Lecture 3 - The vector space $\mathbb{R}^n$

Note Title 1/23/2008

Today we will review stuff we already know about $\mathbb{R}^n$, focusing on what is important for us.

2 Goals:

- Better geometric idea for spaces of sol to systems
- Good understanding of a key example for gen concepts.

Also at the end look at a specific kind of system.

Notation / Defn $\mathbb{R}$ = real numbers

$\mathbb{R}^n$ = ordered ntuples of real numbers

= all points of the form $(x_1, \dots, x_n)$

We will write points in $\mathbb{R}^n$ in 2 ways:

1) row vectors: $(1, 2, \pi)$, $(0, -1, 1, 17)$, etc

2) column vectors: $\begin{bmatrix} 1 \\ 2 \\ \pi \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 17 \end{bmatrix}$, etc

All carries the same info. We'll often use column vectors since they play well with matrices.

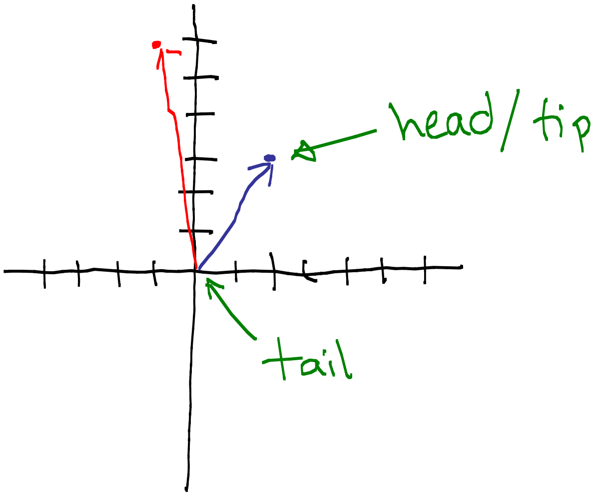
Use geometric language to talk about $\mathbb{R}^n$ and stuff within it

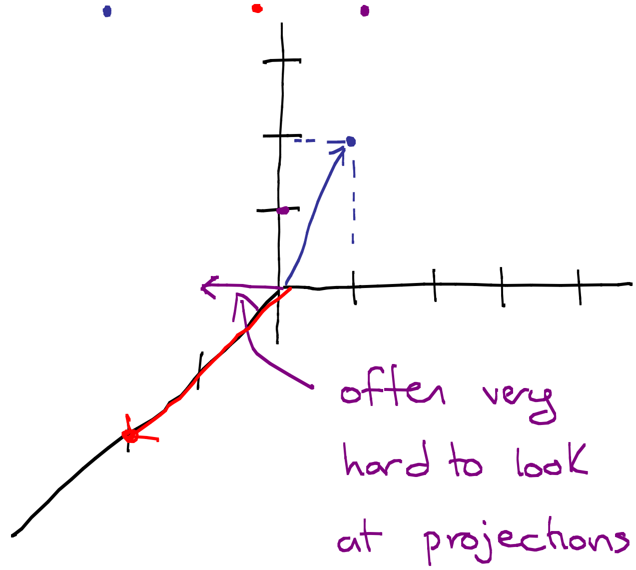
If we can understand something for $\mathbb{R}^2$ & $\mathbb{R}^3$, we can reason out how it works in general.

Want also to think of elements in $\mathbb{R}^n$ as vectors = arrows joining two points.

Ex: Plot some points/ vectors:

$\begin{bmatrix}2\\3\end{bmatrix}$, $\begin{bmatrix}-1\\6\end{bmatrix}$ in $\mathbb{R}^2$; $\begin{bmatrix}0\\1\\2\end{bmatrix}$, $\begin{bmatrix}2\\0\\0\end{bmatrix}$, $\begin{bmatrix}2\\1\\2\end{bmatrix}$ in $\mathbb{R}^3$





For us, the algebraic aspects are very important.

