Lecture 24 - Linear Transformations & Matrices

Today we show how linear transformations become identified with matrices after a choice of basis.

Big result of last time:

<u>Thm</u> If $B = \{\bar{v}_1, \dots, \bar{v}_n\}$ is a basis of V, then the map

$$L_B : \mathbb{R}^n \longrightarrow V$$
$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \longmapsto a_1 v$$

is an isomorphism.

In other words, the assignment $\bar{v} \longleftrightarrow [\bar{v}]_B$ preserves all of the algebraic structure.

If $L : V \longrightarrow W$, and if B is a basis for V, B' a basis for W, then we want to complete the square:

A a lin trans $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\longleftrightarrow$ an $m \times n$ matrix.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, dashed, "A"] \arrow[d, leftrightarrow] & \mathbb{R}^m \arrow[d, leftrightarrow] \\ V \arrow[r, "L"] & W \end{tikzcd}$$

In other words, we want to find an A s.t.

$$[L(\bar{v})]_{B'} = A \cdot [\bar{v}]_B$$

As with $\mathbb{R}^n \longrightarrow \mathbb{R}^m$, it suffices to work with a basis:

<u>Def</u> If $L : V \longrightarrow W$, $B = \{\bar{v}_1, \dots, \bar{v}_n\}$ a basis for V, B' a basis for W, then ${}_{B'}[L]_B$, the <u>matrix of L</u> w.r.t the B & B' bases is the $m \times n$ matrix whose j^{th} column is $[L(\bar{v}_j)]_{B'}$.

Since $L(\bar{v}_j) \in W$, it makes sense to form $[L(\bar{v}_j)]_{B'}$.

<u>Ex</u>: $V = P_2(x)$, $W = P_3(x)$, $L : V \to W$ is

$$L(p) = \int_0^x p(t)\,dt.$$

V basis: $\{1, x, x^2\}$ $\qquad$ W basis: $\{1, x, x^2, x^3\}$

$L(1) = x$

$L(x) = x^2/2$

$L(x^2) = x^3/3$

$$\rightsquigarrow \quad [L] = \begin{array}{cc} & \begin{matrix} 1 & x & x^2 \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2 \\ x^3 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \end{array}$$

We again label the rows and columns to make things a little easier.

Ex 2:

$L: P_2(x) \to P_2(x)$ given by

$L(p) = p'' - 2p' + p.$

Source basis: $\mathcal{E} = \{1, x, x^2\}$

$\mathcal{B} = \{1, x, x^2\}$

$\mathcal{B}' = \{1, x-2, x^2-4x+2\}$

$L(1) = 1$

$L(x) = x - 2$

$L(x^2) = x^2 - 4x + 2$

$$\Rightarrow \quad {}_{B}[L]_{\mathcal{E}} = \begin{array}{cc} & \begin{matrix} 1 & x & x^2 \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2 \end{matrix} & \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \quad \text{while}$$

$${}_{B'}[L]_{\mathcal{E}} \quad \begin{array}{cc} & \begin{matrix} 1 & x & x^2 \end{matrix} \\ \begin{matrix} 1 \\ x-2 \\ x^2-4x+2 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

So choosing our basis correctly makes our matrix easier.

<u>Change of Basis</u>:

${}_{B'}[L]_B$ represents L, taking coordinates from B basis to those in the B' basis.

We can use this to get ${}_{C'}[L]_C$ for any C, C':

<u>Prop</u>: If $L: V \to W$, B, C are bases for V, B', C' bases for W, then

$${}_{B'}[L]_B = {}_{B'}C_{C'}\ {}_{C'}[L]_C\ {}_C C_B$$

"translator" from C' to B'

what we already know

"translator" from B to C

Thus if we know one matrix form, we know all of them.

When $V = W$, we normally choose $B = B'$.

Def Two $n \times n$ matrices A, B are <u>similar</u> if there is a C s.t.

$$B = C^{-1} \cdot A \cdot C.$$

Thus if $\mathcal{B}$ and $\mathcal{B}'$ are bases of V, then ${}_{\mathcal{B}}[L]_{\mathcal{B}}$ and ${}_{\mathcal{B}'}[L]_{\mathcal{B}'}$ are always similar.

Important example: Diagonalization.

Saw that if $A = A^t$, then there is a matrix C s.t.

$$C^{-1} \cdot A \cdot C = D \leftarrow \text{diagonal.}$$

The columns of C are eigenvectors of A, and the entries of D are the eigenvalues.

Stated differently: $L_A : \mathbb{R}^n \to \mathbb{R}^n$, $\bar{v} \mapsto A\bar{v}$ is linear. Let $\mathcal{E}$ be the standard basis, and let $\mathcal{B}$ be a basis of eigenvectors. Then

$${}_{\mathcal{E}}[L_A]_{\mathcal{E}} = A \quad \leftarrow \text{This is always true.}$$

while:

$${}_{\mathcal{B}}[L_A]_{\mathcal{B}} = D \quad \left(\text{since } A \cdot \bar{v}_i = \lambda_i \cdot \bar{v}_i \longleftrightarrow \begin{array}{cc} \begin{matrix} \bar{v}_1 \\ \vdots \\ \bar{v}_i \\ \vdots \\ \bar{v}_n \end{matrix} & \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} \end{array} \right)$$

So
$$D = {}_{\mathcal{B}}[L_A]_{\mathcal{B}} = {}_{\mathcal{B}}C_{\mathcal{E}} \cdot {}_{\mathcal{E}}[L_A]_{\mathcal{E}} \cdot {}_{\mathcal{E}}C_{\mathcal{B}}$$
$$= \left({}_{\mathcal{E}}C_{\mathcal{B}}\right)^{-1} \cdot A \cdot \left({}_{\mathcal{E}}C_{\mathcal{B}}\right)$$
$$= C^{-1} \cdot A \cdot C,$$

where the columns of C are again eigenvectors.

Can always remember by labeling:

$\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ $\qquad \mathcal{C} = \{\bar{c}_1, \ldots, \bar{c}_m\}$

$\mathcal{B}' = \{\bar{b}_1', \ldots, \bar{b}_n'\}$ $\qquad \mathcal{C}' = \{\bar{c}_1', \ldots, \bar{c}_m'\}$

then

$${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{array}{cc} & \bar{b}_1 \cdots \bar{b}_n \\ \begin{matrix} \bar{c}_1 \\ \vdots \\ \bar{c}_m \end{matrix} & \begin{bmatrix} & & \\ & L & \\ & & \end{bmatrix} \end{array} = \begin{array}{cc} & \bar{c}_1' \cdots \bar{c}_m' \\ \begin{matrix} \bar{c}_1 \\ \vdots \\ \bar{c}_m \end{matrix} & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array} \cdot \begin{array}{cc} & \bar{b}_1' \cdots \bar{b}_n' \\ \begin{matrix} \bar{c}_1' \\ \vdots \\ \bar{c}_m' \end{matrix} & \begin{bmatrix} & & \\ & L & \\ & & \end{bmatrix} \end{array} \cdot \begin{array}{cc} & \bar{b}_1 \cdots \bar{b}_n \\ \begin{matrix} \bar{b}_1' \\ \vdots \\ \bar{b}_n' \end{matrix} & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array}$$

The entries of the same color must match up.