17

Equivariant Stable Homotopy Theory

17.1 Introduction

Equivariant stable homotopy theory considers spaces and spectra endowed with the action of a fixed group G. Classically, this group has been taken to be finite or compact Lie, but here we will consider only the case of a finite group acting. Our goal is to produce a broad-strokes overview of the state of equivariant stable homotopy theory, focusing the intuition behind many of the objects and constructions, exploring some of the tools in equivariant algebra, and showing how one can compute with these as easily as one computes classically.

There are many wonderful references for much of the foundational material in equivariant stable homotopy theory. For example [2], [58], [32], [18], [60], and [67, 68] are excellent sources for learning about specific models and their applications. In this, we will focus more on multiplicative and computational aspects, working through various examples along the way.

In all that follows, we work as model independently as possible. We will use the phrase "homotopically meaningful" to signify that a particular functor or construction descends to the underlying ∞-category or lifts to a Quillen functor on appropriate model categories.

Notation and conventions

In this chapter, G will be a finite group. Letters like H and K will most often refer to subgroups of G, and N will refer to normal subgroups. Spaces are always assumed to be compactly generated, weak Hausdorff.

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. 1440140, while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring semester of 2019. The author was also supported by NSF Grant DMS–1811189. The author also thanks Andrew Blumberg, Tyler Lawson, and Haynes Miller for their careful comments on earlier drafts.

17.2 G-spaces and functors between them

17.2.1 The categories of G-spaces

Definition 17.2.1. A G-space is a topological space X together with a continuous map

$$G \times X \longrightarrow X$$

$$(g, x) \longmapsto g \cdot x$$

such that

1. if $e \in G$ is the identity, then for all $x \in X$, $e \cdot x = x$, and
2. for all $g, h \in G$ and $x \in X$, we have $g \cdot (h \cdot x) = (g \cdot h) \cdot x$.

As is common in mathematics, although a G-space is two pieces of data, we will normally denote them only by the name of the underlying space.

Definition 17.2.2. If X and Y are G-spaces then an equivariant map $f\colon X \to Y$ is a continuous map $f\colon X \to Y$ such that for all $g \in G$ and $x \in X$, we have

$$f(g \cdot x) = g \cdot f(x).$$

It is a useful exercise to check that equivariant maps compose and that the identity is equivariant.

Notation 17.2.3. Let Top^G denote the category of G-spaces and equivariant maps.

We have a forgetful functor from G-spaces to spaces which just forgets the action of G.

Notation 17.2.4. Let $i^*_{\{e\}}\colon \mathsf{Top}^G \to \mathsf{Top}$ be the forgetful functor.

This forgetful functor is faithful, since an equivariant map is just a continuous map with the property that it commutes with the action of G. In particular, we can use the natural topological enrichment of Top to produce a topological enrichment on Top^G. Here the topology on the Hom sets is just the subspace topology given by the faithful inclusions from $i^*_{\{e\}}$.

Example 17.2.5. If V is a finite dimensional orthogonal representation of G, then we have several G-spaces attached to V:

1. Let $D(V) = \left\{\vec{v} \in V \mid ||\vec{v}|| \leq 1\right\}$ be the unit disk in V,
2. let $S(V) = \left\{\vec{v} \in V \mid ||\vec{v}|| = 1\right\}$ be the unit sphere in V, and

3. let $S^V = D(V)/S(V)$ be the one point compactification of V, the "V-sphere".

Example 17.2.6. If V is a representation of G, then let

$$a_V \colon S^0 \to S^V$$

be the inclusion of the origin and point at infinity. We call this the Euler class of V, since it is the Euler class of the equivariant bundle $V \to *$.

Our forgetful functor down to spaces is just one of a host of forgetful functors wherein we forget the actions of only some of the elements of G.

Definition 17.2.7. Given any subgroup $H \subseteq G$, let

$$i_H^* \colon \mathsf{Top}^G \to \mathsf{Top}^H$$

be the forgetful functor which forgets the actions of elements of G not in H.

Example 17.2.8. If V is any representation of G, then

$$i_H^* a_V = a_{i_H^* V}.$$

These forgetful functors play an essential role in equivariant unstable and stable homotopy theory. They are categorically very well behaved, commuting with all limits and colimits in the category, and they have both adjoints. The left adjoint is given by a kind of balanced tensor product, while the right adjoint is given by an equivariant function object, just as in ordinary representation theory.

Definition 17.2.9. If X is an H-space, then let

$$G \underset{H}{\times} X = G \times X/\sim,$$

where $\sim$ is the equivalence relation given by $(gh, x) \sim (g, hx)$ for all $g \in G$, $h \in H$, and $x \in X$. This has a G-action given by

$$g \cdot [(g', x)] = [(gg', x)].$$

Definition 17.2.10. If Y is an H-space, then let

$$\mathsf{Top}^H(G, Y)$$

be the space of H-equivariant maps from G (viewed as an H-space with the left action of H on G) to Y. This gets an action of G via the right action of G on itself:

$$(g \cdot f)(g') = f(g'g).$$

Proposition 17.2.11. *The constructions*

$$X \mapsto G \underset{H}{\times} X, \quad Y \mapsto \mathsf{Top}^H(G, Y)$$

extend to functors

$$G \underset{H}{\times} \text{-}, \mathsf{Top}^H(G, \text{-})\colon \mathsf{Top}^H \to \mathsf{Top}^G,$$

called induction and coïnduction respectively.

Induction is left adjoint to the forgetful functor i_H^*, and coïnduction is right adjoint to the forgetful functor i_H^*.

Just as classically, we can test spaces by mapping in points. Here, however, we have to remember at which subgroup our point was "born". Since the point is the terminal object in the category of spaces, there is a unique H-space structure on $\{*\}$: the trivial one. This gives us the G-space

$$G/H \cong G \underset{H}{\times} *.$$

Proposition 17.2.12. *If Y is a G-space, then*

$$\mathsf{Top}^G(G/H, Y) \cong \{y \in Y \mid h \cdot y = y, \forall h \in H\}.$$

Definition 17.2.13. If $H \subseteq G$ and if Y is a G-space, then the H-fixed points of Y are

$$Y^H := \{y \in Y \mid h \cdot y = y, \forall h \in H\}.$$

The key step for Proposition 17.2.12 is that stabilizers of points only grow under an equivariant map. We can use this to describe the left adjoint to the H-fixed points functor.

Notation 17.2.14. Let

$$i_*\colon \mathsf{Top} \to \mathsf{Top}^G$$

be the functor which endows a space X with a trivial G-action: for all $g \in G$ and $x \in X$, $g \cdot x = x$.

The same construction works for other quotient groups.

Notation 17.2.15. Let $Q = G/N$, and let

$$i_*^N\colon \mathsf{Top}^Q \to \mathsf{Top}^G$$

be the functor which views a Q-space as a G-space via the quotient map $G \to Q$.

Underlying this is the observation that every continuous map between spaces with a trivial G-action is equivariant.

Proposition 17.2.16. *The functor i_* is left adjoint to the G-fixed points functor*

$$(\text{-})^G\colon \mathsf{Top}^G \to \mathsf{Top}.$$

More generally, the functor

$$G \underset{H}{\times} i_*\colon \mathsf{Top} \to \mathsf{Top}^G$$

is left adjoint to the H-fixed points functor.

Under the homeomorphisms given by Proposition 17.2.12, maps of orbits correspond to various inclusions and maps between the fixed points for various subgroups of G. This gives us a way to conceptualize a G-space: begin with the fixed points and then begin adding orbits of the form G/H (in families), working our way down the subgroup lattice of G. We shall make this concept increasingly precise.

We also have pointed versions of all of these statements.

Notation 17.2.17. Let Top^G_* be the category of G-spaces equipped with a G-fixed basepoint.

Notation 17.2.18. Let $G_+ \underset{H}{\wedge}$ - and $\mathsf{Top}^H_*(G_+,\text{-})$ be the (pointed) induction and coïnduction.

17.2.2 Equivariant homotopies and CW-complexes

We define homotopies and CW-structures largely in parallel with the classical ones. We first note that the category of G-spaces has a closed symmetric monoidal structure.

Definition 17.2.19. If X and Y are G-spaces, then let $X \times Y$ be endowed with the action

$$g \cdot (x, y) = (gx, gy).$$

Let $\underline{\mathsf{Top}}(X, Y)$ have the action

$$(gf)(x) = g\big(f(g^{-1}x)\big).$$

Essentially the same definitions can be applied in the pointed cases, giving the smash product and pointed mapping spaces.

Proposition 17.2.20. *The Cartesian product and function spaces with conjugation action give a closed symmetric monoidal structure on Top^G.*

The smash product and pointed function spaces with conjugation action give a closed symmetric monoidal structure on Top^G_*.

Remark 17.2.21. We can auto-enrich Top^G, forming a category $\underline{\mathsf{Top}}$. Our notation is chosen to reflect the fact that the former is the fixed points of the latter.

Definition 17.2.22. If $f_0, f_1 \colon X \to Y$ are equivariant maps, then a homotopy from f_0 to f_1 is an equivariant map

$$F \colon X \times I \to Y$$

such that for all $x \in X$ and $i \in \{0,1\}$, $F(x,i) = f_i(x)$. We say that f_0 and f_1 are homotopic if there is a homotopy from one to the other.

In the pointed case, we require that the homotopy be relative to the basepoint.

Notation 17.2.23. If X and Y are G-spaces, let $[X,Y]^G$ denote equivariant homotopy classes of maps from X to Y.

If X and Y are pointed G-spaces, let $[X,Y]^G_*$ denote the equivariant homotopy classes of pointed maps from X to Y.

Example 17.2.24. If V is a representation such that $V^G = \{0\}$, then the Euler class a_V is not homotopic to a constant map. Any such homotopy would, applying fixed points, provide a null-homotopy of the identity map on S^0.

Conversely, if $V^G \neq \{0\}$, then the Euler class a_V is null-homotopic, with nullhomotopy given by tracing along a ray in V^G.

The classical arguments that "homotopic is an equivalence relation" go through without change. Here, however, we can already see more rigidity than in the classical case.

Proposition 17.2.25. *If $f \colon X \to Y$ and $f' \colon Y \to X$ are homotopy inverses, then for all $H \subseteq G$, f and f' induce homotopy equivalences*

$$f^H \colon X^H \rightleftarrows Y^H \colon f'^H.$$

Our notion of homotopy and homotopy equivalence then gives the weak one.

Definition 17.2.26. A map $f \colon X \to Y$ is a weak homotopy equivalence if for all $H \subseteq G$,

$$f^H \colon X^H \to Y^H$$

is a weak homotopy equivalence.

These are the weak equivalences in a model structure on Top^G. The fibrations are also defined relative to the fixed points.

Theorem 17.2.27. *There is a model category structure on* Top^G *in which the weak equivalences are the weak homotopy equivalences, in which the fibrations are those maps* $p\colon E \to B$ *such that for all* $H \subseteq G$,

$$p^H : E^H \to B^H$$

is a [Serre] fibration, and where the cofibrations are what they have to be.

Using Proposition 17.2.12, we can turn our notation of a weak equivalence into a diagramatic statement: a map $f\colon X \to Y$ is a weak homotopy equivalence if for all orbits G/H, the induced map

$$\mathsf{Top}^G(G/H, f)\colon \mathsf{Top}^G(G/H, X) \to \mathsf{Top}^G(G/H, Y)$$

is a weak equivalence of spaces.

Definition 17.2.28. Let Orb^G be the full subcategory of Top^G generated by orbits.

Definition 17.2.29. If X is a G-space, then let

$$\underline{X}\colon (\mathsf{Orb}^G)^{op} \to \mathsf{Top}$$

be the restriction of the Yoneda functor given by

$$\underline{X}(G/H) = \mathsf{Top}^G(G/H, X)$$

to Orb^G.

The Yoneda embedding says that the assignment $X \to \underline{X}$ gives a functor

$$\mathsf{Top}^G \to \mathrm{Fun}\left((\mathsf{Orb}^G)^{op}, \mathsf{Top}\right).$$

Since Top is a cofibrantly generated model category, we have an induced model structure on $\mathrm{Fun}\left((\mathsf{Orb}^G)^{op}, \mathsf{Top}\right)$ in which the weak equivalences and fibrations are determined levelwise.

We also have a functor that turns a diagram of this shape into a G-space.

Notation 17.2.30. Let $J\colon \mathsf{Orb}^G \to \mathsf{Top}^G$ be the inclusion of the orbit category into the category of G-spaces.

Let

$$\text{-} \otimes_{\mathsf{Orb}^G} J\colon\ \mathrm{Fun}\left((\mathsf{Orb}^G)^{op}, \mathsf{Top}\right) \to \mathsf{Top}^G$$

by the coend with J.

Elmendorf's Theorem is that these two functors are inverse Quillen equivalences.

Theorem 17.2.31 ([24])**.** *There is a Quillen equivalence*

$$\underline{(\text{-})}\colon \mathsf{Top}^G \rightleftarrows \mathrm{Fun}\left((\mathsf{Orb}^G)^{op}, \mathsf{Top}\right) : \text{-} \otimes_{\mathsf{Orb}^G} J.$$

Remark 17.2.32. Elmendorf's Theorem shows that homotopically, there is a difference between the category of G-spaces and the functor category

$$\mathrm{Fun}(BG, \mathsf{Top}),$$

where BG is the category with one object and morphism set G.

For example, the G-homeomorphism type of X can be read out of $\underline{X}$. The automorphism group of $G/\{e\}$ as a G-space is G^{op}, and $X \cong \underline{X}(G/\{e\})$. This identification need not be homotopically meaningful, however. For example, the map $EG \to *$ induces an equivalence at level $G/\{e\}$, but not of full diagrams.

17.2.2.1 CW-Structures

Our notion of a G-CW complex: we attach cells with various stabilizers inductively.

Definition 17.2.33. A G-CW structure on a G-space X is a a filtration of X

$$\emptyset = X^{[-1]} \subseteq X^{[0]} \subseteq X^{[1]} \subseteq \dots \subseteq \bigcup_i X^{[i]} = X$$

such that

1. $X^{[0]}$ is a discrete G-set,
2. for each i, there is a discrete G-set T_i and a G-map

$$\theta_i \colon T_i \times S^{i-1} \to X^{[i-1]}$$

   such that we have a pushout diagram

$$\begin{tikzcd} T_i \times S^{i-1} \arrow[r, "\theta_i"] \arrow[d, hook] & X^{[i-1]} \arrow[d, hook] \\ T_i \times D^i \arrow[r] & X^{[i]}, \end{tikzcd}$$

3. X has the direct limit topology induced by the filtration.

There is a pointed version of a G-CW complex defined analogously; we will use both.

Since every G-set decomposes as a disjoint union of orbits, each of our sets T_i can so be decomposed. This means that attaching data in the second condition could equivalently have been written

$$\theta_i \colon \coprod_{j \in \mathcal{I}_i} G/H_j \times S^{i-1} \to X^{[i-1]}.$$

The choices here then hide some of the naturality: we choose a presentation of T_i as a disjoint union of orbits.

Example 17.2.34. If $\underline{X}$ is such that $\underline{X}(G/H)$ is a CW-complex for all H, and all of the maps are cellular, then the G-space produced by Elmendorf's Theorem is a G-CW complex.

Example 17.2.35. If $X_\bullet$ is a simplicial object in G-sets, then the geometric realization of $X_\bullet$ has the natural structure of a G-CW complex.

Theorem 17.2.36 (Equivariant Whitehead Theorem)**.** *A weak equivalence between G-CW complexes is a homotopy equivalence.*

Showing this uses an obstruction theory that records the stabilizers of individual cells as well as a varying target. To get a sense for what happens, consider attaching a single equivariant cell $G_+ \underset{H}{\wedge} D^i$ to a G-space X along a map $\theta\colon G_+ \underset{H}{\wedge} S^{i-1} \to X$. Consider also a map $f\colon X \to Y$. Then an extension of f over $X \cup (G_+ \underset{H}{\wedge} D^i)$ exists if and only if

$$f \circ \theta\colon G_+ \underset{H}{\wedge} S^{i-1} \to Y$$

is null-homotopic. By Proposition 17.2.11, this is null if and only if the adjoint map

$$\widetilde{f \circ \theta}\colon S^{i-1} \to i_H^* Y$$

is null. Since S^{i-1} has a trivial H-action, any H-equivariant map $S^{i-1} \to Y$ or $S^{i-1} \times I \to Y$ must land in Y^H. Thus the obstruction to extending over a cell of the form $G_+ \underset{H}{\wedge} D^i$ is in $\pi_{i-1}(Y^H)$. As H-varies over the cells, so then does the group in which our extensions live. Coefficient systems and Bredon cohomology record exactly what we see.

17.2.3 Coefficient systems and cohomology

17.2.3.1 The Category of Coefficient Systems

Notation 17.2.37. Let Fin^G denote the category of finite G-sets.

Disjoint union gives Fin^G a co-Cartesian monoidal structure; Cartesian product gives the Cartesian monoidal structure.

Definition 17.2.38. Let $\mathcal{C}$ be a category with finite products. A coefficient system of objects of $\mathcal{C}$ is a product preserving functor

$$\underline{M}\colon (\mathsf{Fin}^G)^{op} \to \mathcal{C}.$$

In this definition, we use the fact that since Fin^G has coproducts given by disjoint union, $(\mathsf{Fin}^G)^{op}$ has products given by disjoint union. The condition is then that

1. for all T_1 and T_2, the inclusions $T_1 \hookrightarrow T_1 \amalg T_2 \hookleftarrow T_2$ induce an isomorphism

$$\underline{M}(T_1 \amalg T_2) \cong \underline{M}(T_1) \times \underline{M}(T_2),$$

2. and $\underline{M}(\emptyset) = *$, the terminal object.

Example 17.2.39. Since the wedge product is the coproduct in pointed spaces, Definition 17.2.29 extends to give a coefficient system $\underline{X}$ of spaces for any G-space X:

$$\underline{X}(T) := \mathsf{Top}_*^G(T_+, X).$$

Remark 17.2.40. We can restate Elmendorf's Theorem as being a Quillen equivalence between G-spaces and coefficient systems of spaces. This is a powerful reinterpretation that is essential in the recent ∞-categorical treatments of equivariant homotopy theory.