1) We can add vectors and get a new vector

2) We can scale vectors getting new vectors

If $\bar{v} = (a_1, a_2, \dots, a_n)$, $\bar{w} = (b_1, \dots, b_n)$, then

$$\bar{v} + \bar{w} = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

If $\bar{v} = (a_1, a_2, \dots, a_n)$, then

$$b \cdot \bar{v} = (ba_1, ba_2, \dots, ba_n)$$

Addition of vectors & multiplication by a number are all done coördinatewise.

Def Ordinary numbers are scalars and the mult of a vector by a number is scalar multiplication.

Also a geometric version of these. First need to note one thing:

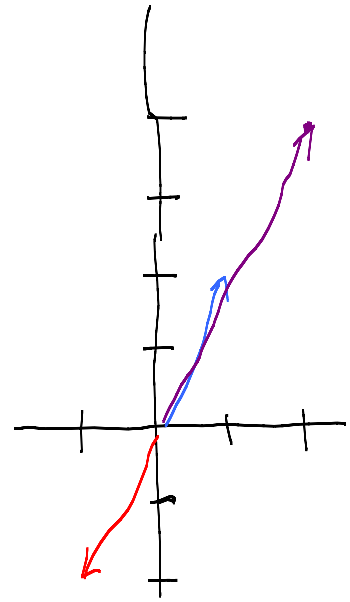
vectors are determined by their length & direction.

We'll talk more about length later. For now, take it as an intuitive notion.

Scalar multiplication just scales the length, leaving the direction fixed.

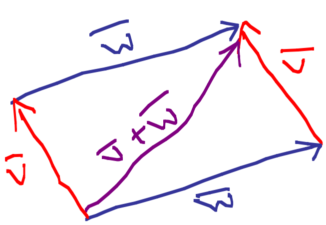
Ex: $\bar{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $2\bar{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $(-1)\bar{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

actually swapped direction, but a different kind of scaling!



So all of the scalar multiples of a vector carve out a line through the origin and the head of the vector.

Addition has a different geometric story:
place the tail of $\bar{v}$ at the head of $\bar{w}$ and draw the vector from the tail of $\bar{w}$ to the head of $\bar{v}$:



The picture shows us that the order in which we add things does not matter:

$$\bar{v} + \bar{w} = \bar{w} + \bar{v}$$

Vector addition and scalar mult are the prototype of the ops in a vector space, one of our course goals.

Focus now on some basic properties:
(vectors have bars, scalars don't)

First a special vector: $\bar{0} = (0, \dots, 0)$ = the origin

Properties:

1) $\bar{v}+(\bar{w}+\bar{u}) = (\bar{v}+\bar{w})+\bar{u}$ — Vector addition is associative

2) $\bar{0}+\bar{v} = \bar{v}+\bar{0} = \bar{v}$ — $\bar{0}$ is a unit for addition

3) $\bar{v}+(-1)\bar{v} = \bar{0} = (-1)\bar{v}+\bar{v}$ — Have additive inverses

4) $\bar{v}+\bar{w} = \bar{w}+\bar{v}$ — addition is commutative

5) $1\cdot\bar{v} = \bar{v}$

6) $a\cdot(b\cdot\bar{v}) = (a\cdot b)\cdot\bar{v}$

7) $(a+b)\cdot\bar{v} = a\cdot\bar{v}+b\cdot\bar{v}$ — distributive property

8) $a\cdot(\bar{v}+\bar{w}) = a\cdot\bar{v}+a\cdot\bar{w}$ — "

How would we show these? By writing out the coordinates.

Ex: 7) $(a+b)\cdot\underbrace{\langle c_1,\dots,c_n\rangle}_{\bar{v}} = \langle (a+b)c_1,\dots,(a+b)c_n\rangle =$

$(ac_1+bc_1,\dots,ac_n+bc_n) = (ac_1,\dots,ac_n)+(bc_1,\dots,bc_n)$

$a(c_1,\dots,c_n)+b(c_1,\dots,c_n) = a\cdot\bar{v}+b\cdot\bar{v}$ ✓

Def A linear combination of vectors is any expression of the form

$$a\bar{v}+b\bar{w}$$

So forming linear combinations is a way to make new vectors from old ones.

