Power Operations and Differentials in Higher Real K-Theory

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Outline

Basic Setup

- Hopkins-Miller Theorem: the Lubin-Tate spectrum E_n is an E_∞ ring spectrum and $\mathbb{G}_n$ acts by E_∞ ring maps.
- Devinatz-Hopkins: $E_n^{h\mathbb{G}_n} = L_{K(n)}S^0$.
- Basic step: Approximate $E_n^{h\mathbb{G}_n}$ by $EO_n(G) = E_n^{hG}$ for G finite.

Homotopy Fixed Points

- $E^{hG} = F_G(EG_+, E)$.
- Skeletal filtration of EG_+ gives a filtration of E^{hG}.
- Associated spectral sequence is the "homotopy fixed point spectral sequence".
- $E_2 = H^s(G; \pi_t(E))$, Adams-Novikov style differentials.
- Shows up here and in homotopy approaches to algebraic K-theory.

Previous Work

- E_1 is p-adic K-theory & $\mathbb{G}_1 = \mathbb{Z}_p^\times$.
- At 2, $E_1^{\mathbb{Z}/2} = KO_2^\wedge$.
- At 2 and 3, can find groups of order 24 in $\mathbb{G}_2$, and E_2^G is essentially TMF.

Proposition

At p, can find cyclic groups of order p^k in $\mathbb{G}_{p^{k-1}(p-1)f}$.

Hopkins-Miller

Theorem

Modulo the image of the transfer,

$$H^*(\mathbb{Z}/p; \pi_*(E_{p-1})) = \mathbb{F}_p[\Delta^{\pm 1}, \beta] \otimes E(h_{1,0}),$$

$|\beta| = (-2, 2)$, $|\Delta| = (2p, 0)$, *and* $|h_{1,0}| = (2p-3, 1)$.
There is a d_{2p-1}*-differential:*

$$d_{2p-1}(\Delta) = h_{1,0}\beta^{p-1}\Delta.$$

Group Action

Theorem

$H^*_{Tate}(\mathbb{Z}/p; \pi_*(E_{f(p-1)})) =$

$$\mathbb{F}_p[\Delta^{\pm 1}, \beta^{\pm 1}][\![\delta_1, \ldots, \delta_{f-1}]\!] \otimes E(h_{1,0}, \ldots, h_{f,0}),$$

where $|\beta| = (-2, 2)$, $|\Delta| = (2p, 0)$, $|\delta_i| = (0, 0)$, *and* $|h_{i,0}| = (2p^i - 3, 1)$.

Actually identify the structure of $\pi_* E_{f(p-1)}$ as a $\mathbb{Z}/p$-module, giving the result.

Differentials

Theorem

We have d_{2p-1}-differentials

$$d_{2p-1}(h_{i,0}) = h_{1,0}h_{i,0}\beta^{p-1}.$$

We have d_{2p^i-1}-differentials

$$d_{2p^i-1}(\Delta^{p^{i-1}}) = h_{i,0}\beta^{p^i-1}\Delta^{p^{i-1}}.$$

Universal Examples

- Sources of differentials: skeleta of Thom spectra over BG.
- Let V be a [virtual] representation of G, and let $S^V \longrightarrow E$ be a G-map.
- Then we get a filtration preserving map $(S^V)^{hG} \longrightarrow E^{hG}$.
- The source is the dual of a Thom spectrum over BG.

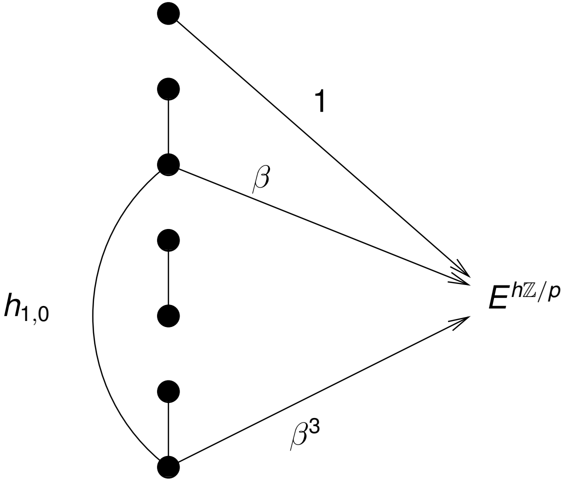
Example

- Canonical example:

$$S^0 \xrightarrow{1} E_{f(p-1)}$$

- $(S^0)^{h\mathbb{Z}/p} = D(B\mathbb{Z}/p_+)$
- The $(-2k)$-cell maps in as β^k.

Differentials on β



Getting Remaining Differentials

Several Strategies:

1. Classical power operation constructions in the E_∞ context.
2. Find orientable bundles for EO-theory$\Rightarrow$ permanent cycles. Ex: $\beta\Delta^{1/(p-1)}$.
3. Comparing to other theories like $T(i)$.
4. Power operation construction in the E_2 context.

Summary

Using geometry, we can find nice, universal families of differentials.
These let us completely described $\pi_*(EO_n(\mathbb{Z}/p))$.