3b) *Solve*

$$xy'' + (2x+3)y' + (x+3)y = 3e^{-x}$$

with initial conditions $y(0) = y'(0) = 0$.

Let $Y = L[y]$. Then since $y(0) = y'(0) = 0$, we have $L[y'] = pL[y]$ and $L[y''] = p^2L[y]$. Since $L[xf] = -\frac{d}{dp}L[f]$ for any f, taking Laplace transforms of the equation to solve, we get

$$\begin{aligned}
\left(-\frac{d}{dp}\right)[p^2Y] + \left(-2\frac{d}{dp}+3\right)[pY] + \left(-\frac{d}{dp}+3\right)[Y] &= \frac{3}{p+1}\\
(-p^2Y' - 2pY) + (-2pY' - 2Y + 3pY) + (-Y' + 3Y) &= \frac{3}{p+1}\\
(-p^2 - 2p - 1)Y' + (p+1)Y &= \frac{3}{p+1}\\
-(p+1)^2Y' + (p+1)Y &= \frac{3}{p+1}\\
Y' - \frac{Y}{p+1} &= \frac{-3}{(p+1)^4}.
\end{aligned}$$

Note that we have to use product rule to expand the derivatives which involve both p and Y. We can solve the above by means of an integrating factor, which we find to be $1/(p+1)$. So we have

$$\begin{aligned}
\frac{Y'}{p+1} - \frac{Y}{(p+1)^2} &= \frac{-3}{(p+1)^4}\\
\left(\frac{Y}{p+1}\right)' &= \frac{-3}{(p+1)^4}\\
\frac{Y}{p+1} &= \frac{1}{(p+1)^3} + C\\
Y &= \frac{1}{(p+1)^2} + C(p+1).
\end{aligned}$$

Recall that the Laplace transform of any function must decay to 0 as $p \to \infty$. Therefore we must have $C = 0$, and so

$$y = L^{-1}\left[\frac{1}{(p+1)^2}\right] = xe^{-x}.$$

5a) If a and b are positive constants, evalutate

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$$

Using formula (12) on p.397 in Simmons, we have

$$
\begin{aligned}
\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx &= \int_0^\infty L[e^{-ax}-e^{-bx}]\,dp \\
&= \int_0^\infty L[e^{-ax}]-L[e^{-bx}]\,dp \\
&= \int_0^\infty \frac{1}{p+a}-\frac{1}{p+b}\,dp \\
&= [\ln(p+a)-\ln(p+b)]_0^\infty \\
&= \ln\left(\frac{p+a}{p+b}\right)\Bigg|_0^\infty \\
&= \ln(1)-\ln\frac{a}{b} \\
&= \ln\frac{b}{a}.
\end{aligned}
$$

Note that it is necessary not to separate the integral into two because we rely on cancellation between the two terms.

2b) Method 1:

We can write the greatest integer function as

$$[x]=\sum_{n=1}^\infty u(x-n),$$

where $u(x)$ is the Heaviside (step) function investigated in part a). It has already been calculated that $L[u(x-a)]=\frac{e^{-ap}}{p}$. Thus we have

$$L[[x]]=L\left[\sum_{n=1}^\infty u(x-n)\right]=\sum_{n=1}^\infty L[u(x-n)]=\sum_{n=1}^\infty \frac{e^{-np}}{p}=\frac{1}{p}\sum_{n=1}^\infty e^{-np}=\frac{1}{p}\frac{e^{-p}}{1-e^{-p}}=\frac{1}{p(e^p-1)},$$

where we have used the formula for the sum of a geometric series. As mentioned in another handout, there's some work to be done to show that the second equality is valid (implicitly we are switching a limit with an integral).

Method 2:

We have $[x]=n$ when $n\le x<n+1$. Therefore we break up the integral in the definition of the Laplace transform into integrals over the intervals $[n,n+1)$, to get

$$
\begin{aligned}
\int_0^\infty e^{-px}[x]\,dx &= \sum_{n=1}^\infty \int_n^{n+1} ne^{-px}\,dx \\
&= \sum_{n=1}^\infty n\int_n^{n+1} e^{-px}\,dx \\
&= \sum_{n=1}^\infty n\,\frac{-e^{-px}}{p}\Bigg|_n^{n+1} \\
&= \frac{1}{p}\sum_{n=1}^\infty (ne^{-pn}-ne^{-p(n+1)}).
\end{aligned}
$$