Ex: $\bar{v} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\bar{w} = \begin{bmatrix}0\\1\\0\end{bmatrix}$. $3\bar{v}+5\bar{w} = \begin{bmatrix}3\\0\\3\end{bmatrix}+\begin{bmatrix}0\\5\\0\end{bmatrix} = \begin{bmatrix}3\\5\\3\end{bmatrix}$, $2\bar{v}-\bar{w} = \begin{bmatrix}2\\0\\2\end{bmatrix}-\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\-1\\2\end{bmatrix}$

$$r\bar{v}+s\bar{w} = \begin{bmatrix}r\\0\\r\end{bmatrix}+\begin{bmatrix}0\\s\\0\end{bmatrix} = \begin{bmatrix}r\\s\\r\end{bmatrix}$$

Linear combinations help us see when subsets of $\mathbb{R}^n$ share the same basic properties.

<u>Ex</u>: • $\mathbb{R}^1$ sits inside $\mathbb{R}^2$ as the x-axis, and we can add vectors on the x-axis and it stays on the x-axis.

• $\mathbb{R}^2$ sits inside $\mathbb{R}^3$ as the (x,y)-plane, and same story.

<u>Non-examples</u>: • The positive real numbers sit inside $\mathbb{R}^1$, and if we add them, it is still in $\mathbb{R}^1$. If we mult by -1, trouble.

• The x & y axis in $\mathbb{R}^2$ can be scaled but not added.

Want to single out those pieces that basically look like $\mathbb{R}^n$.

<u>Def</u> A collection of vectors V is a subspace of $\mathbb{R}^n$ if any linear combination of vectors in V is again in V.

In other words, $\bar{v}, \bar{w} \in V$, then $a\bar{v} + b\bar{w} \in V$.

<u>Ex</u>: Lines & planes through the origin in $\mathbb{R}^3$.

<u>Ex</u>: If $\bar{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and if $V = \{a \cdot \bar{v}\} = \left\{ \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} \middle| \, a \text{ is anything} \right\}$,

then V is a subspace:

$$r\bar{v}, s\bar{v} \in V, \text{ then is } a(r\bar{v}) + b(s\bar{v}) \in V?$$

$a(r\bar{v}) + b(s\bar{v}) = (ar)\bar{v} + (bs)\bar{v} = (ar+bs)\bar{v}$, so yes, this is again in V.

<u>Ex</u>: Is $V = \left\{ \begin{bmatrix} a \\ 1 \end{bmatrix} \right\}$ a subspace of $\mathbb{R}^2$?

No!! $2\begin{bmatrix} a \\ 1 \end{bmatrix} = \begin{bmatrix} 2a \\ 2 \end{bmatrix}$, and since $1 \neq 2$, done.

Lots of subspaces arise in a <u>very</u> natural context.

<u>Def</u> A system of equations

$$\begin{aligned} a_1 x + \cdots &= b_1 \\ &\vdots \\ a_n x + \cdots &= b_n \end{aligned}$$

is homogeneous if $b_1 = b_2 = \cdots = b_n = 0$.

<u>Remark</u> Homogeneous systems are always consistent: the point $(0, \ldots)$ is always a solution.

<u>Thm</u> The set of all solutions to a homogeneous system is a subspace of $\mathbb{R}^n$.

<u>Ex:</u>

$$-3 \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 3 & 7 & 9 & 0 \end{array}\right] -2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] -2$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

x, y lead
z free

$\rightarrow$

$z = s$
$x = -3s$
$y = 0$

solutions all look like $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3s \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, so the line through the particular solution $(-3, 0, 1)$.