Proposition 17.2.41. *Let X be a pointed G-space. Then the assignment*

$$T \mapsto \pi_k\big(\mathsf{Top}_*^G(T_+, X)\big) =: \underline{\pi}_k(X)(T)$$

gives a coefficient system of pointed sets if $k = 0$, of groups if $k = 1$, or of abelian groups if $k \geq 2$.

Definition 17.2.42. Let $\mathtt{Coeff}$ be the category whose objects are coefficient systems of abelian groups and whose morphisms are natural transformations.

Coefficient systems are so-named because they are the natural coefficients for equivariant cohomology.

Example 17.2.43. A C_p-coefficient system $\underline{M}$ is the following data:

1. An abelian group $\underline{M}(*)$,
2. a C_p-module $\underline{M}(C_p)$, and
3. a "restriction map"
$$res_e^{C_p} \colon \underline{M}(*) \to \underline{M}(C_p)$$
which factors through the inclusion of the fixed points
$$\underline{M}(C_p)^{C_p} \subseteq \underline{M}(C_p).$$

Proposition 17.2.44. *The category* $\mathtt{Coeff}$ *is an abelian category with a finite set of projective generators.*

The projective generators can all be chosen to represent the various evaluation functors

$$\underline{M} \mapsto \underline{M}(T),$$

or even simply to represent evaluation at the orbits G/H, by the product preserving property.

Theorem 17.2.45 ([15])**.** *For any coefficient system $\underline{M}$, there is a unique cohomology theory on G-CW pairs for which*

$$H^*(G/H;\underline{M}) \cong \begin{cases} \underline{M}(G/H) & * = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

We build this out of the natural coefficient system of chain complexes we get by composing with Elmemdorf coefficient system with the singular chains functor. Rather than spell this out, we describe one of the main ways we can compute with this: cellular cohomology.

17.2.3.2 Cellular Bredon Homology

The usual suspension axiom shows that the induced spheres $G_+ \underset{H}{\wedge} S^n$ play the role for Bredon cohomology that ordinary spheres play for classical cohomology: they are "Moore spaces" in the sense that the have a single non-vanishing cohomology group. We can now prove the usual results about the cellular cohomology and the corresponding relationship to Bredon cohomology, copying the usual definitions.

Definition 17.2.46. If X is aG-CW complex of finite type, then let

$$C^k_{\text{cell}}(X;\underline{M}) = H^k(X^{[k]}, X^{[k-1]};\underline{M}).$$

Define a boundary map

$$\delta\colon C^k_{\text{cell}}(X;\underline{M}) \to C^{k+1}_{\text{cell}}(X;\underline{M})$$

via the long exact sequence for the triple $(X^{[k+1]}, X^{[k]}, X^{[k-1]})$.

The standard argument then applies here to show that this complex gives Bredon cohomology.

Proposition 17.2.47. *The cohomology of the cellular cochain complex is the Bredon cohomology of X.*

We further unpack this in the case that X is finite type. In this case, for each k

$$X^{[k]}/X^{[k-1]} \cong T_{k+} \wedge S^k,$$

for some finite G-set T_k. Moreover, the boundary map is the map induced by

$$\partial_k\colon T_{k+} \wedge S^k \cong X^{[k]}/X^{[k-1]} \to \Sigma X^{[k-1]} \to \Sigma X^{[k-1]}/X^{[k-2]} \cong T_{k-1+} \wedge S^k.$$

By definition, this map is an element of

$$\pi_k\big(T_{k-1+} \wedge S^k\big)(T_k).$$

When $k \geq 2$. we can easily describe the bottom homotopy coefficient system of an induced sphere like this.

Definition 17.2.48. For each finite G-set T, let $\underline{\mathbb{Z}}[T]$ be the coefficient system defined by

$$\underline{\mathbb{Z}}[T](T') := \mathbb{Z}\big\{\mathsf{Top}^G(T', T)\big\},$$

the free abelian group on the set $\mathsf{Top}^G(T', T)$.

Theorem 17.2.49. *If T is a finite G-set, then for all $n \geq 2$, we have a natural (in T) isomorphism*

$$\underline{\pi}_n(T_+ \wedge S^n) \cong \underline{\mathbb{Z}}[T].$$

Proof. Using the closed monoidal structure on G-spaces, we have a natural isomorphism

$$[T'_+ \wedge S^n, T_+ \wedge S^n]^G \cong [S^n, \mathsf{Top}^G(T', T)_+ \wedge S^n].$$

The right-hand side is non-equivariant homotopy classes of maps, and by the Hurewicz theorem, we have

$$[S^n, \mathsf{Top}^G(T', T)_+ \wedge S^n] \cong H_n\big(\mathsf{Top}^G(T', T)_+ \wedge S^n; \mathbb{Z}\big) \cong \bigoplus_{\mathsf{Top}^G(T', T)} \mathbb{Z}.$$

$\square$

Remark 17.2.50. Since we only test against a finite G-set (and hence compact), Theorem 17.2.49 remains true if T is infinite.

It is helpful to additively enlarge the category Coeff.

Notation 17.2.51. Let $\mathsf{Fin}^G_{\mathbb{Z}}$ be the category with objects finite G-sets and with morphisms the free abelian group on the morphisms in Fin^G:

$$\mathsf{Fin}^G_{\mathbb{Z}}(S, T) := \mathbb{Z}\big\{\mathsf{Top}^G(S, T)\big\}.$$

Proposition 17.2.52. *The natural faithful inclusion $\mathsf{Fin}^G \hookrightarrow \mathsf{Fin}^G_{\mathbb{Z}}$ induces an equivalence of categories between coefficient systems of abelian groups and functors $\big(\mathsf{Fin}^G_{\mathbb{Z}}\big)^{op} \to \mathsf{Ab}$ that take disjoint unions to products and which are linear on Hom objects.*

Remark 17.2.53. The coefficient systems $\underline{\mathbb{Z}}[T]$ are the representable functors in this enlarged diagram category, representing the evaluation at T functor. These are projective generators.

This then allows us to determine the effect of the attaching maps on Bredon cohomology (and hence makes computing equivariant cohomology groups as easy as computing the non-equivariant ones). Any map

$$f \colon T'_+ \wedge S^n \to T_+ \wedge S^n,$$

induces a corresponding map of coefficients systems on $\underline{\pi}_n$:

$$\underline{\mathbb{Z}}[T'] \to \underline{\mathbb{Z}}[T].$$

Thus for any coefficient system $\underline{M}$, by the Yoneda Lemma, we have a "restriction along f" map

$$f^*\colon \underline{M}(T) \to \underline{M}(T')$$

given by mapping out of f. This is exactly the Bredon differential induced by the relative attaching map.

Theorem 17.2.54. *The Bredon cellular cochain complex is the complex $C^*_{cell}(X;\underline{M})$ with*

$$C^k_{cell}(X;\underline{M}) := \underline{M}(T_k),$$

and where the coboundary map is

$$C^{k-1}_{cell}(X;\underline{M}) = \underline{M}(T_{k-1}) \xrightarrow{\underline{M}(\partial_k)} \underline{M}(T_k) = C^k_{cell}(X;\underline{M}).$$

Remark 17.2.55. If we work instead with a covariant functor $\mathsf{Fin}^G \to \mathsf{Ab}$ which takes disjoint union to direct sum, then we can mirror the entire argument to build the Bredon homology and Bredon cellular homology.

Example 17.2.56. We close this section with an example of how to compute Bredon homology. Let $G = C_2$, with generator γ, and let $\underline{M}$ be a coefficient system. Let σ be the 1-dimensional sign representation. We compute $H^*(S^{k\sigma};\underline{M})$ for any k as a functor of $\underline{M}$.

A cell structure for $S^{k\sigma}$ is given by

$$S^0 \cup (C_{2+} \wedge e^1) \cup_{1-\gamma} (C_{2+} \wedge e^2) \cup \cdots \cup (C_{2+} \wedge e^k),$$

where the bottom attaching map is the action map

$$C_{2+} \wedge S^0 \to S^0,$$

which induces the restriction map

$$\underline{M}(*) \to \underline{M}(C_2)$$

on Bredon cellular cochains. The attaching map for the ℓ-cell modulo the $(\ell-2)$-skeleton is the map

$$(1 + (-1)^\ell\gamma),$$

and this induces multiplication by this element in $\underline{M}(C_2)$. Our chain complex is therefore

$$\underline{M}(*) \xrightarrow{res^{C_2}_e} \underline{M}(C_2) \xrightarrow{1-\gamma} \underline{M}(C_2) \to \cdots \to \underline{M}(C_2),$$

and the Bredon cohomology is the cohomology of this cochain complex.

17.2.4 Families and isotropy separation

One of the most useful consequences of Elmendorf's theorem is a way to isolate the contribution of cells with a particular stabilizer. This is called "isotropy separation", and stably, it will provide an explanation for several confusing features.

Definition 17.2.57. A family of subgroups is a set $\mathcal{F}$ of subgroups of G such that

1. if $H \in \mathcal{F}$ and if $K \subseteq G$, then $K \in \mathcal{F}$, and
2. if $H \in \mathcal{F}$, then for all $g \in G$, $gHg^{-1} \in \mathcal{F}$.

Remark 17.2.58. We can repackage the two conditions via the orbit category, as together they say that if $H \in \mathcal{F}$, and if we have a map $G/K \to G/H$ in the orbit category, then $K \in \mathcal{F}$. This shows that a family of subgroups is the same data as a sieve on the orbit category.

Definition 17.2.59. If X is a space, then let

$$\Phi_X = \big\{ H \mid X^H \neq \emptyset \big\}.$$

Then Φ_X is a family, the "geometric isotropy of X". It records which stabilizers can show up in a G-CW decomposition of X.

Associated to a family, there is a universal homotopy type given by Elmendorf's Theorem.

Definition 17.2.60. If $\mathcal{F}$ is a family, then let $\underline{E\mathcal{F}}$ be the coefficient system given by

$$\underline{E\mathcal{F}}(G/H) = \begin{cases} \emptyset & H \notin \mathcal{F} \\ * & H \in \mathcal{F}. \end{cases}$$

Let $E\mathcal{F}$ be the G-space produced by Elmendorf's Theorem from $underline E\mathcal{F}$.

The following proposition is immediate from the coefficient system formulation of the universal space and gives some explanation of the nomenclature.

Proposition 17.2.61. *If X is a G-CW complex, then*

$$[X, E\mathcal{F}]^G = \left\{ \begin{array}{ll} * & \Phi_X \subseteq \mathcal{F} \\ \emptyset & \Phi_X \not\subseteq \mathcal{F}. \end{array} \right.$$

Corollary 17.2.62. *The space $E\mathcal{F}$ is determined by the condition that*

$$(E\mathcal{F})^H \simeq \begin{cases} \emptyset & H \notin \mathcal{F} \\ * & H \in \mathcal{F}. \end{cases}$$

Example 17.2.63. Let $\mathcal{F}_e = \big\{\{e\}\big\}$. The associated space $E\mathcal{F}_e$ is a G-CW complex such that the fixed points for any non-trivial subgroup are empty and such that the underlying space is contractible. This is exactly the homotopical description of EG.

Example 17.2.64. Let $\mathcal{A}ll = \{H \mid H \subseteq G\}$. Then a model for $E\mathcal{A}ll$ is a point.

Example 17.2.65. Let N be a normal subgroup. The collection of subgroups which intersect N trivially forms a family $\mathcal{F}_N$ with associated universal space $E\mathcal{F}_N$.

Example 17.2.66. Let $\mathcal{P} = \{H \mid H \subsetneq G\}$ be the family of proper subgroups. Then a model for the homotopy type of $E\mathcal{P}$ is

$$\operatorname{colim}_n S(n\bar{\rho}_G),$$

where $\bar{\rho}_G$ is the quotient of the regular representation ρ_G by the trivial summand.

Since fixed points commute with products (both being limits), given any family $\mathcal{F}$, we can functorially restrict the isotropies to only be elements of $\mathcal{F}$ by simply crossing with $E\mathcal{F}$.

Proposition 17.2.67. *If X is a G-CW complex, then the geometric isotropy of $X \times E\mathcal{F}$ is given by*

$$\mathcal{F} \cap \Phi_X.$$

Corollary 17.2.68. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two families, then*

$$E\mathcal{F}_1 \times E\mathcal{F}_2$$

is the universal space associated to the family $\mathcal{F}_1 \cap \mathcal{F}_2$.

Example 17.2.69. For any G-CW complex X,

$$E\mathcal{F}_e \times X = EG \times X$$

is the Borel space which frees up the action of G.

Definition 17.2.70. If $\mathcal{F}$ is a family, then let

$$E\mathcal{F}_+ \to S^0$$

be the pointed map which sends $E\mathcal{F}$ to the non-basepoint. Let $\tilde{E}\mathcal{F}$ denote the cofiber.

Example 17.2.71. If $\mathcal{F} = \mathcal{A}ll$, then $\tilde{E}\mathcal{F} \simeq *$.

Example 17.2.72. If $\mathcal{F} = \mathcal{P}$ is the family of proper subgroups, then a model for $\tilde{E}\mathcal{P}$ is

$$\tilde{E}\mathcal{P} = S^{\infty\bar{\rho}_G} = \operatorname{colim}_n S^{n\bar{\rho}_G} = S^0[a_{\bar{\rho}_G}^{-1}],$$

the infinite $\bar{\rho}_G$-sphere.

Definition 17.2.73. For a pointed G-space X, let

$$E\mathcal{F}_+ \wedge X \to X \to \tilde{E}\mathcal{F} \wedge X$$

be the isotropy separation sequence , the result of smashing the defining cofiber sequence for $\tilde{E}\mathcal{F}$ with X.

Considering the fixed points of $E\mathcal{F}$, the following is an immediate, important application of the definitions.

Proposition 17.2.74. *Let X be a pointed G-CW complex and let $\mathcal{F}$ be a family.*

1. *For any $H \in \mathcal{F}$, the map*

$$i_H^*(E\mathcal{F}_+ \wedge X) \to i_H^* X$$

is an H-equivalence and $i_H^*(\tilde{E}\mathcal{F} \wedge X)$ is contractible.

2. *For any $K \notin \mathcal{F}$, the map*

$$X^K \to (\tilde{E}\mathcal{F} \wedge X)^K$$

is an equivalence an $(E\mathcal{F}_+ \wedge X)^K$ is contractible.

Putting this all together, the isotropy separation sequence expresses X as an "extension" of two conceptually simpler spaces:

1. A space $E\mathcal{F}_+ \wedge X$ whose geometric isotropy is contained in $\mathcal{F}$ and
2. a space $\tilde{E}\mathcal{F} \wedge X$ which is contractible when restricted to any of the subgroups in $\mathcal{F}$.

17.3 Stabilization and G-spectra

17.3.1 Conceptual goals for stabilization

There are several different conceptual approaches to stabilization in G-spectra, and somewhat surprisingly, these lead to the same results. There are two dominant themes: one geometric and one algebraic.

Goal (Geometric Stabilization)**.** Have a good theory of Milnor–Spanier–Atiyah duality for G-manifolds.

If we have a manifold on which G acts smoothly, then we can attempt to mirror the Milnor–Spanier–Atiyah explanation of Poincaré duality via an identification of the dual of our manifold with the Thom spectrum of the virtual normal bundle. Almost immediately we run into trouble: if the group action is non-trivial, then we have no equivariant embeddings of our manifold into a Euclidean space with a trivial action. We must instead consider an embedding of our manifold into some representation of G. This gives a kind of S-duality for our manifold, but it requires that we consider suspensions by possibly non-trivial representations.

Goal (Algebraic Stabilization)**.** Universally make pushout diagrams and pullback diagrams agree (and hence finite coproducts should be finite products).

This is one of the usual ∞-categorical formulations of stabilization: we take spaces and universally build a category out of it in which pushout and pullback diagrams agree. In particular, we see that finite coproducts and finite products then necessarily agree. In the equivariant context, we have an added subtlety which is fundamental to the more modern approach to understanding equivariant stable homotopy theory: *the group should be allowed to act on all indexing objects*. For pushouts and pullbacks, this means that the group can act on the indexing diagram, while for finite coproducts being finite products, this includes an identification of induction (the coproduct over G/H) with coïnduction (the product over G/H).

17.3.1.1 The Spanier–Whitehead category

Boardman's stable homotopy category was defined as an extension of the ordinary Spanier–Whitehead category under colimits. The same kind of analysis work equivariantly. We loosely sketch Adams' original treatment of the equivariant Spanier–Whitehead category [2] [62].

Definition 17.3.1. A universe for G is a countably infinite dimensional orthogonal representation U such that

1. the trivial representation $\mathbb{R}\subseteq U$ and
2. if $V\subseteq U$ is a finite dimensional representation, then the infinite orthogonal sum of V with itself also embeds into U.

A complete universe is one which contains all irreducible representations of G.

Equivalently, a complete universe is isomorphic to the countably infinite orthogonal direct sum of copies of the regular representation ρ_G.

Notation 17.3.2. If V is a finite dimensional representation of G, then let

$$\Sigma^V : \mathsf{Top}^G \to \mathsf{Top}^G$$

be the functor

$$X \mapsto S^V \wedge X.$$

Definition 17.3.3. If X and Y are finite, pointed G-CW complexes, then let

$$\{X,Y\}^G = \varinjlim_{V} [\Sigma^V X, \Sigma^V Y]_*^G$$

be the "stable homotopy classes of maps from X to Y", where the direct limit is taken over the poset of finite dimensional representations in a chosen complete G-universe.

Let SW^G denote the category whose objects are finite, pointed G-CW complexes and whose morphisms are these stable homotopy classes of maps.

Proposition 17.3.4. *The smash product of finite, pointed G-CW complexes extends to a symmetric monoidal product on the Spanier–Whitehead category.*

There is a clear extension of Spanier's original notion of "S-duality" to this equivariant Spanier–Whitehead category as described by Spanier–Whitehead [63]. Here, if X is a G-CW complex embedded in the V-sphere, then X is V-dual to the unreduced suspension of its complement. This gives us, for example, that if T is a finite G-set that embeds into V, then T_+ and $S^V \wedge T_+$ are V-dual.

