Lecture 19 – Projections & Gram-Schmidt

Note Title 3/26/2008

It's in general very hard to write a vector as a linear comb of others.

If we have a little extra structure, it is a lot easier.

Recall on $\mathbb{R}^n$, we have the dot product: $\bar{u} = [a_1 \dots a_n]$, $\bar{v} = [b_1 \dots b_n]$,

$$\bar{u} \cdot \bar{v} = a_1 b_1 + \dots + a_n b_n.$$

We also said that $\bar{u}$ and $\bar{v}$ are orthogonal if $\bar{u} \cdot \bar{v} = 0$.

With $\cdot$, it is easier to find the coefficients.

<u>Def</u> A set of vectors $\{\bar{v}_1, \dots, \bar{v}_n\}$ is <u>orthonormal</u> if

1) $\bar{v}_i \cdot \bar{v}_j = 0 \quad i \neq j \quad$ (<u>ortho</u>gonal)
2) $\bar{v}_i \cdot \bar{v}_i = 1 \quad$ (normal)

So orthonormal = orthogonal set of unit vectors.

Why is this nice?

Let $\bar{u} = a_1 \bar{v}_1 + \dots + a_n \bar{v}_n$. Then

$$u \cdot \bar{v}_i = (a_1 \bar{v}_1 + \dots + a_n \bar{v}_n) \cdot \bar{v}_i = (a_1 \bar{v}_1) \cdot \bar{v}_i + \dots + (a_n \bar{v}_n) \cdot \bar{v}_i = a_1 (\bar{v}_1 \cdot \bar{v}_i) + \dots + a_n (\bar{v}_n \cdot \bar{v}_i)$$
$$= a_i (\bar{v}_i \cdot \bar{v}_i) = a_i$$

So knowing that $\{\bar{v}_1, \dots, \bar{v}_n\}$ is orthonormal lets us recover the coefficients!

<u>Ex</u>: $\bar{v}_1 = [1/\sqrt{2}, 1/\sqrt{2}]$, $\bar{v}_2 = [1/\sqrt{2}, -1/\sqrt{2}]$

1) $\{\bar{v}_1, \bar{v}_2\}$ is a basis for $\mathbb{R}^2$: $\begin{vmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{vmatrix} = -1/2 - 1/2 = -1$

$\Rightarrow$ $\bar{v}_1$ & $\bar{v}_2$ span $\mathbb{R}^2$. So given a vector, we can find the coefs:

$$\bar{v} = [2, 0]: \quad \left.\begin{aligned} \bar{v} \cdot \bar{v}_1 &= 2/\sqrt{2} = \sqrt{2} \\ \bar{v} \cdot \bar{v}_2 &= 2/\sqrt{2} = \sqrt{2} \end{aligned}\right\} \Rightarrow \begin{aligned} \bar{v} &= \sqrt{2}\,[1/\sqrt{2}, 1/\sqrt{2}] + \sqrt{2}\,[1/\sqrt{2}, -1/\sqrt{2}] \\ &= \sqrt{2}\,\bar{v}_1 + \sqrt{2}\,\bar{v}_2 \end{aligned}$$

Given arbitrary basis vectors $\bar{v}_1, \bar{v}_2$, it'd be very hard to find the coefs. We'll see soon how to replace a basis with an orthonormal one like this.

<u>Projections</u>

Let $\bar{v} \in \mathbb{R}^n$ be a non-zero vector. We can look at the line through $\bar{v}$.

Def Let $\bar{u} \in \mathbb{R}^n$. The projection of $\bar{u}$ onto $\bar{v}$ is

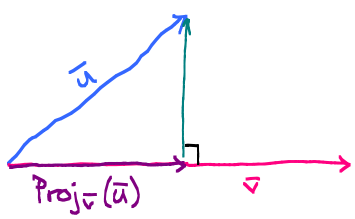
$$\text{Proj}_{\bar{v}} \bar{u} = \frac{\langle \bar{u}, \bar{v} \rangle}{\langle \bar{v}, \bar{v} \rangle} \bar{v}$$

$$= \left\langle \bar{u}, \frac{\bar{v}}{\|\bar{v}\|} \right\rangle \frac{\bar{v}}{\|\bar{v}\|}$$

unit vector
in $\bar{v}$ direction

If $\bar{v}$ is a unit vector:

$$\text{Proj}_{\bar{v}}(\bar{u}) = (\bar{u} \cdot \bar{v}) \bar{v}$$



$\text{Proj}_{\bar{v}}(\bar{u})$ is the part of $\bar{u}$ in the direction of $\bar{v}$

Better said, $\text{Proj}_{\bar{v}}(\bar{u}) \cdot \bar{v} = \bar{u} \cdot \bar{v}$

$\Rightarrow$ $\bar{u} - \text{Proj}_{\bar{v}}(\bar{u})$ and $\bar{v}$ are orthogonal.

This is the heart of the Gram-Schmidt Method:

Given a basis $\{\bar{v}_1, \dots, \bar{v}_n\}$, we'll produce an orthonormal one.

1) $\bar{w}_1 = \bar{v}_1$

$\bar{u}_1 = \bar{w}_1 / \|\bar{w}_1\|$

2) $\bar{w}_2 = \bar{v}_2 - \text{Proj}_{\bar{u}_1} \bar{v}_2 = \bar{v}_2 - \langle \bar{v}_2, \bar{u}_1 \rangle$

$\bar{u}_2 = \bar{w}_2 / \|\bar{w}_2\|$

3) $\bar{w}_3 = \bar{v}_3 - \text{Proj}_{\bar{u}_1} \bar{v}_3 - \text{Proj}_{\bar{u}_2} \bar{v}_3$

$\bar{u}_3 = \bar{w}_3 / \|\bar{w}_3\|$

4) $\vdots$

n) $\bar{w}_n = \bar{v}_n - \text{Proj}_{\bar{u}_1} \bar{v}_n - \dots - \text{Proj}_{\bar{u}_{n-1}} \bar{v}_n$

$\bar{u}_n = \bar{w}_n / \|\bar{w}_n\|$

Then $\{\bar{u}_1, \dots, \bar{u}_n\}$ is an orthonormal basis.

Ex: $\underset{\bar{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\bar{v}_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\bar{v}_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$

$\bar{w}_1 = \bar{v}_1$. $\quad w_2 = \bar{v}_2 - \dfrac{\overset{2}{\langle \bar{w}_1, \bar{v}_2 \rangle}}{\underset{3}{\langle \bar{w}_1, \bar{w}_1 \rangle}}$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$\bar{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\langle \bar{w}_1, \bar{v}_3 \rangle}{\langle \bar{w}_1, \bar{w}_1 \rangle} \bar{w}_1 - \frac{\langle \bar{w}_2, \bar{v}_3 \rangle}{\langle \bar{w}_2, \bar{w}_2 \rangle} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} - \frac{-1/3}{2/3} \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} + \begin{bmatrix} 1/6 \\ 1/6 \\ -1/3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}$$