightsquigarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ where } A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 3 \\ 9 & 14 \end{bmatrix}, \; C = [0], \; D = [1 \;\; 6]$$

are matrices.

Important simplification: Block matrices block multiply:

$$A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}, \quad B = \begin{bmatrix} M & N \\ U & V \end{bmatrix}, \quad AB = \begin{bmatrix} PM + QU & PN + QV \\ RM + SU & RN + SV \end{bmatrix}$$

(when they all make sense)

For this to work: # columns of P & R = # Rows M & N

columns of Q & S = # Rows of U & V

So the column partition of A must agree w/ the row partition of B.