Many of the standard arguments apply without change here.

Proposition 17.3.5. *The equivariant Spanier–Whitehead category is additive: finite wedges and products exist and agree and the morphism sets are naturally abelian group valued. The composition and symmetric monoidal products induce bilinear maps on morphism sets.*

The fact that our universe contains all trivial suspensions also ensures that in the Spanier–Whitehead category, cofiber sequences are also fiber sequences.

Corollary 17.3.6. *For any finite, pointed G-CW complex Y, the functors*

$$X \mapsto \begin{cases} \{X, \Sigma^n Y\}^G & n \geq 0 \\ \{\Sigma^{-n} X, Y\}^G & n \leq 0. \end{cases}$$

give a cohomology theory on finite, G-CW complexes.

The inclusion of G-spaces with a trivial action into G-spaces preserves cofiber sequences and hence by restriction, we have a cohomology theory on CW complexes (viewed as G-CW complexes with a trivial action). We can work a little more generally.

Proposition 17.3.7. *If $Q = G/N$ is a quotient of G, then the pushforward $i_*^N \colon \mathsf{Top}^Q \to \mathsf{Top}^G$ extends to an embedding*

$$i_*^N \colon \mathsf{SW}^Q \to \mathsf{SW}^G$$

which takes a G/N-CW complex X to itself, viewed as a G-CW complex, and which on morphisms, is the map induced on colimits by the inclusion of the subsystem of G/N-representations in all G-representations.

When $N = G$, we have just the ordinary Spanier–Whitehead category, and we can then restate our result about cohomology theories.

Corollary 17.3.8. *For any finite, pointed G-CW complex Y, the functors*

$$X \mapsto \begin{cases} \{X, \Sigma^n Y\}^G & n \geq 0 \\ \{\Sigma^{-n} X, Y\}^G & n \leq 0. \end{cases}$$

give a cohomology theory on finite CW complexes.

Remark 17.3.9. These cohomology theories on ordinary CW-pairs are not simply represented by maps in the ordinary, non-equivariant Spanier–Whitehead category from X to Y^G. We will see an explicit example of this in Example 17.3.21 below.

The non-trivial representations in the universe play a different role than the trivial ones: they provide a good theory of duality for manifolds with non-trivial action, as described above, and they produce transfer maps. We first build a slight extension of the Spanier–Whitehead category. The standard cofinality argument gives the following.

Proposition 17.3.10. *For any finite dimensional representation V, the V-fold suspension gives a fully-faithful embedding of the equivariant Spanier–Whitehead category into itself.*

Definition 17.3.11. Let the extended Spanier–Whitehead category be the category obtained from the equivariant Spanier–Whitehead category by formally adjoining representing objects $\Sigma^{-V} Y$ for the functors

$$X \mapsto \{\Sigma^V X, Y\}^G.$$

Let $\overline{\mathsf{SW}}^G$ denote the extended Spanier–Whitehead category.

This turns the suspension functors into autoequivalences of the Spanier–Whitehead category, and we can replace V-duality with ordinary, categorical duality.

Corollary 17.3.12. *In the extended Spanier–Whitehead category, all objects are dualizable.*

There is a natural map

$$Y \to \Sigma^{-V}(\Sigma^V Y) \tag{17.13}$$

adjoint to the identity map on $\Sigma^V Y$. This map is an equivalence in the extended Spanier–Whitehead category: it represents the suspension map

$$\{X, Y\}^G \to \{\Sigma^V X, \Sigma^V Y\}^G.$$

In particular, we can think of the objects $\Sigma^{-V} Y$ as being the formal desuspension of Y by V. Using these maps, we can extended $\mathbb{Z}$-graded theories.

Definition 17.3.14. Let Y is a pointed G-CW complex and let V and W be representations of G, then we define a functor of finite, pointed G-CW complexes by
$$X \mapsto \{\Sigma^W X, \Sigma^V Y\}^G =: Y^{W-V}(X).$$

The notation hides some of the naturality. We mean here actual pairs of representations, not isomorphism classes. There is also significant naturality in the vector spaces. We first rephrase the full-faithfulness of suspension here.

Proposition 17.3.15. *If V, W, and U are finite dimensional representations, then suspension gives a natural isomorphism*

$$Y^{W-V}(X) \xrightarrow{\cong} Y^{(U\oplus W)-(U\oplus V)}(X).$$

These isomorphisms are compatible in the sense that if U' is another finite dimensional representation, then the map

$$Y^{W-V}(X) \xrightarrow{\cong} Y^{((U'\oplus U)\oplus W)-((U'\oplus U)\oplus V)}(X)$$

is the same as the composite

$$Y^{W-V}(X) \xrightarrow{\cong} Y^{(U\oplus W)-(U\oplus V)}(X) \xrightarrow{\cong} Y^{((U'\oplus U)\oplus W)-((U'\oplus U)\oplus V)}(X).$$

Additionally, given any isomorphism of representations $V \to V'$, we have an associated isomorphism of representation spheres $S^V \to S^{V'}$. Smashing with either X or Y will then give us a natural isomorphisms. These new maps clearly depend only on the stable homotopy type of the map $S^V \to S^{V'}$, however.

Definition 17.3.16. Let JO^s be the maximal subgroupoid of the full subcategory of SW^G spanned by the representation spheres S^V.

Proposition 17.3.17. *Given a pointed G-space Y, the assignment*

$$\big(V, W, X\big) \mapsto Y^{W-V}(X)$$

extends to a functor

$$\mathsf{JO}^{s,op} \times \mathsf{JO}^s \times \mathsf{SW}^{G,op} \to \mathsf{Ab}.$$

This is the prototype of an "RO(G)-graded cohomology theory". The naming is quite misleading, however, since we are indexing here by pairs of representations (See Adams' extended discussion of this point for a stronger warning [2]). The isomorphism type of the abelian group

$$Y^{W-V}(X, A)$$

naturally depends only on the associated virtual representation $W - V \in RO(G)$. The problem is that the representation spheres can have non-trivial automorphisms. Put another way, in the extended Spanier–Whitehead category, representation spheres are all invertible under the smash product, since this is just a restatement of the map in Equation 17.13 applied to $Y = S^0$ being an isomorphism. The description above is a concrete way to describe the grading by the Picard groupoid of this symmetric monoidal category, as in [27]. There is a natural map $\mathrm{RO}(G)$ to the Picard group of the symmetric monoidal category, but in forming this, we have thrown away the information given by the isomorphisms.

17.3.1.2 Change of Groups

To compare the Spanier–Whitehead categories for G and its subgroups, we simply note that the set of representations of a subgroup H which are the restriction of a representation of G are cofinal in all representations of H. This gives the following.

Proposition 17.3.18. *If H is a subgroup of G, then the restriction functor i_H^* in pointed spaces extends to a restriction functor*

$$i_H^* \colon \mathcal{SW}^G \to \mathcal{SW}^H.$$

The induction functor $G_+ \underset{H}{\wedge} (\text{-})$ in pointed spaces extends to an induction functor

$$G_+ \underset{H}{\wedge} (\text{-}) \colon \mathcal{SW}^H \to \mathcal{SW}^G$$

which is left-adjoint to the restriction.

Our second goal is realized in the Spanier–Whitehead category via the additional representations: these give us stable maps in the wrong direction to the ordinary action maps $G/H \to *$. A choice of embedding

$$e \colon G/H \hookrightarrow V$$

of G/H into a representation V gives us a Thom collapse map

$$S^V \to G/H_+ \wedge S^V,$$

and hence a stable homotopy class

$$t_H^G \in \{S^0, G/H_+\}^G.$$

Definition 17.3.19. Let t_H^G be the transfer, the stable homotopy class $\{S^0, G/H_+\}^G$ given by the Thom collapse of any choice of embedding $G/H \hookrightarrow V$.

This definition seems to depend on the choices of embeddings, but by consider larger and larger representations (i.e. looking farther down the colimit defining the stable homotopy classes of maps), we see that the connectivity of the space of such embeddings goes to infinity and hence any choices are stably homotopic.

The "wrong-way" transfer map on orbits provides powerful new tools. In particular, it realizes the algebraic goal and makes the Spanier–Whitehead category behave even more like an equivariant algebraic category like the chain complexes of representations.

Theorem 17.3.20 (Wirthmüller isomorphism [71])**.** *The induction functor* $G_+ \underset{H}{\wedge} (\text{-})$ *is also right-adjoint to the restriction* i_H^**.*

Induction being also the *right* adjoint to the forgetful functor means that we lose a lot of intuition for maps between G-CW complexes. This is what has traditionally made computations in equivariant stable homotopy theory more daunting than their classical counterparts.

Example 17.3.21. For any finite group G, we have an isomorphism

$$\{S^0, G_+\}^G \cong \{S^0, S^0\}^{\{e\}} \cong \mathbb{Z}.$$

In particular, the free G-set G has non-trivial maps from the fixed G-set $*$!

It is possible, however, to give some algebraic information about at least stable maps out of the zero sphere. For this, we exploit some of the additional extra structure.

17.3.2 Mackey functors and Segal–tom Dieck

Combining our adjunctions with ordinary finite sums being finite products also gives us the duality results that we want.

Corollary 17.3.22. *If T is a finite G-set, then T_+ is self-dual in the extended equivariant Spanier–Whitehead category.*

Corollary 17.3.23. *For any finite G-set T, the functors*

$$\begin{aligned}&\{T_+ \wedge (\text{-}), (\text{-})\}^G \colon \mathsf{SW}^{G^{op}} \times \mathsf{SW}^G \to \mathsf{Ab} \textit{ and}\\ &\{(\text{-}), T_+ \wedge (\text{-})\}^G \colon \mathsf{SW}^{G^{op}} \times \mathsf{SW}^G \to \mathsf{Ab}\end{aligned}$$

are naturally isomorphic.

This algebraic structure is encoded in a Mackey functor.

Definition 17.3.24 ([21])**.** A Mackey functor is a pair of functors: one covariant, $\underline{M}_*$, and one contravariant, $\underline{M}^*$, from the category of finite G-sets to abelian groups such that

1. The functors agree on objects:

$$\underline{M}^*(T) = \underline{M}_*(T) =: \underline{M}(T).$$

2. The functor $\underline{M}^*$ is product preserving.
3. We have a Beck–Chevalley condition: if we have a pullback diagram of finite G-sets
$$\begin{tikzcd} T' \arrow[r, "f'"] \arrow[d, "h'"'] & T \arrow[d, "h"] \\ S' \arrow[r, "f"'] & S, \end{tikzcd}$$
then we have a commutative diagram
$$\begin{tikzcd} \underline{M}(T') \arrow[r, "\underline{M}_*(f')"] & \underline{M}(T) \\ \underline{M}(S') \arrow[u, "\underline{M}^*(h')"] \arrow[r, "\underline{M}_*(f)"'] & \underline{M}(S). \arrow[u, "\underline{M}^*(h)"'] \end{tikzcd}$$

The contravariant map $\underline{M}^*(f)$ is called the "restriction" along f, while the covariant map $\underline{M}_*(f)$ is called the "transfer" along f.

Morphisms of Mackey functors are simply collections of homomorphisms of abelian groups that commute with all of the additional structure maps.

Notation 17.3.25. Let $\mathcal{M}ackey^G$ denote the category of G-Mackey functors.

Example 17.3.26. A C_p-Mackey functor is a C_p-coefficient system $\underline{M}$ together with a transfer map

$$tr_e^{C_p} : \underline{M}(C_p/e) \to \underline{M}(C_p/C_p)$$

that factors through the C_p coinvariants of $\underline{M}(C_p/e)$ and such that

$$res_e^{C_p} \circ tr_e^{C_p}(a) = \sum_{g \in C_p} g \cdot a.$$

The push-pull formula, when applied to the case where all of T, S, and S' are orbits, is sometimes called the "double-coset formula". It expresses the various ways we could rewrite

$$\{G/H_+ \wedge X, G/K_+ \wedge Y\}^G$$

as either the set of H-equivariant maps or of K-equivariant maps.

Corollary 17.3.27. *For any finite, pointed G-CW complexes X and Y, the coefficient system*
$$T \mapsto \{T_+ \wedge X, Y\}^G$$
extends to a Mackey functor.

In general, the Beck–Chevalley condition that shows up in the definition of a Mackey functor suggest a reformulation in terms of functors from a correspondence category. This works for Mackey functors too, as shown by Lindner.

Definition 17.3.28. Let $\mathcal{A}_{\mathbb{N}}$ denote the category whose objects are finite G-sets and for which the morphisms from S to T are isomorphisms classes of finite G-sets over $S \times T$:

$$\mathcal{A}_{\mathbb{N}}(S,T) = \{[S \leftarrow U \rightarrow T]\}.$$

Composition is given by pullback.

The identity functor gives an isomorphism $\mathcal{A}_{\mathbb{N}} \cong \mathcal{A}_{\mathbb{N}}^{op}$, and the disjoint union of finite G-sets is both the product and the coproduct in this category. This means that the Hom sets are naturally commutative monoid valued.

Proposition 17.3.29 ([52]). *A Mackey functor is equivalently described as a product preserving functor $\mathcal{A}_{\mathbb{N}} \to \mathsf{Ab}$.*

This makes Mackey functors into a kind of diagram category, and morphisms of Mackey functors are just natural transformations.

Proposition 17.3.30. *The category Mackey^G of G-Mackey functors is an abelian category with enough projectives.*

In fact, the projective generators are easy to describe.

Definition 17.3.31. Let $\underline{A}_T$ be the functor

$$S \mapsto K\big(\mathcal{A}_{\mathbb{N}}(T,S)\big),$$

where $K(\text{-})$ denotes group completion.

When $T = *$, we call $\underline{A} = \underline{A}_*$ the Burnside Mackey functor.

Remark 17.3.32. The abelian group $\underline{A}(*)$ is the Grothendieck group of the category Fin^G. In particular, objects are finite virtual G-sets. This has a ring structure under Cartesian product, and we call it the Burnside ring.

Proposition 17.3.33. *The Mackey functor $\underline{A}_T$ represents the functor*

$$\underline{M} \mapsto \underline{M}(T),$$

and hence is projective.

There is a non-full, product preserving embedding

$$(\mathsf{Fin}^G)^{op} \hookrightarrow \mathcal{A}_{\mathbb{N}}$$

which is also the identity on objects and which sends a map $f\colon S \to T$ to the correspondence

$$T \xleftarrow{f} S \xrightarrow{=} S.$$

Precomposing with this inclusion gives a forgetful functor

$$U\colon \mathsf{Mackey}^G \to \mathsf{Coeff}^G.$$

This forgetful functor just forgets the transfer maps, recording only the restriction maps. The algebra underlying both Mackey functors and a lot of the equivariant intuition is that this is part of a monadic (and comonadic) adjunction.

Proposition 17.3.34. *The forgetful functor $U\colon \mathsf{Mackey}^G \to \mathsf{Coeff}^G$ has a left adjoint $L\colon \mathsf{Coeff}^G \to \mathsf{Mackey}^G$, the "Mackeyfication" of a coefficient system. The category of Mackey functors is the category of algebras over the associated monad.*

The left adjoint L free adjoins missing transfers, placing only those contraints required by the Mackey functor structure.

Example 17.3.35. Consider a C_p-coefficient system $\underline{M}$. Then $L(\underline{M})(C_p/e) = \underline{M}(C_p/e)$, while

$$L(\underline{M})(C_p/C_p) = \underline{M}(C_p/C_p) \oplus \big(\underline{M}(C_p/e)\big)/C_p,$$

where $\underline{M}(C_p/e)/C_p$ is the coinvariants. The transfer map here is composite of the inclusion with the canonical quotient:

$$\underline{M}(C_p/e) \to \big(\underline{M}(C_p/e)\big)/C_p \hookrightarrow L(\underline{M})(C_p/C_p),$$

while the restriction is given by the restriction in $\underline{M}$ on the summand $\underline{M}(C_p/C_p)$ and the trace

$$\big(\underline{M}(C_p/e)\big)/C_p \xrightarrow{a\mapsto\sum ga} \big(\underline{M}(C_p/e)\big)^{C_p} \subseteq \underline{M}(C_p/e).$$

This connects back to equivariant stable homotopy theory in a very transparent way.

Theorem 17.3.36. *If X is a finite, pointed G-CW complex, then we have a natural isomorphism*

$$\underline{\{S^0, X\}} \cong L\big(\underline{\pi}_0^s(X)\big),$$

where $\underline{\pi}_0^s$ is the coefficient system

$$T \mapsto \pi_0^s\big(\mathsf{Top}_*^G(T_+, X)\big) = \{S^0, \mathsf{Top}_*^G(T_+, X)\}^e$$

of non-equivariant stable maps to the fixed points of X.

Since left adjoints are easy to compute on representable functors, we deduce the stable homotopy groups of the zero sphere.

Corollary 17.3.37. *For any finite G, we have*

$$\underline{\{S^0, S^0\}} \cong \underline{A}.$$

Putting this again into words, equivariant stabilization takes our expected value and freely puts in the transfers. These transfer terms arise as summands, isomorphic to Weyl coinvariants of the value of the coefficient system at other orbits.

An incredible theorem of Segal and tom Dieck vastly generalizes this, allowing us to understand what happens more arbitrarily. In particular, they determined the representing object for the cohomology theory given by Corollary 17.3.8.

Theorem 17.3.38 (Segal–tom Dieck splitting)**.** *Let Y be a fixed finite, pointed G-CW complex. If X is a finite, pointed G-CW complex on which G acts trivially, then the functor*

$$X \mapsto \{X, \Sigma^n Y\}^G$$

is represented by the non-equivariant infinite loop space

$$\prod_{(H)} \Omega^\infty \Sigma^n \big(EW_G(H)_+ \underset{W_G(H)}{\wedge} \Sigma^\infty Y^H \big),$$

where the product ranges over all conjugacy classes of subgroups of G and where $W_G(H)$ is the Weyl group of H in G.

