Lecture 22 - Quadratic Forms

Note Title 4/26/2008

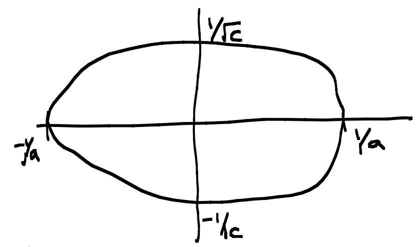
Today we'll look at a geometric application of diagonalization

Def A quadratic form in x & y is an expression of the form

$$Q(x,y) = ax^2 + bxy + cy^2$$

Equations of the form $Q(x,y) = d$ give rise to conic sections. Easy to see when $b=0$, $d=1$.

Case 1: $a, c > 0$: ellipse: $\frac{x^2}{(1/a)} + \frac{y^2}{(1/c)} = 1$

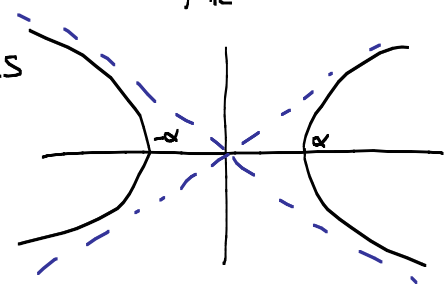


Case 2: $a > 0 > c$: hyperbola hitting x-axis

$\alpha = \sqrt{1/a}$

$\gamma = \sqrt{-1/c}$

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\gamma^2} = 1$$

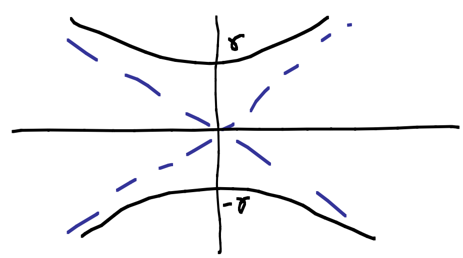


Case 3: $a < 0 < c$: hyperbola hitting y-axis

$\alpha = \sqrt{-1/a}$

$\gamma = \sqrt{1/c}$

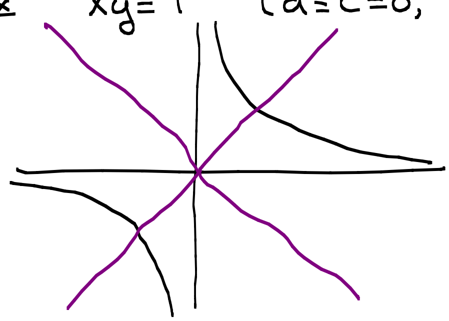
$$\frac{y^2}{\gamma^2} - \frac{x^2}{\alpha^2} = 1$$



If $a, c < 0$, no solutions & if a or $c = 0$, only lines.

If $d \neq 1$ (or 0), then can always divide by d to get an equation of this form.

Ex $xy = 1$ $(a = c = 0,\ b = d = 1)$



This looks like a hyperbola rotated 45°

Our goal is to change coordinates to transform $Q(x,y) = 1$ into $Q'(x', y') = 1$

easy (no b).

Def If $Q(x,y) = ax^2 + bxy + cy^2$, the the associated matrix is $\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$.

notice the $1/2$ part!

This has the property that $Q(x,y) = [x \quad y]\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$= \bar{x}^t A \bar{x}.$$

A is symmetric, so we know we can find an orthogonal matrix C (whose columns are eigenvectors) s.t.

$$C^t A C = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = D$$

In other words, $A = CDC^t$, so

$$\bar{x}^t \cdot A \cdot \bar{x} = \bar{x}^t \cdot (C \cdot D \cdot C^t) \cdot \bar{x} = (\bar{x}^t C) \cdot D \cdot (C^t \cdot \bar{x})$$
$$= (C^t \cdot \bar{x})^t \cdot D \cdot (C^t \cdot \bar{x})$$
$$= (\bar{x}')^t \cdot D \cdot \bar{x}'$$

In other words, with the new coordinates given by the columns of C (our "change of basis matrix" in the language of Lecture 23), Q is exceptionally nice:

$$Q(x', y') = \bar{x}'^t \cdot D \cdot \bar{x}' = \lambda_1 x'^2 + \lambda_2 y'^2.$$

$\Rightarrow$ eigenvalues of A tell us the shape of $Q(x,y) = 1$:

Eigenvalues are...	Graph is ...
both positive	ellipse
one pos, one neg	hyperbola
both neg	(no solutions)
one zero	degenerate (lines)

Now an example:

$$Q(x,y) = 2x^2 + 2xy + 2y^2 = 1$$

So

$$Q(x,y) = \bar{x}^t A \bar{x} \quad \text{where} \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Diagonalize A: $\det(A - \lambda I) = (\lambda - 3)(\lambda - 1)$

we want unit vectors to ensure C is orthogonal.

$\lambda = 1$: $A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, so $(A-I)\bar{v} = \bar{0} \Leftrightarrow \bar{v} = \begin{bmatrix} r \\ -r \end{bmatrix}$, so take $\bar{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\lambda = 3$: $A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, so $(A-3I)\bar{v} = \bar{0} \Leftrightarrow \bar{v} = \begin{bmatrix} r \\ r \end{bmatrix}$, so take $\bar{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

Then $C = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $C^t = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, and $C^t \cdot A \cdot C = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

order here corresponds to the order of the eigenvectors as columns.

So $Q(x,y) = \bar{x}'^t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \cdot \bar{x}'$

$= x'^2 + 3y'^2$

& $Q(x,y) = 1$ is an ellipse: