Lecture 11 - Applications

Note Title 2/20/2008

1st Some examples of finding the matrix for a linear transformation:

Ex 1) $L(x,y) = (3y, 2x+y)$.

$$L(\bar{e}_1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad L(\bar{e}_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow L(\bar{v}) = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} \cdot \bar{v}.$$

2) $L(x,y,z) = (x+y, y+z)$

$$L(\bar{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad L(\bar{e}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad L(\bar{e}_3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow L(\bar{v}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \bar{v}.$$

If $L: \mathbb{R}^n \to \mathbb{R}^m$,

The number of column is the number of variables L takes as arguments $= \dim \mathbb{R}^n$

The number of rows is the number of coördinate functions for L. $= \dim \mathbb{R}^m$

Markov Chains & Stochastic Matrices

We'll need just an intuitive notion of probability:

$0 \le p \le 1$, $p=0$ means "won't happen", $p=1$ means "will happen", others mean "might happen"

Def A stochastic matrix is a square matrix whose entries are probabilites and whose columns sum to 1.

Ie A is stochastic $\Rightarrow$ each column represents all possible future states with their likelyhood.

Ex: Flipping a coin:
$$\begin{array}{cc} \begin{array}{cc} H & T \end{array} & \\ \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} & \begin{array}{c} H \\ T \end{array} \end{array} \qquad \begin{bmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{bmatrix}$$

Thm If A & B are stochastic, then so is AB.

Ex:
$$\begin{bmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{bmatrix} \begin{bmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{bmatrix} = \begin{bmatrix} 4/9 + 1/6 & 1/3 + 1/4 \\ 2/9 + 1/6 & 1/6 + 1/4 \end{bmatrix} = \begin{bmatrix} 11/18 & 7/12 \\ 7/18 & 5/12 \end{bmatrix} \quad \checkmark$$

The stochastic matrix is a way of describing how to transition from one set of states to another

$$\begin{array}{cccc} A & B & C & \\ \left[\begin{matrix} ? & ? & ? \end{matrix}\right. & & \left.\right] & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

1^{st} column is the probabilities of getting A, B, C given A.
2^{nd} column " " " " " " " B.
3^{rd} " " " " " " " " C.

This is an important piece in "Markov Chains" (probabilities of something happening depends only on where you are, not on where you've been).

Lets us also predict outcomes:

Have 2 populations of mice: country & city.

Country mice have a $1/3$ chance of moving to the city.

City mice " " $1/3$ " " " " " country.

$$\begin{array}{c|cc} & \text{city} & \text{country} \\ \hline \text{city} & 2/3 & 1/3 \\ \text{country} & 1/3 & 2/3 \end{array}$$

Let's say we start with 90 city mice & 27 country mice: $\begin{bmatrix} 70 \\ 27 \end{bmatrix}$

Odds are, after 1 year/cycle, we'll have

$$\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 90 \\ 27 \end{bmatrix} = \begin{bmatrix} 60+9 \\ 30+18 \end{bmatrix} = \begin{bmatrix} 69 \\ 48 \end{bmatrix} \begin{matrix} \text{city mice} \\ \text{country mice} \end{matrix}$$

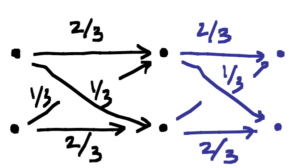
after 1 more year:

$$\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 69 \\ 48 \end{bmatrix} = \begin{bmatrix} 46+16 \\ 23+32 \end{bmatrix} = \begin{bmatrix} 62 \\ 55 \end{bmatrix} \begin{matrix} \text{city} \\ \text{country.} \end{matrix}$$

$$= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}^2 \begin{bmatrix} 90 \\ 27 \end{bmatrix} = \begin{bmatrix} 5/9 & 4/9 \\ 4/9 & 5/9 \end{bmatrix} \begin{bmatrix} 90 \\ 27 \end{bmatrix}$$

The number of mice at some stage depends only on the previous stage!

Can depict this all in a different way: a directed graph



The (i,j) position is the way to get to i from j.

Can run the whole argument backwards:

Given a directed graph, get a matrix:

$$a_{ij} = \begin{cases} 1 & \text{an arrow from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

A carries all of the information the graph does, including paths!

If $B = A^2$, then b_{ij} = # of 2 step paths from i to j.