17.3.3 Fixed points as a Mackey functor

Even though we have not produced any particular models of G-spectra, the Spanier–Whitehead category provides a litany of desiderata. In particular, the abelian group valued cohomology theory given by

$$Y^*(X, A) = \{X/A, \Sigma^n Y\}^G$$

is really just a piece of a Mackey functor valued cohomology theory:

$$\underline{Y}^*(X, A)(T) = \{X/A, \Sigma^n T_+ \wedge Y\}^G.$$

The Segal–tom Dieck splitting describes the representing object for each of these (as T varies):

$$\bigvee_{(H)} \Sigma^\infty EW_G(H)_+ \underset{W_G(H)}{\wedge} (T_+ \wedge Y)^H,$$

and the Yoneda Lemma then gives more structure reflecting the Mackey functor structure on the cohomology groups:

Proposition 17.3.39. *The assignment*

$$T \mapsto \bigvee_{(H)} \Sigma^\infty EW_G(H)_+ \underset{W_G(H)}{\wedge} (T_+ \wedge Y)^H$$

extends to a Mackey functor object in the homotopy category of spectra.

The summand corresponding to G is simply

$$\Sigma^\infty_+ Y^G.$$

This is the one we would expect to have! The others are somewhat more surprising. The Mackey structure gives us a way to interpret these, however: they are the images of the transfers from various subgroups. To see this, note that the assignment

$$Y \mapsto \bigvee_{(H)} \Sigma^\infty EW_G(H)_+ \underset{W_G(H)}{\wedge} Y^H$$

is also functorial in the G-CW complex Y. Moreover, map induced by a map $f\colon Y \to Z$ is the expected one: wedge together the various maps

$$EW_G(H)_+ \underset{W_G(H)}{\wedge} Y^H \xrightarrow{EW_G(H)_+ \underset{W_G(H)}{\wedge} f^H} EW_G(H)_+ \underset{W_G(H)}{\wedge} Z^H.$$

Applying this to the maps $G/H_+ \wedge Y \to Y$ gives us an identification of the various other summands. We spell this out for $H = \{e\}$.

Example 17.3.40. Now let $T = G$ as above and consider the unique map $G \to *$. This gives us the transfer from $\{e\}$ to G in the Mackey functor structure, and in the homotopy category, this is witnessed as the map

$$\bigvee_{(H)} \Sigma^\infty EW_G(H)_+ \underset{W_G(H)}{\wedge} (G_+ \wedge Y)^H \to \bigvee_{(H)} \Sigma^\infty EW_G(H)_+ \underset{W_G(H)}{\wedge} Y^H.$$

Since $G_+ \wedge Y$ has a free G-action, the only nontrivial summand is the one corresponding to $\{e\}$. Thus the transfer is represented in the homotopy category by

$$Y \simeq EW_G(\{e\})_+ \underset{W_G(\{e\})}{\wedge} (G_+ \wedge Y) \to \bigvee_{(H)} \Sigma^\infty EW_G(H)_+ \underset{W_G(H)}{\wedge} Y^H.$$

The appearance of the homotopy orbits, rather than just the underlying space, are also explained by the Mackey functor structure. The transfer map factors through the Weyl coïnvariants, since there is a unique map $G \to *$. The representing object then factors through the homotopy coïnvariants.

Remark 17.3.41. The restriction maps in the Mackey functor structure can also be described, but they are a little less intuitive. When restricting all the way to $H = \{e\}$, the map is just the wedge of the ordinary Becker–Gottleib transfer maps

$$\Sigma^\infty EW_G(H)_+ \underset{W_G(H)}{\wedge} Y^H \to \Sigma^\infty Y^H \to \Sigma^\infty Y$$

Remark 17.3.42. The construction

$$Y \mapsto \bigvee_{(H)} EW_G(H)_+ \underset{W_G(H)}{\wedge} \Sigma^\infty Y^H$$

commutes with filtered colimits, and we can therefore use this to extend to infinite complexes in the usual way.

17.3.4 Categories of G-spectra

There are two main approaches to lifting the homotopical discussions above to a model or ∞-category: extending the geometric, representation theory based spectra or extending the algebra, Mackey functor approach. These all have the same underlying homotopy theory, which is generated by the extended equivariant Spanier–Whitehead category under colimits, just as with the ordinary stable homotopy category.

Continuing our model-independent approach, we list a collection of desired features, which are in some sense defining properties. Any category Sp^G of G-spectra will have these properties.

1. They are closed symmetric monoidal, pointed model / ∞-categories.
2. For every $H \subseteq G$, there is a strong symmetric monoidal restriction functor
$$i_H^*\colon \mathsf{Sp}^G \to \mathsf{Sp}^H$$
which has both adjoints
$$G_+ \underset{H}{\wedge} (\text{-}) \dashv i_H^*(\text{-}) \dashv F_H(G_+, \text{-}).$$
3. The natural map
$$G_+ \underset{H}{\wedge} (\text{-}) \to F_H(G_+, \text{-})$$
descends to an equivalence (the Wirthmüller isomorphism) in the homotopy category.
4. There are functors $\Sigma_+^\infty\colon \mathsf{Top}^G \to \mathsf{Sp}^G$ which commute with the restriction functors.
5. The homotopy functor $\Sigma_+^\infty\colon ho\mathsf{Top}^G \to ho\mathsf{Sp}^G$ factors through the extended Spanier–Whitehead category and induces an equivalence between the extended Spanier–Whitead category and the compact objects in $ho\mathsf{Sp}^G$.
6. The homotopy category is a Brown category [44].

Following the historical designation, we call such a category of G-spectra "genuine". There are many models for a category of genuine G-spectra:

1. Lewis–May–Steinberger consider the clear extension of Adams' original treatment of spectra, viewing G-spectra as sequences of G-spaces, indexed by finite dimensional representations of some fixed G-universe [48].

2. Shimakawa built an equivariant version of Segal's Γ spaces, showing that Γ-G-spaces model connective G-spectra [61]

3. Mandell–May generalized orthogonal spectra to equivariant orthogonal spectra [54, 39].

4. Mandell extended work of Hovey–Shipley–Smith on symmetric spectra to produce equivariant symmetric spectra [56].

5. Blumberg considered continuous functors on G-CW complexes and a version of excision here [10] (See also [20]).

6. Guillou–May took a more algebraic approach, building equivariant spectra as spectral Mackey funtors [35].

7. Barwick produced a genuine ∞-category of spectral Mackey functors in a very general context [8].

Any of these categories will work for our discussion in this section. When discussing multiplicative concerns, there are several subtleties which arise, and only some of the models are known to work well there: equivariant orthogonal spectra [39], equivariant symmetric spectra [38], and the ∞-categorical enrichment for spectral Mackey functors.

17.3.4.1 Equivariant cohomology and Brown representability

Since our category of G-spectra contains full-faithfully the extended Spanier–Whitehead category as a symmetric monoidal subcategory, the Picard groupoid of the extended Spanier–Whitehead category is a sub-groupoid of the Picard groupoid of G-spectra. This gives a natural grading for maps in the category and for homology or cohomology theories. Just as classically, G-spectra represent cohomology theories.

Definition 17.3.43 (See [58, XIII])**.** An RO(G)-graded cohomology theory is a functor

$$E^{(\text{-})}(\text{-})\colon \mathsf{JO}^s \times ho\mathsf{Top}_*^{G,op} \to \mathsf{Ab}$$

together with natural suspension isomorphisms

$$\Sigma^W\colon E^V(X) \to E^{W\oplus V}(\Sigma^W X),$$

for each representation W such that

1. for each fixed V, E^V takes wedges to products and is exact in the middle for cofiber sequences,

2. the suspension isomorphisms are natural in W for maps in JO^s, and
3. for all representations W and U, we have
$$\Sigma^{W\oplus U} = \Sigma^W \circ \Sigma^U.$$

Naturality of the suspension isomorphisms gives us a commutative square for any V and W and any map $f\colon X \to Y$

$$\begin{tikzcd} E^V(Y) \arrow[r, "E^V(f)"] \arrow[d, "\Sigma^W"'] & E^V(X) \arrow[d, "\Sigma^W"] \\ E^{W\oplus V}(\Sigma^W Y) \arrow[r, "E^{W\oplus V}(\Sigma^W f)"'] & E^{W\oplus V}(\Sigma^W X). \end{tikzcd}$$

In particular, the map $E^V(f)$ is completely determined by $E^{W\oplus V}(\Sigma^W f)$.

Proposition 17.3.44. *If E is an* $\mathrm{RO}(G)$*-graded cohomology theory, then E descends to a functor*

$$E^{(\text{-})}(\text{-})\colon \mathsf{JO}^s \times \mathsf{SW}^{G,op} \to \mathsf{Ab}$$

Although we call these $\mathrm{RO}(G)$-graded, as written, these are graded by the actual representations. The suspension isomorphisms allow us to extend to pairs of representations, just as with the Spanier–Whitehead category. Again, the correct approach is to grade on the Picard groupoid. In this more general setting, the $\mathrm{RO}(G)$-graded cohomology theories actually descend to the extended Spanier–Whitehead categories, where we set

$$E^V(\Sigma^{-W}X) := E^{W\oplus V}(X),$$

and the map induced by the map $X \to \Sigma^{-W}\Sigma^W X$ of Equation 17.13 is taken to $(\Sigma^W)^{-1}$.

Notation 17.3.45. Following Hu–Kriz [45], we will use $*$ as a wildcard for grading by $\mathbb{Z}$ and $\star$ for $\mathrm{RO}(G)$.

Proposition 17.3.46. *For any G-spectrum E, the assignment*

$$(V, X) \mapsto [\Sigma^\infty X, S^V \wedge E]^G$$

together with the natural suspension isomorphisms

$$[\Sigma^\infty X, S^V \wedge E]^G \xrightarrow{\cong} [\Sigma^\infty \Sigma^W X, S^{W\oplus V} \wedge E]^G$$

give an $\mathrm{RO}(G)$*-graded cohomology theory.*

The Brown representability theorem allows us to invert this procedure, at least in the homotopy category of spectra. The best reference for this is the treatment in the Alaska notes [58], which uses Lewis–May–Steinberger spectra [48]. However, since the representability takes place in the homotopy category, the result is again model agnostic.

Theorem 17.3.47 ([58, XIII.3])**.** *If $E^\star$(-) is an* RO(G)*-graded cohomology theory, then there is a G-spectrum E which represents it in the homotopy category.*

Example 17.3.48. In general, Bredon cohomology for a coefficient system $\underline{M}$ only gives a $\mathbb{Z}$-graded cohomology theory. However, if $\underline{M}$ is the underlying coefficient system for a Mackey functor, then Lewis–May–McClure show that Bredon cohomology with coefficients in $\underline{M}$ has a natural extension [47]. The resulting G-spectrum is the Eilenberg–Mac Lane spectrum $H\underline{M}$ and it has the defining property that the homotopy Mackey functors are given by

$$\underline{\pi}_k H\underline{M} = \begin{cases} \underline{M} & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

To get a sense of why the transfers are needed here, consider the group $G = C_2$. If Bredon cohomology with coefficients in $\underline{M}$ extends to an $\mathrm{RO}(C_2)$-graded theory, then we can take the Bredon cohomology of any virtual representation sphere. Recall that the sign sphere S^σ has a cell structure

$$S^0 \cup C_{2+} \wedge e^1,$$

where the attaching map is the action map

$$C_{2+} \wedge S^0 \to S^0.$$

The dual of the fold map is the transfer map $S^0 \to C_{2+}$, giving a fiber sequence

$$S^{-\sigma} \to S^0 \to C_{2+}.$$

Mapping out of this gives a formula for the $-\sigma$th cohomology of a point with coefficients in $\underline{M}$, and we see that we must have a map

$$\underline{M}(C_2) \to \underline{M}(*)$$

which restricts to the ordinary fold map non-equivariantly. This is equivalent to the data of a C_2-Mackey functor.

As Lewis observed, working with this larger grading allows us to see more structure than we might have just using integral gradings [49].

Example 17.3.49. Let $G = C_2$, and let $\underline{\mathbb{Z}}$ be the constant Mackey functor $\mathbb{Z}$. Then

$$H_*(\mathbb{C}P^1; \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ 0 & \textit{otherwise}, \end{cases}$$

while

$$H_\star(\mathbb{C}P^1;\underline{\mathbb{Z}}) = H_\star(pt;\underline{\mathbb{Z}}) \oplus H_{\star-\rho}(pt;\underline{\mathbb{Z}}).$$

Here $\mathbb{C}P^1$ is a C_2-space via complex conjugation, and ρ is the regular representation of C_2.

Example 17.3.50. Consider the equivariant K-theory functor which assigns to a G-space the Grothendieck group of complex, equivariant vector bundles over X. Equivariant Bott periodicity says that for every complex representation V, we have a natural isomorphism

$$\tilde{K}^0_G(X) \cong \tilde{K}^0_G(\Sigma^V X).$$

These isomorphisms allow us to extend to an $\mathrm{RO}(G)$-graded theory. There is an analogous story for real K-theory KO_G or Atiyah's Real K-theory $K_{\mathbb{R}}$.

Building on this example, we can construct homotopical versions of bordism theories via Thom spectra.

Example 17.3.51. Consider the space $BO_G(n)$ which classifies n-dimensional equivariant bundles. There is a universal n-plane bundle over this, and associated Thom space. These assemble to give the Thom spectrum MO_G of homotopy equivariant unoriented bordism.

There is a simply story for equivariant complex bordism; here we use the spaces $BU_G(n)$ which classify n-dimensional complex equivariant bundles. This gives a spectrum MU_G.

There is also a Real version of bordism, where we take the spaces $BU(n)$ as C_2 spaces with the C_2-action by complex conjugation. Here, the associated Thom spectrum is $\mathrm{MU}_{\mathbb{R}}$, the Real bordism spectrum of Fujii–Landweber [26, 46].

These theories are not the geometrically defined bordism theories one would expect from the non-equivariant case. The geometrically defined theories are only $\mathbb{Z}$-graded theories.

17.3.5 Fixed, homotopy fixed, and geometric fixed points

We turn now some of the basic properties and constructions.

17.3.5.1 Fixed points

Proposition 17.3.52. *For any normal subgroup N, there is a strong symmetric monoidal, faithful functor*

$$i^N_* \colon \mathsf{Sp}^{G/N} \to \mathsf{Sp}^G$$

lifting and extending the functor on the extended Spanier–Whitehead categories of Proposition 17.3.7.

When $N = G$, we will also write this just as i_*.

Remark 17.3.53. The point-set models for the pushforward are all strong symmetric monoidal functors. However, we have no effective control over the homotopical behavior of commutative monoids under it. In Example 17.4.48 below, we show how badly this can go.

Proposition 17.3.54. *The functor i_*^N has a right adjoint*

$$(\text{-})^N \colon \mathsf{Sp}^G \to \mathsf{Sp}^{G/N},$$

the N-fixed point functor.

In general, this fixed point functor is difficult to understand. Our conditions on the relationship with the Spanier–Whitehead category then describes this on finite G-CW complexes.

Theorem 17.3.55 (Segal–tom Dieck Splitting)**.** *If X is a finite G-CW complex, then*

$$\left(\Sigma_+^\infty X\right)^G \simeq \bigvee_{(H)} EW_G(H)_+ \underset{W_G(H)}{\wedge} \Sigma_+^\infty X^H.$$

The zero sphere S^0 is in the image of the pushforward, so combining the pushforward with induction, we see that the spectrum of maps out of

$$G/H_+ \cong G_+ \underset{H}{\wedge} S^0$$

is the H-fixed points. Mapping out of the maps $G/H_+ \to G/K_+$ for various subgroups H and K then give us restriction and conjugation maps

$$E^K \to E^H.$$

The Wirthmüller isomorphism says that induction is also the right adjoint to the restriction, homotopically, and this gives us transfer maps the other way

$$E^H \to E^K.$$

Proposition 17.3.56. *For any G-spectrum E, the fixed points E^H as H ranges over the subgroups of G assemble into a Mackey functor object in the homotopy category of spectra.*

This proposition can be viewed as a more general one, building on the closed symmetric monoidal structure.

Proposition 17.3.57. *For any G-spectra E and E', the homotopy classes of maps $E \to E'$ assemble into a Mackey functor*

$$T \mapsto [T_+ \wedge E, E']^G$$

which when evaluated at G/H, records the homotopy classes of H-equivariant maps.

This is the key idea in G-spectra: stabilization made our category equivariantly additive. We can not only add maps as usual in a stable setting but also form "twisted" sums, where the group acts on the source. These are the transfer maps. All of the algebraic invariants which we are used to are naturally Mackey functor valued in equivariant stable homotopy.

In particular, our homotopy groups naturally assemble into homotopy Mackey functors. The fully-faithful inclusion of the extended Spanier–Whitehead category into the homotopy category allows us to compute this for the sphere spectrum.

Corollary 17.3.58. *There are natural Mackey enrichments of Bredon cohomology with coefficients in $\underline{M}$ and of the equivariant K-theories.*

Just as classically, spheres are appropriately connected.

Proposition 17.3.59. *For all $n < 0$, we have*

$$\underline{\pi}_n(T_+ \wedge \Sigma^\infty S^0) = 0,$$

while

$$\underline{\pi}_0(T_+ \wedge \Sigma^\infty S^0) = \underline{A}_T.$$

The connectivity of the sphere spectrum then allows us to mirror the classical formation of the Postnikov tower: the long exact sequence in stable homotopy shows that we can kill homotopy groups above some fixed dimension k for a G-spectrum by transfinitely coning off all maps from equivariant spectra of the form $T_+ \wedge S^\ell$ with $\ell \geq k$.

Proposition 17.3.60. *There is a t-structure on G-spectra, where $\tau_{\geq 0}^{Post}$ consists of all spectra E with $\underline{\pi}_k(E) = 0$ for $k < 0$ and similarly for $\tau_{\leq -1}^{Post}$. The heart of this t-structure is the category of Mackey functors.*

Corollary 17.3.61. *Every G-spectrum E has a Postnikov tower: there is a functorial tower under E*

$$\cdots \to P^n_{Post}(E) \to P^{n-1}_{Post}(E) \to \dots$$

in which the homotopy Mackey functors of $P^n_{Post}(E)$ vanish for $k > n$ and agree with those of E for $k \leq n$. The fibers are Eilenberg–Mac Lane spectra for the homotopy Mackey functors of E.

Remark 17.3.62. The tower under E dual to the Postnikov tower is the Whitehead tower, where here we approximate E by appropriately connective objects.

