Lecture 4 - Basis, Dimension & Dot Product

At the end of last time, we defined linear combinations of a pair of vectors. Today we will use similar ideas to get a way to describe arbitrary vectors in terms of ones we like.

Example of what we want:

$$\{\bar{e}_1=(1,0,0),\ \bar{e}_2=(0,1,0),\ \bar{e}_3=(0,0,1)\} \text{ in } \mathbb{R}^3$$

1) Every vector (a,b,c) is a linear comb of these:

$$(a,b,c) = a\bar{e}_1 + b\bar{e}_2 + c\bar{e}_3$$

2) If $a\bar{e}_1+b\bar{e}_2+c\bar{e}_3 = d\bar{e}_1+f\bar{e}_2+g\bar{e}_3$, then $a=d,\ b=f,\ c=g$. ie each choice of coefs gives a different vector.

<u>Def</u> A set of vectors $S=\{v_1,\dots\}$ spans W if every vector in W is a linear comb of vectors of S.

In other words, if $\bar{w}\in W$, then there are coefficients $a_1,\dots$ s.t. $\bar{w}=a_1\bar{v}_1+\dots$

<u>Ex</u>: • $\{\bar{e}_1,\bar{e}_2,\bar{e}_3\}$ spans $\mathbb{R}^3$ (part 1)

• $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ spans the solutions to

$$x+y+z=0: \quad \longleftrightarrow \left[\begin{array}{ccc|c}1&1&1&0\end{array}\right]$$

$\Rightarrow$ x is lead, y,z free $\Rightarrow$ $y=s,\ z=t$

$x=-s-t$ $\Rightarrow$ sol are of the form

$$(-s-t,\ s,\ t)$$

$$\begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

- $\{(2,1,1), (0,2,1), (0,0,2)\}$ spans $\mathbb{R}^3$: Have to show that we can find r,s,t s.t. $r(2,1,1)+s(0,2,1)+t(0,0,2) = (a,b,c)$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & a \\ 1 & 2 & 0 & b \\ 1 & 1 & 2 & c \end{array}\right] \begin{array}{c} \div 2 \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a/2 \\ 0 & 2 & 0 & b-a/2 \\ 0 & 1 & 2 & c-a/2 \end{array}\right] \begin{array}{c} \\ \div 2 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a/2 \\ 0 & 1 & 0 & b/2-a/4 \\ 0 & 0 & 2 & c-b/2-a/4 \end{array}\right] \begin{array}{c} \\ \\ \div 2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a/2 \\ 0 & 1 & 0 & b/2-a/4 \\ 0 & 0 & 1 & c/2-b/4-a/8 \end{array}\right] \Rightarrow \begin{array}{l} r = a/2 \\ s = b/2 - a/4 \\ t = c/2 - b/4 - a/8 \end{array}.$$

So "spans" means the system with our vectors as columns has a solution.

- $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, (1,1,1)\}$ spans $\mathbb{R}^3$

Spanning means we have enough vectors to reach every point in our space. Doesn't say anything about uniqueness.

<u>Def</u> A set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is linearly independent if

$$a\bar{v}_1 + b\bar{v}_2 + c\bar{v}_3 = \bar{0} \Rightarrow a = b = c = 0.$$

(Same def for bigger sets).

Put in words, a set is lin ind if there is a unique way to write zero a a linear comb of the vectors.

This actually implies that there is a unique way to write any vector in the span.

If $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is linearly independent, thm

$$a\bar{v}_1 + b\bar{v}_2 + c\bar{v}_3 = d\bar{v}_1 + f\bar{v}_2 + g\bar{v}_3 \Rightarrow a = d, \ldots$$

Why? Just subtract the right hand side from the left: $(a-d)\bar{v}_1 + (b-f)\bar{v}_2 + (c-g)\bar{v}_3 = 0$

Ex: • $\{(1,0), (0,1)\}$ is linearly independent:

$$a(1,0) + b(0,1) = (a,b) = (0,0) \Leftrightarrow a=b=0$$

• $\{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$ is not linearly independent:

$$\bar{e}_1 + \bar{e}_2 + \bar{e}_3 - (1,1,1) = (0,0,0).$$

linear dependence relation

can rewrite this as

$\bar{e}_1 + \bar{e}_2 + \bar{e}_3 = (1,1,1)$ so

A set is not linearly independent when one vector is a linear combination of the others.

Put together:

Def A set $\mathcal{B}$ is a basis if it is both linearly independent and spans.

Ex: • $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ is a basis for $\mathbb{R}^3$. The standard basis.

• $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the solutions to $x+y+z=0$.

Already saw it spans. Why linearly independent?

$$a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -a-b \\ b \\ a \end{bmatrix} \Rightarrow a=b=0$$

Def The size of any basis is the dimension of the space.

