# Lecture 13 - Determinant & Inverses



Last time finished with properties of the determinant. In particular, we saw easy rules to understand what happens when we apply elementary operations. Two important consequences:

1) If we repeat a row or column, the det = 0.

2) If one row is a scalar multiple of another, then the det = 0.

Ex:

$$A = \begin{bmatrix} 73 & 81 & 9 & 6 \\ 14 & 97 & -1 & 0 \\ 6 & 8 & 5 & 3 \\ 12 & 16 & 10 & 6 \end{bmatrix} \qquad \text{Row}_4 = 2\,\text{Row}_3 \Rightarrow \det(A) = 0.$$

Why? 1) → 2) by factoring out the scalar.

1) Follows because swapping the equal rows leaves the matrix fixed while swapping the sign of the det.

Can further simplify our lives by performing a few row operations:

If $U$ is upper triangular, then $\det(U)$ = product of diagonal entries.

So we can find $\det(A)$ by row reducing down to an upper triangular matrix:

- adding rows to other rows does nothing
- swapping rows swaps sign:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & -2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$|A| = |\,''\,| = -|\,''\,| = -(1\cdot(-2)\cdot(-2)) = -4$$

This is a relatively useful method for finding $|\cdot|$.

Det has 1 more property: "linear in each row":

$$A = \begin{bmatrix} \vec{a}^1 \\ \vdots \\ \vec{a}^n \end{bmatrix}.$$

If $B = \begin{bmatrix} \overline{b}^1 \\ \vdots \\ \overline{b}^i + \overline{a}^i \\ \vdots \\ \overline{b}^n \end{bmatrix}$, then $|B| = \begin{vmatrix} \overline{b}^1 \\ \vdots \\ \overline{b}^i \\ \vdots \\ \overline{b}^n \end{vmatrix} + \begin{vmatrix} \overline{b}^1 \\ \vdots \\ \overline{a}^i \\ \vdots \\ \overline{b}^n \end{vmatrix}$.

This, together with "repeated rows give zero" and $|I| = 1$ determine det!

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Last time finished with: $A$ invertible $\Rightarrow$ $\det(A) \neq 0$.

Today we will show the converse and give a formula for $A^{-1}$ if $\det(A) \neq 0$!

<u>Def</u> If $A$ is $n \times n$, the <u>matrix of cofactors</u> is the ... matrix of cofactors of elements of $A$:

$$C = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}$$

The <u>adjoint</u> of $A$, $\mathrm{adj}(A)$ is defined by $\mathrm{adj}(A) = C^t$.

The importance of $\mathrm{adj}(A)$ is the following:

$$A \cdot \mathrm{adj}(A) = |A| \cdot I$$

<u>Proof</u>:

The $(i,j)^{th}$ entry of $A \cdot \mathrm{adj}(A)$ is $\mathrm{row}_i(A) \cdot \mathrm{column}_j(\mathrm{adj}(A))$.

$$= [a_{i1} \; \cdots \; a_{in}] \cdot \begin{bmatrix} C_{j1} \\ \vdots \\ C_{jn} \end{bmatrix} = a_{i1} C_{j1} + \cdots + a_{in} C_{jn}.$$

For $i = j$, this is $a_{i1} C_{i1} + \cdots + a_{in} C_{in} = |A|$.

For $i \neq j$, this is the determinant of the matrix we get by replacing row $j$ with a copy of row $i$. $\Rightarrow$ for $i \neq j$, this sum is $0$! ▨

What do we get?

If $|A| \neq 0$, then $A^{-1} = \frac{1}{|A|} \mathrm{adj}(A)$.

This is a beautiful, theoretical result that holds very generally! In particular, if the entries of $A$ are integers and $|A| = \pm 1$, then the entries of $A^{-1}$ are integers!

<u>Ex</u>: $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$  $|A| = 3 - 2 = 1$.  $C_{11} = 1$, $C_{12} = -1$, $C_{21} = -2$, $C_{22} = 3$ $\Rightarrow$

$$C = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \Rightarrow \mathrm{adj}(A) = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Also gives another test for linear independence:

$\{\overline{v}_1, \ldots, \overline{v}_n\}$ in $\mathbb{R}^n$ is lin ind if and only if $\left| \overline{v}_1 \cdots \overline{v}_n \right| \neq 0$.

Better stated:

$A\bar{x} = \bar{b}$ has a unique solution iff $|A| \neq 0$.

There is actually a formula for finding the inverse in terms of the entries of A:

Cramer's Rule.

Let $B_i$ be the matrix we get by replacing column $i$ of A with $\bar{b}$. Then

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{where} \quad x_i = \frac{|B_i|}{|A|}.$$