Proposition 17.3.63. *The t structure is compatible with the symmetric monoidal structure in the sense that if $E \in \tau_{\geq n}^{Post}$ and and $E' \in \tau_{\geq m}^{Post}$, then*

$$E \wedge E' \in \tau_{\geq (n+m)}^{Post}.$$

This is important for knowing the multiplicativity of the Atiyah–Hirzebruch spectral sequence below.

We close this subsection with a comment about describing fixed points for more general spectra. For trivial desuspensions of finite G-CW complexes, we can still understand the fixed points. For more general desuspensions, we have little understanding.

Example 17.3.64. Let $G = C_2$ and let σ denote the sign representation of C_2. The fixed points of $\Sigma^{-\sigma}\Sigma^\infty X$ sit in a fiber sequence

$$\big(\Sigma^{-\sigma}\Sigma^\infty X\big)^{C_2} \to \Sigma^\infty X^{C_2} \vee (\Sigma^\infty EC_{2+} \underset{C_2}{\wedge} X) \to \Sigma^\infty X,$$

where the rightmost map is the restriction in the homotopy Mackey functor for X.

17.3.5.2 Borel and coBorel, free and cofree

The space EG is a universal space on which G acts freely. Using this, we can free up or co-free up the action on any G-space or spectrum. This has the effect of weakening the amount of information we need to remember.

Definition 17.3.65. If E is a G-spectrum, then let

$$E_h := EG_+ \wedge E$$

be the Borel construction on E.

Even though the action on EG is free, we have non-trivial fixed points stably. This is another consequence of the transfer, generalizing what we saw in Example 17.3.21.

Theorem 17.3.66 (Adams' Isomorphism [2])**.** *For any G-spectrum E, the transfer induces an equivalence*

$$EG_+ \underset{G}{\wedge} E \to \big(EG_+ \wedge E\big)^G.$$

More generally, if E is a spectrum on which a normal subgroup N acts freely, then the transfer induces an equivalence of G/N-spectra

$$E_N \to E^N,$$

and the fixed points also become the left adjoint to the pushforward.

Remark 17.3.67. Smashing with the universal space $E\mathcal{F}_{N+}$ of Examples 17.2.65 always frees up the N-action.

Definition 17.3.68. If E is a G-spectrum, then let

$$E^h := F(EG_+, E)$$

be the cofree G-spectrum of pointed maps from EG_+ to E.

Lemma 17.3.69. *The map $EG_+ \to S^0$ gives us canonical maps*

$$E_h \to E \text{ and } E \to E^h,$$

and the map

$$E_h \to E$$

is the left-most map from the isotropy separation sequence smashed with E.

By construction, both the Borel construction and the cofree spectrum care only about underlying equivalences.

Proposition 17.3.70. *If $f\colon E \to E'$ is an equivariant map such that $i_e^* f$ is an equivalence, then*

$$f^h\colon E^h \to {E'}^h \text{ and } f_h\colon E_h \to E'_h$$

are equivariant equivalences.

Definition 17.3.71. The homotopy fixed points of E are the G-fixed points of E^h:

$$E^{hG} := \big(F(EG_+, E)\big)^G.$$

Since $i_H^* EG = EH$ for any subgroup H,

$$E^{hH} \simeq \big(F(EG_+, E)\big)^H,$$

and the notation is unambiguous.

Example 17.3.72. Let $\underline{M}$ be a Mackey functor. By construction, the homotopy Mackey functors of the spectrum

$$H\underline{M}^h = F(EG_+, H\underline{M})$$

are the Bredon cohomology Mackey functors of EG with coefficients in $\underline{M}$. The standard cellular complex for EG_+ then allows us to determine these via cellular Bredon cohomology. In particular, we see that the homotopy Mackey functors are the ordinary Mackey functors associated to group cohomology, in this case, with coefficients in the G-module $\underline{M}(G)$:

$$\underline{\pi}_{-k}(HM^h)(G/K) \cong H^k(K; \underline{M}(G)).$$

The cofree construction can be viewed in another way, connecting G-spectra to the perhaps more expected "G-objects in spectra". Remark 17.2.32 points out that in spaces, the corresponding categories of spaces are homotopically distinct, and here we see something similiar. The cofree construction gives us a way to compare them.

Definition 17.3.73. A G-object in spectra is a functor $BG \to \mathsf{Sp}$. The category of G-objects in spectra is the functor category $\mathrm{Fun}(BG, \mathsf{Sp})$.

Just as with spaces, a G-object in spectra is an ordinary spectrum E together with an action map of spectra $G_+ \wedge E \to E$ that satisfies the usual axioms for an action. There is an evident forgetful functor which forgets the additional structure.

Proposition 17.3.74. *The restriction map*

$$\mathsf{Sp}^G \to \mathsf{Sp}$$

lifts along the forgetful functor $Fun(BG, \mathsf{Sp}) \to \mathsf{Sp}$ *to give a functor*

$$\mathsf{Sp}^G \to \mathrm{Fun}(BG, \mathsf{Sp}).$$

The easiest way to see this is actually via the Wirthmüller isomorphisms. The restriction of E can be identified with the fixed points of $G_+ \wedge E$, via induction's equivalence with coinduction. As we have used before, $G^{op} \cong G$ is the automorphism group of G as a G-set, and by naturality of the tensoring operation of G-spaces on G-spectra, this gives an action.

Proposition 17.3.75. *The cofree G-spectrum gives a homotopical functor*

$$F(EG_+, \text{-})\colon \mathrm{Fun}(BG, \mathsf{Sp}) \to \mathsf{Sp}^G$$

from G-objects in spectra to G-spectra.

This allows us to view any G-object in spectra as an actual G-spectrum in a homotopically meaningful way.

Example 17.3.76. The Goerss–Hopkins–Miller theorem says that for any perfect field k and formal group law Γ over k, the Lubin–Tate spectrum $E(k, \Gamma)$ admits the structure of an E_∞ ring spectrum and that the Morava stabilizer group $\mathrm{Aut}(\Gamma)$ acts on this via E_∞ ring maps. Thus for any finite subgroup G of $\mathrm{Aut}(\Gamma)$, $E(k, \Gamma)$ can be viewed as a G-spectrum.

Example 17.3.77. If M is an abelian group, then the cofree spectrum

$$HM^h = F(EG_+, i_*^G HM)$$

represents Borel cohomology:

$$X \mapsto H^*(EG_+ \underset{G}{\wedge} X; M).$$

Remark 17.3.78. The cofree construction is also lax monoidal, so it preserves various kinds of structured products. More surprisingly, it takes E_∞ ring spectra to G-E_∞-ring spectra (See Proposition 17.3.97 below).

Since the map

$$E \to E^h$$

is an underlying equivalence, the map

$$E_h \to (E^h)_h$$

is always an equivalence. If we smash the isotropy separation sequence for the family $\mathcal{F}_e$ with E^h and take fixed points, then we get a cofiber sequence of spectra. The cofiber was described by Greenlees–May.

Definition 17.3.79 ([33])**.** If E is a spectrum with G-action, then let

$$E^{tG} = \big(\tilde{E}G \wedge F(EG_+, E)\big)^G$$

be the Tate spectrum of E.

Again, since cofree spectra care only about the underlying spectrum with a G-action, it makes no difference if we consider genuine G or these less strict ones.

Corollary 17.3.80 (Tate sequence)**.** *For any G-spectrum E, we have a cofiber sequence*

$$E_{hG} \to E^{hG} \to E^{tG},$$

where the first map is a lift to spectra of the trace map from group homology to group cohomology.

Remark 17.3.81. The trace map

$$E_{hG} \to E^{hG}$$

is often called the "norm map", especially in trace methods literature. We call it the trace to avoid confusion with the multiplicative norm functors described below.

We can interpret the homotopy fixed points as providing a way to isolate the contribution of the underlying homotopy. This is the key feature of the Segal conjecture, proved by Carlsson: maps out of EG_+ can be viewed as a completion at the ideal given by the restriction to the trivial subgroup [17].

Theorem 17.3.82 (Segal Conjecture [17])**.** *If X is a finite, pointed G-CW spectrum, then the projection map $X_h \to X$ induces a natural isomorphism*

$$\pi^*(X)^\wedge_I \to \pi^*(EG_+ \wedge X),$$

where $I \subseteq \underline{A}(*)$ is the ideal of the Burnside ring consisting of all elements which restrict to 0 in $\underline{A}(G)$.

It can be helpful here to work more generally.

Proposition 17.3.83. *If $\mathcal{F}$ is a family, then the homotopy type of $F(E\mathcal{F}_+, E)$ depends only on the H-equivariant homotopy type of E for all $H \subseteq G$. Moreover, if we have an inclusion $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then we have a canonical map*

$$F(E\mathcal{F}_{2+}, E) \to F(E\mathcal{F}_{1+}, E).$$

This map is an equivalence when restricted to any $H \in \mathcal{F}_1$.

Applying this to $\mathcal{F}_2 = \mathcal{A}ll$ gives immediately a special case.

Corollary 17.3.84. *For any family $\mathcal{F}$, we have a natural map*

$$E \to F(E\mathcal{F}_+, E).$$

When E is a (commutative, associative, etc) ring spectrum, this is a map of (commutative, associative, etc) ring spectra.

Theorem 17.3.85 ([3])**.** *If $\mathcal{F}$ is a family of subgroups, and if $I(\mathcal{F}) \subseteq \underline{A}(*)$ is the ideal of all elements of the Burnside ring which restrict to zero in $\underline{A}(G/H)$ for all $H \in \mathcal{F}$, then the projection $E\mathcal{F}_+ \wedge X \to X$ induces an isomorphism*

$$\pi^*(X)^{\wedge}_{I(\mathcal{F})} \to \pi^*(E\mathcal{F}_+ \wedge X).$$

17.3.5.3 Geometric fixed points

The categorical fixed points are in many ways poorly behaved: they are not strong symmetric monoidal and they do not do what we expect on suspension spectra. In both cases, the underlying problem is that the fixed points for proper subgroups contribute to the fixed points for all of G via the transfer maps. The geometric fixed points fix both of these problems.

Definition 17.3.86. Let

$$\Phi^G(E) = \left(\tilde{E}\mathcal{P} \wedge E\right)^G$$

be the geometric fixed points of E, where again $\mathcal{P}$ is the family of proper subgroups.

Definition 17.3.87. For a G-spectrum E, the isotropy separation sequence for E is the cofiber sequence

$$E\mathcal{P}_+ \wedge E \to E \to \tilde{E}\mathcal{P} \wedge E$$

we get by smashing E with the defining cofiber sequence for $\tilde{E}\mathcal{P}$.

By functoriality of the fixed points functor, the following is immediate.

Proposition 17.3.88. *We have a natural map*

$$E^G \to \Phi^G(E).$$

By definition of $\tilde{E}\mathcal{P}$, if H is a proper subgroup, then

$$\Phi^G(G/H_+) = \big(\tilde{E}\mathcal{P} \wedge G/H_+\big)^G \simeq *,$$

since $i_H^*\tilde{E}\mathcal{P}$ is equivariantly contractible. Now if X is a G-CW complex, then the inclusion of the fixed points $X^G \hookrightarrow X$ is the inclusion of a subcomplex with the property that the quotient X/X^G is built entirely out of cells with a proper stabilizer. This gives use the first desired property.

Proposition 17.3.89. *The geometric fixed points of a suspension spectrum is the suspension spectrum of the fixed points:*

$$\Sigma^\infty_+(X^G) \xrightarrow{\simeq} \Phi^G\Sigma^\infty_+(X),$$

where the map is induced by the inclusion of the fixed points of X.

Remark 17.3.90. We can take the observation about the contractbility of the geometric fixed points for proper subgroups as the *defining feature* of the geometric fixed points. There is a smashing localization which nullifies any G-CW complex built out of cells with proper stabilizers, and the result of applying this to S^0 is just $\tilde{E}\mathcal{P}$. The fixed points on the category of local objects in then an equivalence.

The geometric fixed points is also strong symmetric monoidal.

Proposition 17.3.91. *For G-spectra E and E', there is a natural equivalence*

$$\Phi^G(E \wedge E') \simeq \Phi^G(E) \wedge \Phi^G(E').$$

The intuition behind property is most easily seen via the model of $\tilde{E}\mathcal{P}$ as the infinite reduced regular representation sphere. In this case, we see that we have an equivalence

$$\tilde{E}\mathcal{P} \simeq S^0[a_{\bar{\rho}_G}^{-1}],$$

where the class

$$a_{\bar{\rho}_G} \colon S^0 \to S^{\bar{\rho}_G}$$

is the Euler class of the representation $\bar{\rho}_G$. The right-hand side of this equivalence has the structure of an E_∞ ring spectrum that is solid in the sense that the multiplication map

$$S^0[a_{\bar{\rho}_G^{-1}}] \wedge S^0[a_{\bar{\rho}_G^{-1}}] \to S^0[a_{\bar{\rho}_G^{-1}}]$$

is an equivalence.

These two properties also describe the geometric fixed points for virtual representation spheres.

Example 17.3.92. If V is a virtual representation of G, then

$$\Phi^G S^V \simeq S^{V^G}.$$

Since the geometric fixed points is a strong symmetric monoidal functor, for each conjugacy class of subgroup H, we have a ring map

$$[S^0, S^0]^G \xrightarrow{\Phi^H} [\Phi^H S^0, \Phi^H S^0] \cong [S^0, S^0] \cong \mathbb{Z}.$$

Definition 17.3.93. The mark homomorphism (also called the ghost coordinates) is the map

$$\underline{A}(*) \to \prod_{(H)} \mathbb{Z}$$

given by the product of the various geometric fixed points maps.

This homomorphism also has a purely algebraic description: it assigns to a virtual G-set T the virtual cardinality of T^H.

Theorem 17.3.94 ([21])**.** *The mark homomorphism is an injective ring map which is a rational isomorphism.*

Example 17.3.95. For $G = C_2$ and the spectrum $\mathrm{MU}_{\mathbb{R}}$, we have

$$\Phi^{C_2}\mathrm{MU}_{\mathbb{R}} \simeq MO.$$

17.3.5.4 Tate squares

We can combine the isotropy separation sequence with the maps on generalized cofree spectra to inductively build G-spectra out of pieces with prescribed isotropy. If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ are families, then we have a natural map of G-spectra

$$F(E\mathcal{F}_{2+}, E) \to F(E\mathcal{F}_{1+}, E).$$

If $\mathcal{F}_0$ is a third family, then we can smash this map with the isotropy separation sequence for $\mathcal{F}_0$, getting a diagram

$$\begin{array}{ccccc} E\mathcal{F}_{0+} \wedge F(E\mathcal{F}_{2+}, E) & \longrightarrow & F(E\mathcal{F}_{2+}, E) & \longrightarrow & \tilde{E}\mathcal{F}_0 \wedge F(E\mathcal{F}_{2+}, E) \\ \downarrow & & \downarrow & & \downarrow \\ E\mathcal{F}_{0+} \wedge F(E\mathcal{F}_{1+}, E) & \longrightarrow & F(E\mathcal{F}_{1+}, E) & \longrightarrow & \tilde{E}\mathcal{F}_0 \wedge F(E\mathcal{F}_{1+}, E) \end{array} \tag{17.96}$$

The map

$$F(E\mathcal{F}_{2+}, E) \to F(E\mathcal{F}_{1+}, E).$$

is an H-equivalence for any $H \in \mathcal{F}_1$, so in particular, if $\mathcal{F}_0 \subseteq \mathcal{F}_1$, then the left-most map is an equivalence. This is the generalized Tate diagram.

Proposition 17.3.97. *If $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$ are families, then we have a natural pullback diagram*

$$\begin{tikzcd}
F(E\mathcal{F}_{2+}, E) \arrow[r] \arrow[d] & \tilde{E}\mathcal{F}_0 \wedge F(E\mathcal{F}_{2+}, E) \arrow[d] \\
F(E\mathcal{F}_{1+}, E) \arrow[r] & \tilde{E}\mathcal{F}_0 \wedge F(E\mathcal{F}_{1+}, E).
\end{tikzcd}$$

If E is a ring spectrum, then all maps in the diagram are maps of ring spectra.

Definition 17.3.98. The fixed points of $\tilde{E}\mathcal{F}_0 \wedge F(E\mathcal{F}_{1+}, E)$ are generalized Tate spectra.

Example 17.3.99. Let $G = C_p$. The only interesting case for families is $\mathcal{F}_0 = \mathcal{F}_1 = \mathcal{F}_e$, and $\mathcal{F}_2 = \mathcal{A}ll$. Our diagram becomes

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \tilde{E}C_p \wedge E \arrow[d] \\
F(EC_{p+}, E) \arrow[r] & \tilde{E}C_p \wedge F(EC_{p+}, E).
\end{tikzcd}$$

The family $\mathcal{F}_e$ is also the family of proper subgroups, so $\tilde{E}C_p \wedge E$ is the data of an ordinary spectrum: the geometric fixed points. The bottom row depends only on the data of E as a functor $BG \to \mathsf{Sp}$, not as a genuine G-spectrum. So we have reduced the problem of describing a G-spectrum to that of a G-object in spectra, and ordinary spectrum (the geometric fixed points), and a map. This has been generalized to larger finite groups by Abram–Kriz, Glasman, and Ayala–Mazel-Gee–Rozenblyum [1], [28], [7].

17.4 Equivariant commutative ring spectra and norms

The category of spectra has a symmetric monoidal product, the smash product. It makes sense then to ask about monoids and commutative monoids in the category of spectra and to ask about these in a homotopy coherent way. In particular, we can ask for G-spectra R together with maps

$$R^{\wedge n} \to R$$

that satisfy the usual associativity or commutativity diagrams, either strictly or up to coherent homotopy. However, just as additively, we should follow the mandate that in equivariant homotopy, the group should be allowed to act on all indices in a diagram. In particular, we should be able to have the group action on $R^{\wedge n}$ intertwine the action on R and a permutation action.

If R has a strictly commutative multiplication, then the multiplication will extend naturally to any of these twisted products. Homotopically, the question becomes more subtle. This introduces a different flavors of commutative ring spectra, the N_∞-ring spectra, where we always have a (coherently commutative) multiplication and some collections of these twisted products.

