APMA 308 - Lecture 1: Systems & Matrices

Note Title 1/16/2008

Introduction

Mike Hill mikehill@virginia.edu

Kerchof 213 OH: TBA

Syllabus & course policy are online. A few things of note:

1) Computational Homeworks are online via Webwork
2) In the next few weeks, we'll roll out online resources
   - class 'blog - you can log on & make posts
   - class wiki - your versions of def'ns, etc from class
3) Homeworks, exams, etc are the same across all sections.

Linear Systems & Matrices

<u>Def</u>: A linear system (= a system of equations) in 3 variables is a collection of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$\vdots$$
$$a_n x + b_n y + c_n z = d_n$$

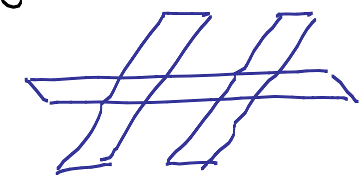
A solution is a point (x, y, z) making every equation true.

Geometric story: $ax + by + cz = d$ is the equation of a plane in $\mathbb{R}^3$. So points making this true = points on the plane.

$\Rightarrow$ solutions to a system are those points sitting on every plane in the system. = points where they intersect.

3 cases:

1) They don't all simultaneously intersect

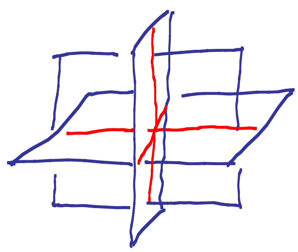


3 planes, 2 are parallel
$\Rightarrow$ don't intersect

Say the system is inconsistent

Otherwise the system is consistent

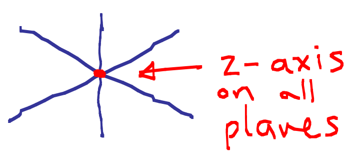
2) They intersect at exactly one point



Say the system is independent

3) They intersect in a line or a plane

(from the top)



Say the system is underdetermined/dependent

Note that x, y, z, etc are basically place holders
$x =$ first spot, $y = 2^{nd}$ spot, etc.
To make things easier on ourselves (and comps), we make some new notation.

Def A matrix is a rectangular array of numbers.

$$\begin{array}{cc} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} & \text{rows} \\ \text{columns} & \end{array}$$

To a system of equations, we can associate a matrix of coefficients by just dropping the variables:
(If a variable doesn't occur, its coef is 0)

$$\begin{aligned} 3x + 17y + 21z &= 146 \\ 2x + \quad\quad 13z &= 19 \\ y + z &= 5 \end{aligned} \implies \begin{bmatrix} 3 & 17 & 21 \\ 2 & 0 & 13 \\ 0 & 1 & 1 \end{bmatrix}$$

Can also associate an augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 17 & 21 & 146 \\ 2 & 0 & 13 & 19 \\ 0 & 1 & 1 & 5 \end{array}\right]$$

(added the col of "=" to the coef matrix)

Moral: Anything we can say about a system is reflected in the augmented matrix.

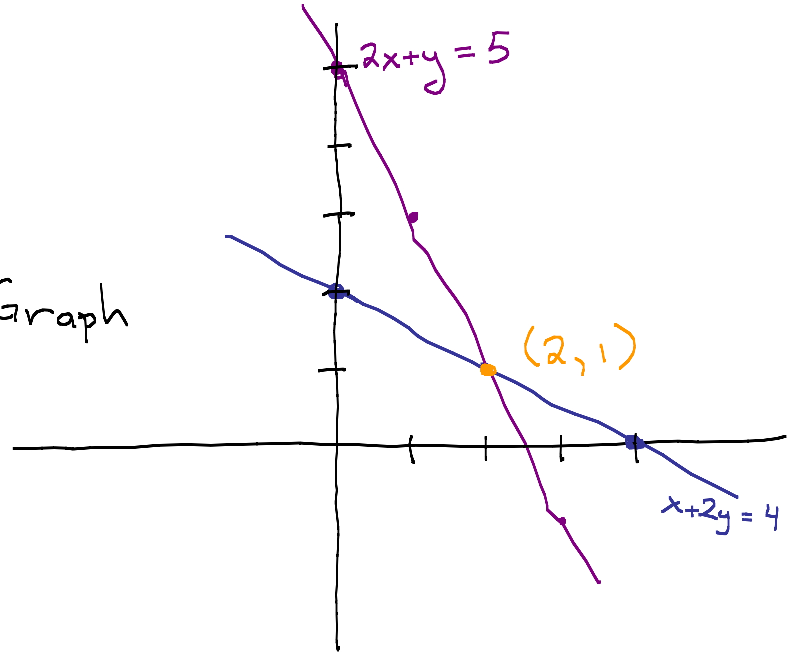
Q: How do we find solutions?
Lots of ways:

1) Graph everything
2) Substitute in
3) Make the system easier

Ex

$$x + 2y = 4 \quad (1)$$
$$2x + y = 5 \quad (2)$$

Graph



Substitute:

$(1) \Leftrightarrow x = 4 - 2y$

$\Rightarrow (2) \Rightarrow 2(4-2y) + y = 5$

$\Rightarrow 8 - 3y = 5$

$\Rightarrow y = 1 \Rightarrow x = 2$

what makes a system easy?

a) each variable occurs once

b) its coefficient is 1.

a) each row is mostly zero

b) the non-zero term = 1

ie
$$\begin{aligned} x \quad\quad\quad &= 3 \\ y \quad &= 5 \\ z &= 7 \end{aligned}$$
is <u>very</u> easy to solve

What can we do? Elementary operations

1) Can reorder the eqns / row

2) Can scale an eqn / row by a non-zero number.

3) Can add a multiple of one eqn/row to another
(why does this not change things?)

Example: system above: $\begin{matrix} x + 2y = 4 \\ 2x + y = 5 \end{matrix} \longleftrightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 1 & 5 \end{array}\right]$

$\Downarrow$

add $(-2)\cdot \text{eqn}_1$ to eqn_2: $\left\{\begin{aligned} x + 2y &= 4 \\ -3y &= -3 \end{aligned}\right.$ $\quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -3 & -3 \end{array}\right] \div 3$

divide eqn_2 by -3: $\left\{\begin{aligned} x + 2y &= 4 \\ y &= 1 \end{aligned}\right.$ $\quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \end{array}\right] + \cdot(-2)$

add $(-2)\cdot \text{eqn}_2$ to eqn_1: $\left\{\begin{aligned} x \quad &= 2 \\ y &= 1 \end{aligned}\right.$! $\quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$

Def A matrix is in row echelon form if

1) each row starts with more zeros than the previous
2) the 1st non-zero entry in each row is a $\underline{1}$.

Ex

- $\begin{bmatrix} 1 & 2 & 3 & 19 \\ 0 & 1 & 4 & 6 \end{bmatrix}$ is in RE form
- $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ is not (fails 1))
- $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ is not (fails 2))

Def A matrix is in reduced row echelon form if in addition

3) the 1st non-zero entry in each row is the only non-zero entry in its column.

Ex: Don't matter!

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Augmented matrices in RRE Form are <u>very</u> easy to solve:

- variables corresponding to the start of rows occur exactly once
- can solve for these in terms of the remaining variables

In row echelon form, it is easy to see how many solutions a system has:

Ex:

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right] \begin{array}{l} \leftarrow \text{tells us } x\text{, uniquely} \\ \leftarrow \text{tells us } y\text{, uniquely} \\ \leftarrow \text{tells us } z\text{, uniquely} \end{array}$$

General case: If the RE form has 1s along the diagonal (those entries where the row is the same as the column), then there is a unique solution to the system.

<u>Remark</u> The condition that each row has more leading zeros than the previous forces the 1st possible non-zero position to be on the diagonal.

<u>Ex</u>: $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & a \end{array}\right] \leftarrow$ this translates to

$$0 \cdot x + 0 \cdot y + 0 \cdot z = a$$

So we see that if $a \neq 0$, then the system is inconsistent! The RE form shows us this in general with a row like

$$[0 \cdots 0 \mid a], \quad a \neq 0.$$