# Lecture 12 - Determinants

Note Title 2/25/2008

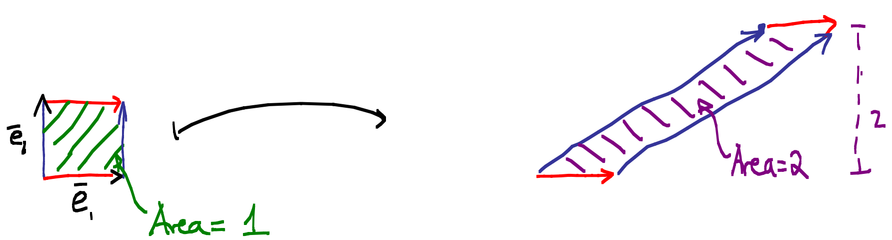
Today we're starting a very useful tool for seeing when a matrix is invertible: determinant.

**Def:** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

**Ex:**

- $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = -2$
- $\begin{vmatrix} 0 & 0 \\ 6 & 18 \end{vmatrix} = 0\cdot 18 - 0\cdot 6 = 0$
- $\begin{vmatrix} 1 & 6 \\ 3 & 18 \end{vmatrix} = 1\cdot 18 - 3\cdot 6 = 0$

What does $|\cdot|$ measure? Scaling $\quad A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \quad |A| = 2.$




For $n\times n$ matrices, we work inductively.

**Def** If $A$ is $n\times n$, let $A_{i,j}$ be the $(n-1)\times(n-1)$ matrix we get by removing row $i$ and col $j$.

- The minor of $a_{i,j}$ is $M_{i,j} = |A_{i,j}|$
- The cofactor is $C_{i,j} = (-1)^{i+j}|A_{i,j}| = (-1)^{i+j} M_{i,j}$

**Ex:** $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \\ 13 & 21 & 34 \end{bmatrix}$

$M_{2,1} = \begin{vmatrix} 1 & 2 \\ 21 & 34 \end{vmatrix} = 34 - 42 = -8 \qquad M_{1,3} = \begin{vmatrix} 3 & 5 \\ 13 & 21 \end{vmatrix} = 63 - 65 = -2$

$C_{2,1} = (-1)^{2+1} M_{2,1} = 8 \qquad C_{1,3} = (-1)^{1+3} M_{1,3} = -2$

**Def** If $A$ is $n\times n$, then $|A| = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$

$$= a_{11}C_{11} + a_{21}C_{21} + \ldots + a_{n1}C_{n1}$$

**Ex:** $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \\ 13 & 21 & 34 \end{bmatrix}$

$$|A| = a_{11}\underline{C_{11}} + a_{12}\underline{C_{12}} + a_{13}\underline{C_{13}}$$

$$= \left(1\cdot(-1)^{1+1}\begin{vmatrix} 5 & 8 \\ 21 & 34 \end{vmatrix}\right) + \left(1\cdot(-1)^{1+2}\begin{vmatrix} 3 & 8 \\ 13 & 34 \end{vmatrix}\right) + \left(2\cdot(-1)^{1+3}\begin{vmatrix} 3 & 5 \\ 13 & 21 \end{vmatrix}\right)$$

$$= (170 - 168) + (-(102 - 104)) + (2\cdot(-2)) = 0.$$

Thm Could use any row or column:

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

$$= a_{1j}C_{1j} + \dots + a_{nj}C_{nj}.$$

So always start with the row or column with the most zeros!

Ex: $A = \begin{bmatrix} 1 & 0 & 87 & 2 \\ 18 & 1 & 93 & 17 \\ 0 & 0 & 3 & 0 \\ 3 & 0 & 12 & 4 \end{bmatrix}$ $\quad |A| = 3(-1)^{3+3}\begin{vmatrix} 1 & 0 & 2 \\ 18 & 1 & 17 \\ 3 & 0 & 4 \end{vmatrix} = 3(-1)^{3+3}\left(1(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}\right)$

$$= 3 \cdot 1 \cdot (4-6) = -6$$

For 3x3 have another way:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

$+$ blues

$-$ reds

Determinant has very nice properties: Behaves well w.r.t. elementary transformations

1) Scaling a row scales the det (expand by that row).
2) Swapping 2 rows changes the sign.
3) Adding a multiple of one row to another does nothing!

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ $\quad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 2 & 4 \end{vmatrix} = 1\cdot 0\cdot 4 + 2\cdot 5\cdot 0 + 0\cdot 2\cdot 3 -$

$$(0\cdot 2\cdot 4 + 0\cdot 0\cdot 3 + 2\cdot 5\cdot 1) = -10$$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} \rightsquigarrow \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix} = 1\cdot 1\cdot 5 + 0\cdot 0\cdot 3 + 0\cdot 2\cdot 2 -$

$$(0\cdot 2\cdot 5 - 0\cdot 1\cdot 3 - 0\cdot 2\cdot 1) = 5.$$

$\Rightarrow$ Can find determinants by simplifying to a nicer matrix and remembering the various steps.

Gives us something much better:

Thm 1) Repeating a row gives det 0.

2) If one row is a scalar multiple of another, det $= 0$.

3) $\det(A\cdot B) = \det(A)\cdot\det(B)$.

In particular, if $A$ is invertible, then $\underset{\substack{\| \\ \det(A)\cdot\det(A^{-1})}}{\det(A\cdot A^{-1})} = \det(I) = 1$

$\Rightarrow \det(A^{-1}) = \det(A)^{-1}$!

This is a huge check for us. If $\det(A)=0$, then $A$ is not invertible!