Loosely speaking, we get cancellation from successive terms, so that this becomes $1/p$ times the sum of e^{-pn}. However, to be rigorous, we write this with a limit:

$$\frac{1}{p}\lim_{N\to\infty}\sum_{n=1}^{N}(ne^{-pn} - ne^{-p(n+1)}).$$

Then we have

$$\begin{aligned}
&\sum_{n=1}^{N}(ne^{-pn} - ne^{-p(n+1)})\\
&\quad = (e^{-p} - e^{-2p}) + (2e^{-2p} - 2e^{-3p}) + (3e^{-3p} + e^{-4p}) + \cdots + (Ne^{-pN} - Ne^{-p(N+1)})\\
&\quad = e^{-p} + (2e^{-2p} - e^{-2p}) + (3e^{-3p} - 2e^{-3p}) + \cdots + (Ne^{-Np} - (N-1)e^{-Np}) - Ne^{-p(N+1)}\\
&\quad = (e^{-p} + e^{-2p} + e^{-3p} + \cdots + e^{-Np}) - Ne^{-p(N+1)}.
\end{aligned}$$

Note that aside from the terms e^{-pn} we get a remainder term $Ne^{-p(N+1)}$, which goes to 0 when we take the limit as $N \to \infty$, but in theory could have messed things up. In any case, taking the limit, the above is equal to

$$\sum_{n=1}^{\infty} e^{-pn} = \frac{e^{-p}}{1-e^{-p}} = \frac{1}{e^p - 1}.$$

Thus we have

$$L[[x]] = \frac{1}{p(e^p - 1)}.$$

As an example of why the remainder term could make a difference, if we ignored such things and abused dots all the time, we could write

$$\begin{aligned}
1 + (2-1) + (3-2) + (4-3) + \cdots &= (1-1) + (2-2) + (3-3) + \cdots\\
&= 0 + 0 + 0 + \cdots\\
&= 0.
\end{aligned}$$

Taking truncated sums, we would see that

$$\begin{aligned}
1 + (2-1) + (3-2) + \cdots + ((N+1) - N) &= (1-1) + (2-2) + (3-3) + \cdots + (N-N) + N + 1\\
&= (0 + 0 + 0 + \cdots + 0) + N + 1\\
&= N + 1,
\end{aligned}$$

so there's a remainder term which does not die off as $N \to \infty$ (and makes all the difference).

2c) Find the Laplace transform of $x - [x]$.

By linearity we have

$$L[x - [x]] = L[x] - L[[x]] = \frac{1}{p^2} - \frac{1}{p(e^p - 1)} = \frac{e^p - 1 - p}{p^2(e^p - 1)}.$$

2d) Find the Laplace transform of $f(x)$, where

$$f(x) = \begin{cases} \sin(x), & 0 \le x \le \pi \\ 0, & x \ge \pi \end{cases}.$$

We have

$$L[f(x)] = \int_0^\infty e^{-px}\sin(x)\,dx = \int_0^\pi \sin(x)\,dx.$$

We can do the integral directly (integrate by parts twice to express the relevant integral as something minus itself, and then solve algebraically), but more interestingly it is possible to avoid this. We have

$$L[f(x)] = \int_0^\infty e^{-px}\sin(x)\,dx - \int_\pi^\infty e^{-px}\sin(x)\,dx$$

The first integral is equal to $L[\sin(x)]$, while the second is similar because of the symmetry of the exponential and sine functions. Specifically, since $\sin(x+\pi) = -\sin(x)$ and $e^{-p(x+\pi)} = e^{-p\pi}e^{-px}$, we can use a change of variables $x = y + \pi$ to write

$$\begin{aligned}\int_\pi^\infty e^{-px}\sin(x)\,dx &= \int_0^\infty e^{-p(y+\pi)}\sin(y+\pi)\,dy\\ &= e^{-\pi p}\int_0^\infty e^{-py}(-\sin(y))\,dy\\ &= -e^{-\pi p}L[\sin(y)].\end{aligned}$$

Therefore

$$\begin{aligned}L[f(x)] &= L[\sin(x)] - (-e^{-\pi p})L[\sin(x)]\\ &= (1+e^{-\pi p})L[\sin(x)]\\ &= \frac{1+e^{-\pi p}}{p^2+1}.\end{aligned}$$