17.4.1 Norms

We start by discussing the general construction for the smash products in which the group acts also on the smash factors. Since the group G is finite, any subgroup is of finite index. We can then form the analogue of "tensor induction" in spectra with the smash product. Heuristically, given an H-spectrum E, we smash together G/H-copies of E and have the group act as in induction, permuting the factors and acting.

Theorem 17.4.1 ([39, Appendix B])**.** *If $H \subseteq G$, then there are strong symmetric monoidal, homotopically meaningful "norm" functors*

$$N_H^G \colon \mathsf{Sp}^H \to \mathsf{Sp}^G$$

such that

1. *the norm commutes with sifted colimits,*
2. *if X is an H-space, then*
$$N_H^G\big(\Sigma_+^\infty X\big) \simeq \Sigma_+^\infty\big(\mathsf{Top}^H(G,X)\big),$$
3. *if V is a virtual representation of H, then*
$$N_H^G S^V \simeq S^{Ind_H^G V}.$$

The properties listed for the norm allow us to compute it for any H-spectrum. Since suspensions of finite G-CW complexes generate the category under colimits, we can resolve any H-spectrum as a colimit (in fact, directed) of shifts of suspension spectra. The norm then commutes with the colimit and has the listed formulae for the other pieces.

Remark 17.4.2. The norm is not an additive functor. However, considering the case of X is a discrete H-space above, we can easily describe a formula for the norm of wedges as wedges of norms.

Example 17.4.3. Let $G = C_2$. Then we have

$$N_e^{C_2}(S^0 \vee S^0) \simeq N_e^{C_2}\big(\Sigma_+^\infty\{a,b\}\big) \simeq \Sigma_+^\infty\big(\mathsf{Top}(C_2,\{a,b\})\big) \simeq S^0 \vee S^0 \vee C_{2+}.$$

The two copies of S^0 correspond to the norms of the two summands. The C_{2+} here is collectively the first sphere smashed with the second and the second sphere smashed with the first (i.e. the two different orders in which to do this).

There is an internal version of the twisted smash powers which can be defined identically. Here we

Definition 17.4.4. If H is a subgroup of G, then let

$$N^{G/H}(E) = N_H^G i_H^* E.$$

If T is a finite G-set, then let

$$N^T(E) = \bigwedge_T E = \bigwedge_{G/H \subseteq T} N^{G/H}(E)$$

be the smash product over the orbits in T of the norm for that orbit.

Proposition 17.4.5. *The functors $N^T \colon \mathsf{Sp}^G \to \mathsf{Sp}^G$ are strong symmetric monoidal, homotopical functors.*

The geometric fixed points of norms are surprisingly easy to describe. This is essentially the "Tate diagonal" used in trace methods. For $G = C_p$, this is a key part of the Segal conjecture and of the topological Singer construction [53].

Proposition 17.4.6. *The diagonal map induces an equivalence*

$$\Phi^H E \to \Phi^G N_H^G(E).$$

In general, fixed and homotopy fixed points of norms are very difficult to understand. We have some connectivity estimates, however.

Proposition 17.4.7. *If E is in $\tau_{\geq k}^{Post}$, then $N_H^G E$ is in $\tau_{\geq k}^{Post}$ as well.*

In general, this is the best we can do for a connectivity estimate. Since we are smashing together several $(k-1)$-connected things, one might expect to see the connectivity scale, just as with the smash product. This need not be the case.

Example 17.4.8. Consider the sphere S^k, which is in $\tau_{\geq k}^{Post}$ for $H = \{e\}$. The norm to G is the regular representation sphere $S^{k\rho_G}$, where $\rho_G = \mathbb{R}[G]$ is the regular representation. The map

$$a_{\bar{\rho}_G} \colon S^k \to S^{k\rho_G}$$

is essential, since it induces an isomorphism on geometric fixed points. This shows that $\underline{\pi}_k(S^{k\rho_G}) \neq 0$.

17.4.2 N_∞-ring spectra

17.4.2.1 N_∞ operads

One of the key benefits for the point-set models of spectra is having good, homotopically meaningful categories commutative ring spectra and their modules. In EKMM style S-modules, a commutative ring spectrum is the same data as an E_∞ ring spectrum, and the same is true for symmetric or orthogonal spectra [25, 55].

In all cases, a key computation is that for a nice spectrum E, the action of Σ_n on the nth smash powers of E is actually free, and hence the natural map

$$E\Sigma_{n+} \underset{\Sigma_n}{\wedge} E^{\wedge n} \to E^{\wedge n}/\Sigma_n$$

is a weak equivalence. The homotopy symmetric powers assemble to give the free E_∞ algebra on E, while the actual ones give the free commutative algebra, and this provides a comparison between the two concepts.

Equivariantly, we have choices for exactly what the homotopy symmetric powers mean, since we can also include an action of G on the Σ_n-free spaces. We begin with two important examples.

Definition 17.4.9. Let $E\Sigma_n$ be $i_*^G E\Sigma_n$. This is $E\Sigma_n$ with a $G \times \Sigma_n$-action via the projection $G \times \Sigma_n \to \Sigma_n$. This is the universal space for the family

$$\mathcal{F}_{\Sigma_n}^{tr} = \{H \times \{e\} \mid H \subseteq G\}$$

of subgroups of G.

Definition 17.4.10. Let $E_G\Sigma_n = E\mathcal{F}_{\Sigma_n}$ be the universal space the family of subgroups Γ such that

$$\Gamma \cap (\{e\} \times \Sigma_n) = \{(e, e)\}.$$

Example 17.4.11. If $G = C_2$, then the space $E_G\Sigma_2$ can be modeled as an infinite sphere:

$$E_G\Sigma_2 \simeq S(\infty\rho \otimes \tau),$$

where τ is the sign representation of Σ_2 and ρ is the regular representation of C_2.

The space $E\Sigma_2$ can be modeled as a different infinite sphere:

$$E\Sigma_2 \simeq S(\infty\tau).$$

The C_2-fixed points of $E_G\Sigma_2/\Sigma_2$ are $\mathbb{R}P^\infty \amalg \mathbb{R}P^\infty$, while those of $B\Sigma_2$ are just $\mathbb{R}P^\infty$.

We generalize the classical notion of an E_∞ operad by allowing any such space, building a general class of N_∞ operads [11].

Definition 17.4.12. An N_∞ operad is a symmetric operad $\mathcal{O}$ in G-spaces such that for each n, the $G \times \Sigma_n$-space $\mathcal{O}_n$ is equivalent to $E\mathcal{F}_n(\mathcal{O})$ for some family $\mathcal{F}_n(\mathcal{O})$ with

$$\mathcal{F}_{\Sigma_n}^{tr} \subseteq \mathcal{F}_n(\mathcal{O}) \subseteq \mathcal{F}_{\Sigma_n}.$$

An N_∞ operad $\mathcal{O}$ is a G-E_∞ operad if for each n, $\mathcal{O}_n \simeq E\mathcal{F}_{\Sigma_n}$.

The family $\mathcal{F}_{\Sigma_n}$ is the largest family such that Σ_n acts freely on the universal space. The conditions $\mathcal{F}_n(\mathcal{O})\subseteq\mathcal{F}_{\Sigma_n}$ then means Σ_n acts freely. The conditions that $\mathcal{F}^{tr}_{\Sigma_n}\subseteq\mathcal{F}_n(\mathcal{O})$ will guarantee enough G-fixed points (which will be necessary for us to have a coherent product in the operadic algebras).

Many of the classical constructions of operads immediately give N_∞ operads.

Example 17.4.13. If U is a universe, then the linear isometries operad defined by

$$\mathcal{L}_n = \mathrm{Isom}(U^{\oplus n}, U)$$

is an N_∞ operad.

Example 17.4.14. If U is a universe, then the little disks operad for U is an N_∞ operad.

Example 17.4.15. If $\mathcal{O}$ is an ordinary E_∞ operad, then endowing it with the trivial action, we get an N_∞-operad we can $\mathcal{O}^{tr}$. This corresponds to the family $\mathcal{F}^{tr}_{\Sigma_n}$.

Just as non-equivariantly, there are model categories of N_∞-ring spectra and of commutative ring spectra.

Definition 17.4.16. Let Comm^G be the category of commutative ring spectra in Sp^G.

If $\mathcal{O}$ is an N_∞-operad, let $\mathcal{O}$-Alg be the category of $\mathcal{O}$-algebras in Sp^G.

Proposition 17.4.17. *The categories* Comm^G *and* $\mathcal{O}$-Alg *are enriched in spaces, are complete and co-complete, and are tensored and cotensored over* Top^G. *They can also be enriched in* Top^G.

Theorem 17.4.18. *The categories* Comm^G *and* $\mathcal{O}$-Alg *admit model structures such that the forgetful functor*

$$\mathsf{Comm}^G \to \mathsf{Sp}^G \text{ and } \mathcal{O}\text{-}\mathsf{Alg} \to \mathsf{Sp}^G$$

are homotopical right adjoint and compatible with the enrichments.

Definition 17.4.19. If $\mathcal{O}$ is an N_∞ operad, then let

$$\mathbb{P}_{\mathcal{O}}\colon \mathsf{Sp}^G \to \mathcal{O}\text{-}\mathsf{Alg}$$

be the free $\mathcal{O}$-algebra functor.

Moreover, the categories of commutative ring spectra and G-E_∞ ring spectra are closely connected. For any G-E_∞ operad $\mathcal{O}$, there is a canonical map of operads $\mathcal{O} \to \mathit{Comm}$. The operadic base-change gives a Quillen equivalence in nice cases.

Theorem 17.4.20. *The categories* Comm^G *and of* G-E_∞ *ring spectra in* Sp^G *are Quillen equivalent.*

Finally, we have a somewhat surprising additional feature of the cofree construction.

Proposition 17.4.21. *If R is an E_∞ ring spectrum on which G acts by E_∞ ring maps, then $F(EG_+, R)$ naturally has the structure of a G-E_∞-ring spectrum.*

17.4.3 Multiplicative norms

17.4.3.1 Norms in operadic algebras

The subgroups which arise in an N_∞ operad parameterize various kinds of twisted products. To see this, we unpack the conditions a little.

Let $\Gamma \subseteq G \times \Sigma_n$ be in $\mathcal{F}_n$. Since the intersection with Σ_n is trivial, the projection onto G, when restricted to Γ is an injection. The image is some subgroup H, and we observe that Γ is then the graph of a homomorphism $f\colon H \to \Sigma_n$. Any homomorphism $H \to \Sigma_n$ defines an H-set structure on the set $\{1, \ldots, n\}$, and we see that the subgroups correspond to presentations of H-set structures of cardinality n. This gives a name to these subgroups: graph subgroups.

Definition 17.4.22. Let $\mathcal{O}$ be an N_∞ operad. A finite H-set T of cardinality n is admissible if the graph of a homomorphism $H \to \Sigma_n$ defining it is in $\mathcal{F}_n(\mathcal{O})$.

Example 17.4.23. For any N_∞-operad $\mathcal{O}$ and for any cardinality n, the set $\{1, \ldots, n\}$ with a trivial G action is admissible. This is the G-set corresponding to the subgroup $G \times \{e\}$.

Since for all n, $\mathcal{F}_n(\mathcal{O})$ is a universal space, we have an equivalent formulation.

Proposition 17.4.24. *Let T be a finite H set of cardinality n and let Γ be the corresponding graph subgroup. Then T is admissible if and only if*

$$\mathsf{Top}^{G \times \Sigma_n}\big(G \times \Sigma_n/\Gamma, \mathcal{O}_n\big) \neq \emptyset.$$

Now let R be an $\mathcal{O}$-algebra. We can then consider the effect of composing any such map with the operadic structure map, getting a contractible space of maps

$$\big(G \times \Sigma_n/\Gamma\big)_+ \underset{\Sigma_n}{\wedge} R^{\wedge n} \to \mathcal{O}_{n+} \underset{\Sigma_n}{\wedge} R^{\wedge n} \to R.$$

In the source, the presence of Γ intertwines the Σ_n action on the indexing set for the smash powers with the H action in G: every $h \in H$ acts both on the individual factors of R and on the indexing set via $H \to \Sigma_n$. This is therefore an example of the norm.

Proposition 17.4.25. *Let T be a finite H-set of cardinality n, and let $\Gamma \subseteq G \times \Sigma_n$ be the graph of a homomorphism $H \to \Sigma_n$ presenting T. Then for any G-spectrum E, we have a G-equivariant equivalence*

$$\left(G \times \Sigma_n/\Gamma\right)_+ \underset{\Sigma_n}{\wedge} E^{\wedge n} \simeq G_+ \underset{H}{\wedge} N^T E.$$

Putting this all together, we see exactly what kinds of structure we see.

Theorem 17.4.26. *Let $\mathcal{O}$ be an N_∞ operad and let R be an $\mathcal{O}$-algebra. For each admissible H-set T, the operad gives a contractible space of maps*

$$N^T i_H^* R \to i_H^* R.$$

When $T = * \amalg *$, this is the contractible space of multiplications, describing an underlying E_∞ ring structure.

Since they come from the operadic structure, these are natural for maps of operadic algebras.

Theorem 17.4.26 used only the operadic action on R and said nothing about the compatibility or constraints imposed by the operadic composition. In fact, this puts huge constraints on the collection of admissible sets, allowing for a complete classification of the homotopy category of N_∞ operads.

Since for each n, $\mathcal{F}_n(\mathcal{O})$ is a family, we deduce several consequences.

Proposition 17.4.27. *Let $\mathcal{O}$ be an N_∞ operad.*

1. *If T is an admissible H-set and if $q\colon G/K \to G/H$ is a map of G-sets, then q^*T is an admissible K-set.*
2. *If T is an admissible H-set and T' is isomorphic to T, then T' is an admissible H-set.*
3. *The trivial H-set of cardinality n is admissible for all n.*

The operadic composition connects the admissible sets for various cardinalities.

Proposition 17.4.28. *Let $\mathcal{O}$ be an N_∞ operad.*

1. *If $T = T_1 \amalg T_2$, then T_1, T_2 are admissible H-sets if and only if T is.*
2. *If K/H is an admissible K-set and T is an admissible H-set, then $K \underset{H}{\times} T$ is an admissible K-set.*

Definition 17.4.29. An indexing system is a collection of full subcategories $\mathcal{O}(H) \subseteq \mathtt{Fin}^H$ for each $H \subseteq G$, such that for each H, the objects of $\mathcal{O}(H)$ satisfy the conditions of Propositions 17.4.27 and 17.4.28.

There is a poset of indexing systems, ordered by inclusion for all H, and the map which sends an N_∞ operad to its admissibles gives a functor to this poset. Blumberg–Hill showed this is a fully-faithful inclusion on the homotopy category, and Gutiérrez–White showed it is essentially surjective.

Theorem 17.4.30 ([11, 36])**.** *There is an equivalence of categories between the homotopy category of N_∞-operads and the poset of indexing systems.*

There are several additional approaches to building N_∞ operads out of the combinatorial data. Rubin considers explicit categorical models [59], while Bonventre–Pereira describe a genuine equivariant extension of operads [13].

Remark 17.4.31. Instead of working multiplicatively, we could have asked for a classification of the possible extension of coefficient systems that include only some transfers. These are again parameterized by indexing systems [12]. This us ways to talk about variants of G-spectra interpolating between the $\mathbb{Z}$-graded theory built out of coefficient systems and the genuine one considered so far here.

17.4.3.2 Norms in commutative rings

For commutative ring spectra, we can make even more intuitive statements. The smash product is the coproduct in Comm^G, and the norm N_H^G is induction built out of the smash product. In particular, it is a strong symmetric monoidal functor, and hence lifts to a functor on Comm^H.

Theorem 17.4.32. *The norm*

$$N_H^G\colon \mathsf{Comm}^H \to \mathsf{Comm}^G$$

is left adjoint to the forgetful functor $i_H^*\colon \mathsf{Comm}^G \to \mathsf{Comm}^H$.

Corollary 17.4.33. *If R is a commutative ring spectrum, then for each H, we have a natural map of equivariant commutative ring spectra*

$$N_H^G i_H^* R \to R.$$

The identification of the adjunction also demonstrates some unexpected behavior for commutative ring spectra, showing that these are more than just naive E_∞ ring spectra. As a consequence of the corollary, we have two features:

1. For each $H\subseteq G$, the G-geometric fixed points of the left-adjoint to the forgetful functor are
$$\Phi^G N_H^G i_H^* R \simeq \Phi^H(R),$$
via the diagonal, and
2. we have a map of (ordinary) E_∞-rings
$$\Phi^H(R) \to \Phi^G(R).$$

The first of these can fail quite spectacularly for naive E_∞-rings.

Example 17.4.34. The left-adjoint L_H^G to the forgetful functor $i_H^*\colon \mathcal{O}\text{-}\mathsf{Alg}^G \to \mathcal{O}\text{-}\mathsf{Alg}^H$ can be computed easily on frees by the universal property:

$$L_H^G \mathbb{P}_{\mathcal{O}^{tr}}(E) \simeq \mathbb{P}_{\mathcal{O}^{tr}}\Big(G_+ \underset{H}{\wedge} E\Big).$$

Since G acts trivially on $\mathcal{O}_n^{tr}$ for all n, we have that the unit map gives an equivalence

$$S^0 \to \Phi^G \mathbb{P}_{\mathcal{O}^{tr}}\Big(G_+ \underset{H}{\wedge} E\Big).$$

This in turn shows that the geometric fixed points of the left adjoint is just the constant functor S^0.

Using the fact that the smash product is the coproduct in Comm^G, we have

Corollary 17.4.35. *If R is an equivariant commutative ring spectrum, then the assignment*

$$T \mapsto N^T R$$

extends to a functor from Fin^G to Comm^G.

17.4.4 Tambara functors

17.4.4.1 Norms in equivariant algebra

These norm maps endow the homotopy Mackey functors of an $\mathcal{O}$-algebra with extra structure. Since these are also the groups associated to a homology theory, stable homotopy, and since equivariant homology theories are RO(G)-graded, we have an RO(G)-graded extension.

