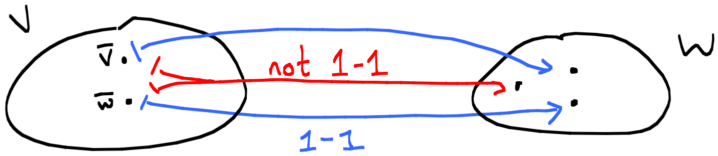
Lecture 21 - 1-1, Onto, & Systems of Equations

Note Title 4/7/2008

Def A linear transformation is 1-1 (or injective) if

$$L(\bar{v}) = L(\bar{w}) \Rightarrow \bar{v} = \bar{w}.$$



The vector space structure makes it easier for us to check this:

Prop $L: V \to W$ is 1-1 $\longleftrightarrow$ $\ker(L) = \{\bar{0}\}$.

Pf L 1-1 $\Rightarrow$ If $\bar{v} \in \ker(L)$, then $L(\bar{v}) = \bar{0} = L(\bar{0}) \Rightarrow \bar{v} = \bar{0}$ (by 1-1).

If $L(\bar{v}) = L(\bar{w})$, then $\bar{0} = L(\bar{v}) - L(\bar{w}) = L(\bar{v} - \bar{w}) \Rightarrow \bar{v} - \bar{w} \in \ker(L)$

So if $\ker(L) = \{\bar{0}\}$, then $L(\bar{v}) = L(\bar{w}) \Rightarrow \bar{v} - w = \bar{0} \Rightarrow \bar{v} = \bar{w}$. $\boxtimes$

The argument actually shows that any two $\bar{v}, \bar{w}$ s.t. $L(\bar{v}) = L(\bar{w})$ differ by an element of $\ker(L)$. We'll return to this.

Ex: $L_2: P_2(x) \to P_3(x)$, $L_2(p) = \int_0^x p(t)\,dt$ from last time. Saw $\ker(L_2) = \{\bar{0}\}$, so L_2 is injective. Also see by direct computation that $\{1, x, x^2\} \longmapsto \{x, \frac{x^2}{2}, \frac{x^3}{3}\}$ which is lin. ind, though not a basis.

1-1 transforms preserve lin ind:

Prop If L is 1-1 & $\{\bar{x}_1, \dots, \bar{x}_k\}$ is lin ind, then $\{L(\bar{x}_1), \dots, L(\bar{x}_k)\}$ is lin. ind.

Pf Look at

$$a_1 L(\bar{x}_1) + \dots + a_k L(\bar{x}_k) = \bar{0}$$
$$\|$$
$$L(a_1\bar{x}_1 + \dots + a_k\bar{x}_k)$$

$$L \text{ 1-1} \Rightarrow a_1\bar{x}_1 + \dots + a_k\bar{x}_k = \bar{0}$$

$$\{\bar{x}_1, \dots, \bar{x}_k\} \text{ lin. ind} \Rightarrow a_1 = \dots = a_k = 0.$$ $\boxtimes$

So a 1-1 transform preserves information in V.

Companion topic is onto.

Def $L: V \to W$ is onto (surjective) if $\text{Range}(L) = W$.

Ex: $L_1 : P_2(x) \to P_1(x)$, $L_1(p) = p'(x)$. Then L_1 is surjective. Also see that
$\{1, x, x^2\} \longmapsto \{0, 1, 2x\}$ which isn't lin ind ($0 \in$ set), but does span.

Then we have a companion to the previous prop:

Prop If $L: V \to W$ is onto & $\{\bar{x}_1, \ldots, \bar{x}_k\}$ spans V, then $\{L(\bar{x}_1), \ldots, L(\bar{x}_k)\}$ spans W.
An onto transformation sees all of the information in W.

Together they say that V and W are essentially the same.

Def A linear transformation $S: V \to W$ is invertible if there is a $T: W \to V$ s.t.

$$S(T(\bar{w})) = \bar{w} \quad \text{for all } \bar{w} \in W \text{ and}$$

$$T(S(\bar{v})) = \bar{v} \quad \text{for all } \bar{v} \in V.$$

This is also called an isomorphism and V & W are said to be isomorphic.

Ex: $T: \mathbb{R}^2 \to P_1(x)$, $(a,b) \longmapsto ax+b$ is invertible. Let $S: P_1(x) \to \mathbb{R}^2$, $ax+b \longmapsto (a,b)$.

Then $S(T(a,b)) = S(ax+b) = (a,b)$;
$T(S(ax+b)) = T(a,b) = ax+b$.

Prop T is invertible iff T is 1-1 and onto.
So an isomorphism is an identification of V with W in a way that preserves all structure.

General Systems of equations.
If $L: V \to W$ is a linear transformation, then any equation of the form

$$L(\bar{x}) = \bar{b}$$

is a system of equations.

This is consistent with the $\mathbb{R}^n$ case. Here

$$L(\bar{x}) = \bar{b} \longleftrightarrow A\bar{x} = \bar{b} \quad \text{some } A \text{ specified by } L.$$

Our intuition from this case tells us what we should expect in general.

Prop: Any two solutions to $L(\bar{x}) = \bar{b}$ differ by an element of $\ker(L)$.
Pf if $L(\bar{v}) = \bar{b}$ and $L(\bar{w}) = \bar{b}$, then $L(\bar{v} - \bar{w}) = L(\bar{v}) - L(\bar{w}) = \bar{b} - \bar{b} = \bar{0}$. $\square$

This helps us make sense of material from other classes like:

Ex Let $D: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be differentiation. Then any solution to a system like

$(D^2 - 2D - 3Id)(f) = 2x$ is of the form
$\overset{\shortparallel}{f''(x) - 2f'(x) - 3f(x)}$

$y = y_h + y_p$, where

y_p is some fixed sol & y_h is a sol

to $f'' - 2f' - 3f = 0$.

(In this case $\ker(D^2 - 2D - 3Id) = \text{Span}(e^{-x}, e^{3x})$, and a choice of y_p is given by $-(2/3)x + 4/9$)

Our previous prop. shows this holds in general.

Prop If $\bar{v}$ is a solution to $L(\bar{x}) = \bar{b}$, then $\bar{v} = \bar{v}_h + \bar{v}_p$ for some fixed sol $\bar{v}_p$ and some $\bar{v}_h \in \ker(L)$.

Thus all solutions are of the form $\bar{v}_p + \bar{v}_h$, so $\{\text{Sols}\} = \{\bar{v}_p + \bar{v}_h \mid \bar{v}_h \in \ker(L)\} \underset{\substack{\uparrow \\ \text{by def}}}{=} \bar{v}_p + \ker(L)$.