Ex: • $\dim \mathbb{R}^3 = 3$

• $\dim(\{x+y+z=0\}) = 2$

Remark: This fits our intuitive, geometric notion of dimension: 1 dim things are lines, 2 dim things are planes, etc.

So if $\mathcal{B}$ is a basis, then any vector is a linear combination of vectors in $\mathcal{B}$ (and uniquely so). Finding the coefficients is tough.

Dot Product

Def If $\bar{u} = (u_1, \ldots, u_n)$, $\bar{v} = (v_1, \ldots, v_n)$, then

$$\bar{u} \cdot \bar{v} = u_1 v_1 + \ldots + u_n v_n$$

Ex: • $(1,2) \cdot (3,4) = 1(3) + 2(4) = 11$

• $(1,8,16) \cdot (2,0,1) = 1 \cdot 2 + 8 \cdot 0 + 16 \cdot 1 = 18$

Properties of $\cdot$

1) $\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$

2) $\bar{u} \cdot (\bar{v} + \bar{w}) = \bar{u} \cdot \bar{v} + \bar{u} \cdot \bar{w}$

3) $(c\bar{u}) \cdot \bar{v} = c(\bar{u} \cdot \bar{v})$

4) $\bar{u} \cdot \bar{u} \geq 0$, w/ equality iff $\bar{u} = \bar{0}$.

How to see these? Expand out:

i) $\bar{u} \cdot (\bar{v} + \bar{w}) = u_1(v_1 + w_1) + \ldots + u_n(v_n + w_n)$

$= u_1 v_1 + u_1 w_1 + \ldots + u_n v_n + u_n w_n$

$= (u_1 v_1 + \ldots + u_n v_n) + (u_1 w_1 + \ldots + u_n w_n) = \bar{u} \cdot \bar{v} + \bar{u} \cdot \bar{w}$

Part 4) says that we can use $\cdot$ to get lengths:

Def The length / norm of $\bar{u}$, $|\bar{u}|$ is

$$|\bar{u}| = \sqrt{\bar{u} \cdot \bar{u}} = \sqrt{u_1^2 + \ldots + u_n^2}$$

Ex: • $\bar{u} = (3,4)$, $\quad |\bar{u}| = \sqrt{9+16} = 5$

• $\bar{v} = (3,-4,12)$ $\quad |\bar{v}| = \sqrt{9+16+144} = \sqrt{169} = 13$

Def A unit vector is any vector of length 1.

Given any vector, we can normalize to get a unit vector: $\bar{u}_{\bar{v}} = \frac{\bar{v}}{|\bar{v}|}$

Ex: $\bar{v} = (3,-4,12)$, $\quad |\bar{v}| = 13 \Rightarrow \bar{u}_{\bar{v}} = \left(\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}\right)$

The unit vectors measure just direction, rather than length.

Can get more geometry from •:

Def The angle between two vectors $\bar{u}$ and $\bar{v}$ is defined by

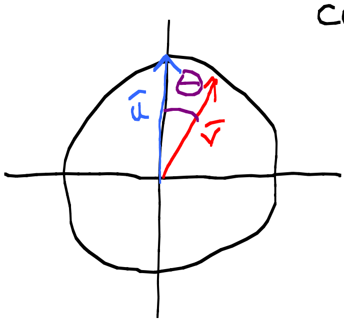
$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}|\cdot|\bar{v}|}, \quad 0 \le \theta \le \pi$$

Def Two vectors are orthogonal if $\theta = \pi/2 \longleftrightarrow \bar{u}\cdot\bar{v} = 0$

Ex: •) $\bar{u} = (3,1)$, $\bar{v} = (-1,3)$ are orthogonal

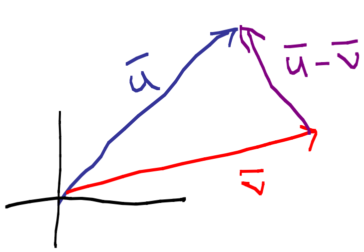
•) $\bar{u} = (0,1)$, $\bar{v} = (1/2, \sqrt{3}/2)$

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}|\cdot|\bar{v}|} = \frac{(\sqrt{3}/2)}{1\cdot 1} = \sqrt{3}/2 \Rightarrow \theta = \pi/6$$



With these notions, the linear algebra captures many of the ideas we had from ordinary geometry:

Distance between points: $d(\bar{u},\bar{v}) = |\bar{u}-\bar{v}| = \sqrt{(u_1-v_1)^2+\dots}$



This gives us things like circles:

$\{\bar{x} \mid d(\bar{x},\bar{a}) = r\}$