Notation 17.4.36. If V and W are representations, let

$$\pi_{V-W}(E) = \pi_{V-W}^G(E) = [S^V, \Sigma^W E]^G.$$

Let

$$\underline{\pi}_{V-W}(E) = \big(T \mapsto [T_+ \wedge S^V, \Sigma^W E]^G\big)$$

be the obvious Mackey enrichment.

Smashing together representative maps gives the following.

Proposition 17.4.37. *If R is an equivariant commutative ring spectrum, then $\pi_\star(R)$ is an equivariant graded commutative ring.*

Here the adjective "equivariant" for the graded commutativity refers to the fact that $\pi_0 S^0 = A(G)$ can have more units than just ± 1, and these units can show up in the formulae for moving representation spheres past each other. In general, the swap map

$$S^V \wedge S^W \to S^W \wedge S^V$$

is a unit in $\pi_0 S^0$; this is the element which controls commutativity.

Example 17.4.38. For $G = C_2$, the sign sphere S^σ swaps with itself by the element $1 - [C_2] \in A(C_2)$. We can see this by checking against all geometric fixed points: the underlying map is the interchange of the 1-sphere with itself (so -1) while the C_2-geometric fixed points is the interchange of the 0-sphere with itself (so 1).

Remark 17.4.39. This technique works quite generally: the ghost coordinates of Definition 17.3.93 give us an injective map

$$[S^{V\oplus W}, S^{W\oplus V}] \to \prod_H [S^{V^H \oplus W^H}, S^{W^H\oplus V^H}] \cong \prod_H \mathbb{Z}.$$

For each coordinate, we have a non-equivariant computation of the degree of moving the W^H-sphere past the V^H-sphere, and this is either ± 1. The corresponding unit in the Burnside ring is the one corresponding to this sequence of units in $\mathbb{Z}$.

Here, we have only used the multiplication in the homotopy category. If R is an $\mathcal{O}$-algebra, then we have more structure.

Proposition 17.4.40. *Let $\mathcal{O}$ be an N_∞ operad. If R is an equivariant commutative ring spectrum, and if G/H is an admissible G-set for $\mathcal{O}$, then we have norm maps*

$$N_H^G \colon \pi_V^H(R) \to \pi^G_{Ind_H^G V}(R)$$

defined by composing the norm functor with the norm in R as an $\mathcal{O}$-algebra:

$$\big(f \colon S^V \to i_H^* R\big) \mapsto \left(S^{Ind_H^G V} \xrightarrow{N_H^G f} N_H^G i_H^* R \to R\right).$$

Remark 17.4.41. Note that in the formation of the norms, we did not need V to be a representation of G: the norm fixed this, producing a representation of G for its index. This suggests an even further extension of the $\mathrm{RO}(G)$-grading: indexing on pairs consisting of subgroups together with a virtual representation for that subgroup. This was used in [39] and was developed more fully in [4].

17.4.4.2 Green and Tambara functors

When $V = \{0\}$, then the grading issues become much simpler: inducing the zero representation gives the zero representation and adding the zero representation to itself gives the zero representation. In particular, our homotopy Mackey functors will have extra structure.

Proposition 17.4.42. *The category of Mackey functors is admits a closed symmetric monoidal structure with the symmetric monoidal product the box product -$\square$-, and internal Hom $\underline{\mathrm{Hom}}$. The symmetric monoidal unit is the Burnside Mackey functor $\underline{A}$.*

The description of Mackey functors as a diagram category on the Burnside category shows how to build the box product: we form the Day convolution product of the tensor product on abelian groups with the Cartesian product in the Burnside category.

Definition 17.4.43. A [commutative] Green functor is a commutative monoid for the box product.

Since the zero sphere commutes with itself on-the-nose, we have a ordinary commutativity for the multiplication on $\underline{\pi}_0$.

Proposition 17.4.44. *If R in an equivariant spectrum with a homotopy commutative and associative multiplication, then $\underline{\pi}_0(R)$ is a commutative Green functor.*

The norm maps on an $\mathcal{O}$-algebra given multiplicative maps on $\underline{\pi}_0$.

Proposition 17.4.45. *If R is an $\mathcal{O}$ algebra and if H/K is an admissible H-set, then we have a multiplicative norm map*

$$\underline{\pi}_0(R)(G/K) \to \underline{\pi}_0(R)(G/H).$$

When $H = G$, these are the expected maps

$$\underline{\pi}_0(R)(G/H) \to \underline{\pi}_0(R)(*).$$

More generally, for admissible H-sets, we identified the corresponding groups in the G-Mackey functor $\underline{\pi}_0(R)$ with the corresponding groups in the H-Mackey functors associated to the restriction $i_H^* \underline{R}$.

These norm maps are not linear, but satisfy instead a kind of twisted distributive law, identical to that studied by Tambara in his original formulation of TNR functors [66] (See also [64]). These formulae for the norms of sums or of transfers are built out of decomposing a coinduced G-set into constituent pieces, and this gives an externalized version of the axioms Tambara considered [40].

Example 17.4.46. For $G = C_2$, we have

$$N_e^{C_2}(a+b) = N_e^{C_2}(a) + N_e^{C_2}(b) + tr_e^{C_2}(a \cdot \gamma(b)),$$

where γ is the non-trivial element of C_2 acting on $\underline{R}(C_2)$.

We can generalize Tambara's construction, considering only certain norms, namely those arising from an indexing system. This gives a category of incomplete Tambara functors.

Theorem 17.4.47 ([16, 11])**.** *If R is an $\mathcal{O}$-algebra, then Mackey functor $\underline{\pi}_0(R)$ naturally has the structure of an incomplete Tambara functor with norms for the indexing system of admissibles for $\mathcal{O}$.*

Because of this, Angeltveit–Bohmann call the resulting structure an RO-graded Tambara functor.

Example 17.4.48. With the Tambara structure, we can show that the pushforward from trivial spectra to C_2-spectra cannot be homotopically compatible with the commutative ring structure. Consider the Eilenberg–Mac Lane spectrum $H\mathbb{F}_2$. Then the pushforward has

$$\underline{\pi}_0(i_* H\mathbb{F}_2) \cong \underline{A}/2$$

is the reduction of the Burnside ring modulo 2. There is no way to put a norm map on this making it into a Tambara functor.

17.5 Computational techniques

We close with several methods for computing in equivariant stable homotopy theory.

17.5.1 Classical spectral sequences

17.5.1.1 Atiyah–Hirezebruch

Just as classically, there is an Atiyah–Hirzebruch spectral sequences computing the homotopy classes of maps between two G-spectra.

Theorem 17.5.1. *There is a multiplicative spectral sequence with E_2 term*

$$E_2^{p,q} = H^p\big(X; \underline{\pi}_q(E)\big)$$

and converging to $[\Sigma^{q-p}X, E]^G$. The d_r differential changes degree by $(r, 1-r)$.

We get this filtration by either filtering the source by the skeletal filtration or the target by the Postnikov filtration. The latter also shows that all of the differentials are secondary cohomology operations, just as classically.

More generally, since all of our algebra invariants are naturally Mackey functors, we have a natural Mackey enrichment.

Theorem 17.5.2. *There is a spectral sequence of Green functors with E_2-term*

$$\underline{E}_2^{p,q} = \underline{H}^p\big(X; \underline{\pi}_q(E)\big)$$

and converging to the homotopy Mackey functors $\underline{E}^{p-q}(X)$.

Remark 17.5.3. In general, even if E is a commutative ring spectrum, this will only be a spectral sequence of graded Green functors. This means that the differentials satisfy the usual Leibnitz rule for products, but we have no control over the norms of classes. The multiplicativity of the spectral sequence is the essential heart of Proposition 17.3.63, while the failure of this to be compatible with norms is Proposition 17.4.7.

As a particular example, we have the homotopy fixed points spectral sequence computing the homotopy groups of the homotopy fixed points E^{hG}. We classically describe this via the skeletal filtration of the source EG_+, but we can equivalently filter the spectrum E by its Postnikov tower.

Theorem 17.5.4. *There is a multiplicative spectral sequence*

$$E_2^{s,t} \cong H^s\big(G; \pi_t(i_e^* E)\big) \Rightarrow \pi_{t-s} E^{hG}.$$

These homotopy fixed points spectral sequences are one of the primary tools used in the trace methods approach to algebra K-theory, together with the generalized Tate diagrams of Proposition 17.3.97. Here the right-hand side will be known by induction (using the cyclotomic structure) while the bottom row will be built out of this homotopy fixed points spectral sequence.

More generally, this can also be done in Mackey functors.

Theorem 17.5.5. *There is a multiplicative spectral sequence of Mackey functors*

$$\underline{E}_2^{s,t}(G/K) \cong H^s\big(K; \pi_t(i_e^* E)\big) \Rightarrow \underline{\pi}_{t-s} E^h.$$

17.5.1.2 Künneth and Universal Coefficients

Lewis and Mandell proved several versions of a Künneth and universal coefficients spectral sequence [50]. To describe the spectral sequences, we have to do more homological algebra in Mackey functors.

Definition 17.5.6. If $\underline{R}$ is a commutative Green functor, then an $\underline{R}$-module is a Mackey functor $\underline{M}$ together with an action map

$$\underline{R} \Box \underline{M} \to \underline{M}$$

making the usual module associativity and unitality diagrams commute. Let $\underline{R}$-Mod be the category of $\underline{R}$-modules. .

The standard arguments about commutative monoids in a symmetric monoidal category show the following.

Proposition 17.5.7. *If $\underline{R}$ is a commutative Green functor, then the closed symmetric monoidal structure on Mackey functors induces a closed symmetric monoidal structure on $\underline{R}$-modules.*

Since the category of Mackey functors has enough projectives (Proposition 17.3.30), the category of $\underline{R}$-modules has enough projectives for any $\underline{R}$. We can therefore consider derived functors.

Definition 17.5.8. Let $\underline{\mathrm{Tor}}^{\underline{R}}_{-s}(\underline{M},\underline{N})$ be the $-s$th derived functor of $\underline{M}\square_{\underline{R}}\underline{N}$, and let $\underline{\mathrm{Ext}}^s_{\underline{R}}(\underline{M},\underline{N})$ be the sth derived functor of $\underline{\mathrm{Hom}}_{\underline{R}}(\underline{M},\underline{N})$.

Theorem 17.5.9 ([50])**.** *Let R be an E_∞ ring spectrum and let M and N be R-modules.*

There is a spectral sequence

$$E_2^{s,t} = \underline{\mathrm{Tor}}^{-s,t}_{\underline{\pi}_* R}(\underline{\pi}_* M, \underline{\pi}_* N) \Rightarrow \underline{\pi}_{t-s}\big(M \underset{R}{\wedge} N\big).$$

There is a spectral sequence

$$E_2^{s,t} = \underline{\mathrm{Ext}}^{s,t}_{\underline{\pi}_* R}(\underline{\pi}_* M, \underline{\pi}_* N) \Rightarrow \underline{\pi}_{t-s}\big(\mathrm{Hom}_R(M,N)\big).$$

These are both Adams graded.

Lewis–Mandell also work with $\mathrm{RO}(G)$-graded versions.

Theorem 17.5.10 ([50])**.** *Let R be an E_∞ ring spectrum and let M and N be R-modules.*

There is a spectral sequence

$$\underline{E}_2^{s,V} = \underline{\mathrm{Tor}}^{-s,V}_{\underline{\pi}_\star R}(\underline{\pi}_\star M, \underline{\pi}_\star N) \Rightarrow \underline{\pi}_{V-s}\big(M \underset{R}{\wedge} N\big).$$

There is a spectral sequence

$$\underline{E}_2^{s,V} = \underline{\mathrm{Ext}}^{s,V}_{\underline{\pi}_\star R}(\underline{\pi}_\star M, \underline{\pi}_\star N) \Rightarrow \underline{\pi}_{V-s}\big(\mathrm{Hom}_R(M,N)\big).$$

These are both Adams graded.

Even for ordinary Bredon homology or cohomology, so $R = H\underline{A}$, this is much trickier than the classical, non-equivariant cases of homology over a PID. Greenlees has shown that the category of Mackey functors has projective dimension 0, 1, or ∞ [30], so the E_2-terms for the ordinary Künneth and universal coefficients spectral sequence in general have infinitely many lines.

Example 17.5.11. Let $G = C_p$, and consider the constant Mackey functor $\underline{\mathbb{Z}}$. Then we have an exact sequence

$$0 \to \underline{\mathbb{Z}} \to \underline{A}_{C_p} \xrightarrow{1-\gamma} \underline{A}_{C_p} \xrightarrow{1} \underline{A} \xrightarrow{p-[C_p]} \underline{A} \to \underline{\mathbb{Z}} \to 0.$$

In particular, we have a periodic resolution of $\underline{\mathbb{Z}}$ with period 4, and we see that having infinitely many Tor or Ext groups will be generic here.

Many of the recent computations in algebraic topology have focused on computations with the constant Mackey functor $\underline{\mathbb{Z}}$ for cyclic groups. Here, the category of modules has projective dimension 3 [6] (See also [14]), and computations are simpler. Zeng has used this to determine the $\mathrm{RO}(G)$-graded homology of a point for cyclic 2-groups [72].

17.5.2 Adams and Adams–Novikov spectral sequences

When we study free G-spectra, then we can equivalently consider Borel theories.

Definition 17.5.12. Let

$$HM_c = EG_+ \wedge HM^h$$

be the freed up Borel cohomology spectrum. This represents "co-Borel cohomology".

Proposition 17.3.70 shows that Borel and co-Borel cohomologies agree on free G-spectra. Greenlees has used these theories to build an Adams spectral sequence which is computable out of essentially non-equivariant information.

Theorem 17.5.13 ([29])**.** *If G is a finite p-group and Y is a free G-spectrum with which p-complete, bounded below, and locally finite, then there is a convergent Adams spectral sequence for any X:*

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{(H\mathbb{F}_p^h)^*(H\mathbb{F}_p^h)}\big((H\mathbb{F}_p^h)^*(Y), (H\mathbb{F}_{p_c})^*(X)\big) \Rightarrow [X,Y]^G_{t-s}.$$

Moreover, we have an isomorphism

$$(H\mathbb{F}_p^h)^*(H\mathbb{F}_p^h) \cong H^*(BG;\mathbb{F}_p)\tilde{\otimes}\mathcal{A}_p,$$

where $\mathcal{A}_p$ is the mod-p Steenrod algebra, and where $\tilde{\otimes}$ refers to the fact that we twist the multiplication in the tensor product by the natural action of $\mathcal{A}_p$ on $H^*(BG;\mathbb{F}_p)$.

Even though this looks like it gives us information only for free spectra, this actually gives much more control over even non-free spectra. The Segal conjecture links these. Moreover, for G a finite p-group, the natural map

$$\pi^*(X)^\wedge_p \to \big(\pi^*(X)^\wedge_I\big)^\wedge_p$$

is an isomorphism. So Greenlees' spectral sequence can be used also to just compute stable cohomotopy, and more generally, to compute maps into any finite, p-complete G-spectrum Y (free or not). This was used by Szymik to compute equivariant stable stems in low degrees [65].

17.5.2.1 Real versions

For $G = C_2$, there is another form of the Adams spectral sequence due to Hu–Kriz [45].

Definition 17.5.14. Let $\underline{\mathbb{F}}_2$ be the constant Mackey functor with value $\mathbb{F}_2$. Let $\mathbb{M}_2$ be $\pi_\star H(\underline{\mathbb{F}}_2)$.

The ring $\mathbb{M}_2$ was originally computed by Stong (unpublished), and a complete description can be found in [19]. A very nice way to determine this ring is given by Greenlees [31]. This is a non-Noetherian ring, but can be easily described. C. May has recently shown that this ring is injective as a module over itself, simplifying many structural questions [57].

Proposition 17.5.15. *We have*

$$H_1(S^\sigma; \underline{\mathbb{F}_2}) \cong \mathbb{F}_2,$$

generated by a class u_σ.

As a module over $\mathbb{F}_2[a_\sigma, u_\sigma]$, we have

$$\mathbb{M}_2 \cong \mathbb{F}_2[a_\sigma, u_\sigma] \oplus \theta \cdot \mathbb{F}_2[a_\sigma^{\pm 1}, u_\sigma^{\pm 1}]/\mathbb{F}_2[a_\sigma, u_\sigma].$$

The class θ is in degree $2\sigma - 2$ and squares to zero.

Remark 17.5.16. The ring $\mathbb{M}_2$ is isomorphic to the cohomology ring of an $\mathrm{RO}(G)$-weighted $\mathbb{P}^1$ with coefficients in the powers of the canonical bundle. The generators are a_σ and u_σ, and the class θ is the Serre dual class. Work of Greenlees–Meier generalizes this [34].

Hu–Kriz computed the $\mathrm{RO}(C_2)$-graded Hopf algebroid of stable cooperations for Bredon homology with coefficients in $\underline{\mathbb{F}}_2$, showing that in fact, this algebra is a free module over the $\mathrm{RO}(C_2)$-graded homology of a point. This is a striking example of the power of the $\mathrm{RO}(C_2)$-grading, since the structure of the algebra with just $\mathbb{Z}$-grading is overly complicated.

Theorem 17.5.17 ([45])**.** *We have an isomorphism*

$$\mathcal{A}_\star := H(\underline{\mathbb{F}}_2)_\star H(\underline{\mathbb{F}}_2) \cong \mathbb{M}_2[\bar{\xi}_1, \dots][\bar{\tau}_0, \dots]/\big(\bar{\tau}_i^2 = a_\sigma \bar{\tau}_{i+1} + (u_\sigma + a_\sigma \bar{\tau}_0)\bar{\xi}_{i+1}\big).$$

The degrees of the elements are

$$|\bar{\xi}_i| = (2^i - 1)\rho_2 \quad |\bar{\tau}_i| = (2^i - 1)\rho_2 + 1,$$

and the coproducts of the generators are the classical coproducts. The left and right units on a_σ are the obvious inclusions. The right unit on u_σ is $u_\sigma + a_\sigma \bar{\tau}_0$.

Theorem 17.5.18 ([45])**.** *There is an Adams spectral sequence*

$$E_2^{s,V} = \mathrm{Ext}_{\mathcal{A}_\star}^{s,V}\big(H_\star(X), H_\star(Y)\big) \Rightarrow [X, Y_I^\wedge],$$

where I is the augmentation ideal of the Burnside ring.

Moreover, Araki showed that the Fujii–Landweber spectrum of Real bordism $\mathrm{MU}_\mathbb{R}$ is flat in the sense that

$$\mathrm{MU}_{\mathbb{R}\star}\mathrm{MU}_\mathbb{R} = \pi_\star \mathrm{MU}_\mathbb{R}[\bar{b}_1, \dots],$$

where $|\bar{b}_i| = i\rho_2$ [5]. Hu–Kriz analyzed this as a Hopf algebroid, producing their Real Adams–Novikov spectral sequence.

Theorem 17.5.19 ([45])**.** *There is a spectral sequence*

$$E_2^{s,V} = \mathrm{Ext}^{s,V}_{(\mathrm{MU}_{\mathbb{R}\star},\mathrm{MU}_{\mathbb{R}\star}\mathrm{MU}_{\mathbb{R}})}\big(\mathrm{MU}_{\mathbb{R}\star}(X),\mathrm{MU}_{\mathbb{R}\star}(Y)\big) \Rightarrow [X, Y_I^\wedge].$$

For other cyclic p-groups, determining the structure of the dual Steenrod algebra and even of the algebraic category in which we should work is ongoing work.

17.5.3 Slice spectral sequence

Building on motivic intuition and on work of Dugger, Hill–Hopkins–Ravenel produced an equivariant refinement of the Postnikov tower that uses various induced representation spheres in place of ordinary trivial ones [23], [39]. We describe here the version using regular representations first described by Ullman [69].

Definition 17.5.20 ([22])**.** A localizing subcategory is a full subcategory closed under homotopy colimits and such that for any cofiber sequence

$$X \to Y \to Z,$$

if X and either Y or Z is in the subcategory, then the third is.

Definition 17.5.21. Let $\tau_{\geq n}$ be the smallest localizing subcategory containing all

$$G_+ \underset{H}{\wedge} S^{k\rho_H},$$

where ρ_H is the regular representation of H and where $k \cdot |H| \geq n$.

By construction, these are nested: if $m \geq n$, then

$$\tau_{\geq m} \subseteq \tau_{\geq n}.$$

Definition 17.5.22. Let $P^n\colon \mathsf{Sp}^G \to \mathsf{Sp}^G$ be the endofunctor of Sp^G which nullifies the localizing category $\tau_{\geq n}$. The slice tower for E is the tower of spectra under E:

$$\dots P^{n+1}(E) \to P^n(E) \to \dots.$$

The fiber $P^n_n(E)$ of $P^n(E) \to P^{n-1}(E)$ is the n-slice of E.

Recent work of Hill–Yarnall and of Wilson has given an equivalent description of the filtration in terms of the geometric fixed points for various subgroups [43] [70]. This gives an interpretation of this tower as running the Postnikov towers for the geometric fixed points of various subgroups at speeds proportionate to the index of the subgroup.

The slice tower of E gives a spectral sequence computing the homotopy Mackey functors of E, or more generally the Mackey functor of maps from any spectrum X into E.

Proposition 17.5.23. *If E is a G-spectrum and X is a finite G-CW complex, then there is a strongly convergent, Adams graded spectral sequence*

$$\underline{E}_2^{s,V} = \underline{\pi}_{V-s} F(X, P_{\dim V}^{\dim V}(E)) \Rightarrow \underline{\pi}_{V-s} F\big(X, E\big) = \underline{E}^{s-V}(X).$$

Restricted to (-1)-connected G-spectra, the slice tower has another, essentially defining feature. The sequence of categories of slice $\geq n$ spectra is the smallest sequence of localizing subcategories $\mathcal{C}_n$ such that

1. All finite G-sets are in the zeroth category,
2. S^1 is in the first category,
3. if $X \in \mathcal{C}_n$ and $Y \in \mathcal{C}_m$, then $X \wedge Y \in \mathcal{C}_{n+m}$, and
4. if $X \in \mathcal{C}_n$, then for all finite H-sets T
$$(G_+ \underset{H}{\wedge} N^T i_H^* X) \wedge Y \in \mathcal{C}_{|T|n+m}.$$

The first three properties are shared by the Whitehead filtration for the Postnikov tower, which can be reinterpreted as the smallest sequence of localizing subcategories satisfying just these. These conditions plus the last guarantee that the slice spectral sequence has unexpectedly strong multiplicative properties.

Proposition 17.5.24. *If R is a commutative ring spectrum, then the slice tower computing the* $\mathrm{RO}(G)$*-graded homotopy of R is a spectral sequence of graded Tambara functors.*

In general, it can be difficult to describe the slice associated graded for a spectrum. For the group C_2 (or more generally, for C_p), we have complete control of the slices, and they depend only on the homotopy Mackey functors in the dimensions of regular representation spheres or regular representations spheres plus 1.

Theorem 17.5.25 ([39, 41])**.** *If E is a C_2-spectrum, then the slices are given by*

$$P_{2k}^{2k}(E) = \Sigma^{k\rho} H \underline{\pi}_{k\rho}(E) \text{ and } P_{2k+1}^{2k+1}(E) = \Sigma^{k\rho+1} H \mathcal{P}^0 \underline{\pi}_{k\rho+1}(E),$$

where $\mathcal{P}^0$ is the endofunctor on C_2 Mackey functors that kills the kernel of the restriction.

More generally, for cyclic 2-groups we understand the slices for the norms of $\mathrm{MU}_{\mathbb{R}}$. This was the key computational step in the solution to the Kervaire invariant one problem [39].

Theorem 17.5.26 ([39])**.** *If $E = N_{C_2}^{C_{2^n}} \mathrm{MU}_{\mathbb{R}}$, then the odd slices of E are contractible, while the even slices are wedges of* $\mathrm{RO}(C_{2^n})$*-graded suspensions of $H\underline{\mathbb{Z}}$, where $\underline{\mathbb{Z}}$ is the constant Mackey functor with value $\mathbb{Z}$.*

More recently, the slice spectral sequence has been applied to questions about the Hopkins–Miller higher real K-theory spectra $EO_n(G)$ for G a cyclic 2-group. Recall from Example 17.3.76 that if G is a finite subgroup of $\mathrm{Aut}(\Gamma)$ for some formal group Γ of height n over a perfect field, then we can view E_n as a G-spectrum by the Hopkins–Miller theorem. Restricting attention to the prime 2, there is a canonical order 2 automorphism of any formal group (in fact over any ring): inversion. Thus every Lubin–Tate spectrum is canonically a C_2-spectrum.

Theorem 17.5.27 ([37])**.** *Let E_n be the Lubin–Tate spectrum for a height n formal group over a perfect field of characteristic 2. Then there is a Real orientation*

$$\mathrm{MU}_{\mathbb{R}} \to E_n$$

lifting an underlying orientation.

Now if G is a finite subgroup of $\mathrm{Aut}(\Gamma)$ that contains C_2, then E_n is also a G-spectrum. Since it is the cofreed up spectrum for an E_∞-ring spectrum, it is canonically a G-E_∞-ring spectrum, and hence has norms.

Corollary 17.5.28. *If G is a finite subgroup of $\mathrm{Aut}(\Gamma)$ that contains C_2, then any Real orientation of E_n gives a G-equivariant map*

$$N_{C_2}^{G} \mathrm{MU}_{\mathbb{R}} \to E_n.$$

The slice machinery can then be used to understand the Lubin–Tate spectra computationally as G-spectra, showing how to unpack the Hurewicz image [51], how to compute the homotopy groups [42], and how to describe the Picard group [9].

Bibliography

[1] William C. Abram and Igor Kriz. The equivariant complex cobordism ring of a finite abelian group. *Math. Res. Lett.*, 22(6):1573–1588, 2015.

[2] J. F. Adams. Prerequisites (on equivariant stable homotopy) for Carlsson's lecture. In *Algebraic topology, Aarhus 1982 (Aarhus, 1982)*, volume 1051 of *Lecture Notes in Math.*, pages 483–532. Springer, Berlin, 1984.

[3] J. F. Adams, J.-P. Haeberly, S. Jackowski, and J. P. May. A generalization of the Segal conjecture. *Topology*, 27(1):7–21, 1988.

[4] Vigleik Angeltveit and Anna Marie Bohmann. Graded Tambara functors. *J. Pure Appl. Algebra*, 222(12):4126–4150, 2018.

[5] Shôrô Araki. Orientations in τ-cohomology theories. *Japan. J. Math. (N.S.)*, 5(2):403–430, 1979.

[6] James E. Arnold, Jr. Homological algebra based on permutation modules. *J. Algebra*, 70(1):250–260, 1981.

[7] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum. A naive approach to genuine g-spectra and cyclotomic spectra. arxiv.org:1710.06416, 2017.

[8] Clark Barwick. Spectral Mackey functors and equivariant algebraic K-theory (I). *Adv. Math.*, 304:646–727, 2017.

[9] Agnes Beaudry, Irina Bobkova, Michael Hill, and Vesna Stojanoska. Invertible $K(2)$-local E-modules in C_4-spectra. 1901.02109, 2019.

[10] Andrew J. Blumberg. Continuous functors as a model for the equivariant stable homotopy category. *Algebr. Geom. Topol.*, 6:2257–2295, 2006.

[11] Andrew J. Blumberg and Michael A. Hill. Operadic multiplications in equivariant spectra, norms, and transfers. *Adv. Math.*, 285:658–708, 2015.

[12] Andrew J. Blumberg and Michael A. Hill. Incomplete Tambara functors. *Algebr. Geom. Topol.*, 18(2):723–766, 2018.

[13] Peter Bonventre and Luis A. Pereira. Genuine equivariant operads. arxiv.org: 1707.02226, 2017.

[14] Serge Bouc, Radu Stancu, and Peter Webb. On the projective dimensions of Mackey functors. *Algebr. Represent. Theory*, 20(6):1467–1481, 2017.

[15] Glen E. Bredon. *Equivariant cohomology theories*. Lecture Notes in Mathematics, No. 34. Springer-Verlag, Berlin-New York, 1967.

[16] M. Brun. Witt vectors and equivariant ring spectra applied to cobordism. *Proc. Lond. Math. Soc. (3)*, 94(2):351–385, 2007.

[17] Gunnar Carlsson. Equivariant stable homotopy and Segal's Burnside ring conjecture. *Ann. of Math. (2)*, 120(2):189–224, 1984.

[18] Gunnar Carlsson. A survey of equivariant stable homotopy theory. *Topology*, 31(1):1–27, 1992.

[19] Jeffrey L. Caruso. Operations in equivariant $\mathbf{Z}/p$-cohomology. *Math. Proc. Cambridge Philos. Soc.*, 126(3):521–541, 1999.

[20] Emanuele Dotto. Higher equivariant excision. *Adv. Math.*, 309:1–96, 2017.

[21] Andreas W. M. Dress. *Notes on the theory of representations of finite groups. Part I: The Burnside ring of a finite group and some AGN-applications*. Universität Bielefeld, Fakultät für Mathematik, Bielefeld, 1971. With the aid of lecture notes, taken by Manfred Küchler.

[22] E. Dror Farjoun. Cellular inequalities. In *The Čech centennial (Boston, MA, 1993)*, volume 181 of *Contemp. Math.*, pages 159–181. Amer. Math. Soc., Providence, RI, 1995.

[23] Daniel Dugger. An Atiyah-Hirzebruch spectral sequence for KR-theory. *K-Theory*, 35(3-4):213–256 (2006), 2005.

[24] A. D. Elmendorf. Systems of fixed point sets. *Trans. Amer. Math. Soc.*, 277(1):275–284, 1983.

[25] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[26] Michikazu Fujii. Cobordism theory with reality. *Math. J. Okayama Univ.*, 18(2):171–188, 1975/76.

[27] David Gepner and Tyler Lawson. Brauer groups and Galois cohomology of commutative ring spectra. arxiv.org: 1607.01118, 2016.

[28] Saul Glasman. Stratified categories, geometric fixed points and a generalized Arone–Ching theorem. arxiv.org:1507.01976, 2017.

[29] J. P. C. Greenlees. Stable maps into free G-spaces. *Trans. Amer. Math. Soc.*, 310(1):199–215, 1988.

[30] J. P. C. Greenlees. Some remarks on projective Mackey functors. *J. Pure Appl. Algebra*, 81(1):17–38, 1992.

[31] J. P. C. Greenlees. Four approaches to cohomology theories with reality. In *An alpine bouquet of algebraic topology*, volume 708 of *Contemp. Math.*, pages 139–156. Amer. Math. Soc., Providence, RI, 2018.

[32] J. P. C. Greenlees and J. P. May. Equivariant stable homotopy theory. In *Handbook of algebraic topology*, pages 277–323. North-Holland, Amsterdam, 1995.

[33] J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. *Mem. Amer. Math. Soc.*, 113(543):viii+178, 1995.

[34] J. P. C. Greenlees and Lennart Meier. Gorenstein duality for real spectra. *Algebr. Geom. Topol.*, 17(6):3547–3619, 2017.

[35] Bertrand J. Guillou and J. Peter May. Equivariant iterated loop space theory and permutative G-categories. *Algebr. Geom. Topol.*, 17(6):3259–3339, 2017.

[36] Javier J. Gutiérrez and David White. Encoding equivariant commutativity via operads. *Algebr. Geom. Topol.*, 18(5):2919–2962, 2018.

[37] Jeremy Hahn and XiaoLin Danny Shi. Real orientation of Lubin–Tate spectra. 1707.03413, 2017.

[38] Markus Hausmann. G-symmetric spectra, semistability and the multiplicative norm. *J. Pure Appl. Algebra*, 221(10):2582–2632, 2017.

[39] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

[40] Michael A. Hill and Kristen Mazur. An equivariant tensor product on Mackey functors. *Journal of Pure and Applied Algebra*, 2019.

[41] Michael A. Hill and Lennart Meier. The C_2-spectrum $\mathrm{Tmf}_1(3)$ and its invertible modules. *Algebr. Geom. Topol.*, 17(4):1953–2011, 2017.

[42] Michael A. Hill, XiaoLin Danny Shi, Guozhen Wang, and Zhouli Xu. The slice spectral sequence of a c_4-equivariant height-4 Lubin–Tate theory. arxiv.org: 1811.07960, 2018.

[43] Michael A. Hill and Carolyn Yarnall. A new formulation of the equivariant slice filtration with applications to C_p-slices. *Proc. Amer. Math. Soc.*, 146(8):3605–3614, 2018.

[44] Mark Hovey, John H. Palmieri, and Neil P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610):x+114, 1997.

[45] Po Hu and Igor Kriz. Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence. *Topology*, 40(2):317–399, 2001.

[46] Peter S. Landweber. Conjugations on complex manifolds and equivariant homotopy of MU. *Bull. Amer. Math. Soc.*, 74:271–274, 1968.

[47] G. Lewis, J. P. May, and J. McClure. Ordinary $RO(G)$-graded cohomology. *Bull. Amer. Math. Soc. (N.S.)*, 4(2):208–212, 1981.

[48] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[49] L. Gaunce Lewis, Jr. The $R\mathrm{O}(G)$-graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbf{Z}/p$ actions. In *Algebraic topology and transformation groups (Göttingen, 1987)*, volume 1361 of *Lecture Notes in Math.*, pages 53–122. Springer, Berlin, 1988.

[50] L. Gaunce Lewis, Jr. and Michael A. Mandell. Equivariant universal coefficient and Künneth spectral sequences. *Proc. London Math. Soc. (3)*, 92(2):505–544, 2006.

[51] Guchuan Li, XiaoLin Danny Shi, Guozhen Wang, and Zhouli Xu. Hurewicz images of real bordism theory and real Johnson-Wilson theories. *Adv. Math.*, 342:67–115, 2019.

[52] Harald Lindner. A remark on Mackey-functors. *Manuscripta Math.*, 18(3):273–278, 1976.

[53] Sverre Lunø e Nielsen and John Rognes. The topological Singer construction. *Doc. Math.*, 17:861–909, 2012.

[54] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and S-modules. *Mem. Amer. Math. Soc.*, 159(755):x+108, 2002.

[55] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. *Proc. London Math. Soc. (3)*, 82(2):441–512, 2001.

[56] Michael A. Mandell. Equivariant symmetric spectra. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 399–452. Amer. Math. Soc., Providence, RI, 2004.

[57] Clover May. A structure theorem for $RO(C_2)$-graded Bredon cohomology. arxiv.org: 1804.03691, 2018.

[58] J. P. May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by

the American Mathematical Society, Providence, RI, 1996. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.

[59] Jonathan Rubin. Combinatorial N_∞ operads. arxiv.org:1705.03585, 2019.

[60] Stafan Schwede. Lectures on equivariant stable homotopy theory. Available on the author's homepage.

[61] Kazuhisa Shimakawa. Infinite loop G-spaces associated to monoidal G-graded categories. *Publ. Res. Inst. Math. Sci.*, 25(2):239–262, 1989.

[62] E. H. Spanier and J. H. C. Whitehead. A first approximation to homotopy theory. *Proc. Nat. Acad. Sci. U. S. A.*, 39:655–660, 1953.

[63] E. H. Spanier and J. H. C. Whitehead. Duality in homotopy theory. *Mathematika*, 2:56–80, 1955.

[64] Neil Strickland. Tambara functors. arXiv:1205.2516, 2012.

[65] Markus Szymik. Equivariant stable stems for prime order groups. *J. Homotopy Relat. Struct.*, 2(1):141–162, 2007.

[66] D. Tambara. On multiplicative transfer. *Comm. Algebra*, 21(4):1393–1420, 1993.

[67] Tammo tom Dieck. *Transformation groups and representation theory*, volume 766 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[68] Tammo tom Dieck. *Transformation groups*, volume 8 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.

[69] John Ullman. On the slice spectral sequence. *Algebr. Geom. Topol.*, 13(3):1743–1755, 2013.

[70] Dylan Wilson. On categories of slices. arxiv.org: 1711.03472, 2017.

[71] Klaus Wirthmüller. Equivariant homology and duality. *Manuscripta Math.*, 11:373–390, 1974.

[72] Mingcong Zeng. Equivariant Eilenberg–Mac lane spectra in cyclic p-groups. arxiv.org: 1710.01769, 2017.

Index