Areas between curves

Section 6.3 concerns finding the area of a region bounded by several curves. The general idea is that, since $\int_a^b f(x)\,dx$ is the (signed) area between the x-axis and the graph of f on the interval $[a, b]$, the area of a region bounded above by a curve $y = f(x)$ (the "top function") and below by a curve $y = f(x)$ (the "bottom function"), and between $x = a$ and $x = b$, is given by

$$\int_a^b f(x) - g(x)\,dx.$$

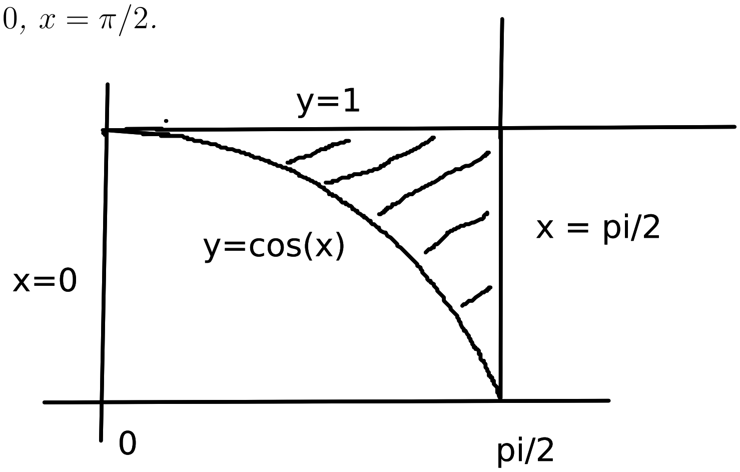
The main steps in the relevant problems are usually to find appropriate limits of integration and a formula (top function) - (bottom function).

Notes:

- It can be difficult to set up the integral if you can't sketch the region. Practice doing so, but also note that you can use algebra and calculus to determine roughly what the picture looks like. Start by finding the points of intersection of the boundary lines and curves.
- It may be necessary to find the points where several curves intersect to get the limits of integration.
- The top and bottom functions may change. In this case, split up the integral into pieces where you have a consistent formula for each piece.
- Sometimes it is easier to integrate in y, using (right function) - (left fucntion) instead of top - bottom. (See #6 below.)

Section 6.3, #4)

$y = \cos x,\ y = 1,\ x = 0,\ x = \pi/2.$



The area is

$$\int_0^{\pi/2} 1 - \cos(x)\,dx.$$

Note that $\cos(x) \geq 0$ for all x between 0 and $\pi/2$, so there is a consistent formula (top function) - (bottom function) $= \cos(x) - 0$. $x = 0$ isn't really a boundary of the region – they gave it to make it a little easier to set up the limits of integration.

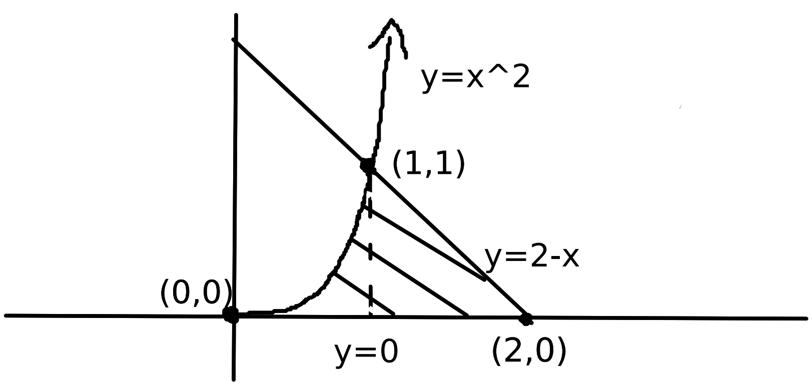
Section 6.3, #6)

$y = x^2$, $y = 2 - x$, $y = 0$.

First find the points of intersection:

- $y = x^2$ intersects $y = 0$ at (0,0)
- $y = 2 - x$ intersects $y = 0$ at (2,0)
- $y = x^2$ intersects $y = 2 - x$ when $x^2 = 2 - x \implies x^2 + x - 2 = 0 \implies x = -2, 1$, hence at the points (-2,4) and (1,1).

The corners of the region are at (0,0), (2,0), and (1,1):



We have two options:

1. Integrate in x:

   There isn't a consistent formula for (top function) - (bottom function), so we split up the region between $x = 0$ and $x = 1$, and between $x = 1$ and $x = 2$. On [0,1], the top function is x^2 and the bottom function is 0, and on [1,2] the top function is $2 - x$ and the bottom function is 0. Hence the area is

$$\begin{aligned}\int_0^1 x^2 - 0\,dx + \int_1^2 (2-x) - 0\,dx &= \frac{1}{3}x^3\Big|_0^1 + (2x - \frac{1}{2}x^2)\Big|_1^2 \\ &= \frac{1}{3} + [(4-2) - (2 - \frac{1}{2})] \\ &= \frac{5}{6}.\end{aligned}$$

2. Alternatively, we can integrate in y. Instead of (top function) - (bottom function) the formula becomes (right function) - (left function). We should solve for x to get formulas for the two curves:

- $y = x^2 \implies x = \sqrt{y}$ (note that $x > 0$, so it's the positive square root)
- $y = 2 - x \implies x = 2 - y$.

The right function is $2 - y$, and the left is $\sqrt{y}$. The area is thus

$$\int_0^1 (2 - y) - \sqrt{y}\, dy = 2y - \frac{1}{2}y^2 - \frac{2}{3}y^{3/2}\Big|_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}.$$

- When taking square roots, be mindful of whether you need the positive or negative root. You ought to be able to tell which sign it should have. In this example, since we're in the first quadrant and therefore $x > 0$, we know to set $x = \sqrt{y}$ instead of $x = -\sqrt{y}$.

Differential Equations

A differential equation (DE) is an equation involving a function and its derivatives. Most of the ones we deal with will look like $\frac{dy}{dx} = f(x, y)$ for some function $f(x, y)$. When the right hand side does not depend on x, the equation is called *autonomous*.

To find the *general solution* means to find all possible solutions. To find a *particular solution* means to find a single function, usually determined by requiring that the function satisfy some initial data, which the book usually denotes (x_0, y_0). This means that we want the solution $y = y(x)$ to satisfy $y(x_0) = y_0$.

Equations of the form $\frac{dy}{dx} = f(x)$

All we have to do is find an antiderivative for $f(x)$, and if appropriate use the initial data to find a value for the additive constant.

Section 8.1, # 2) $\frac{dy}{dx} = e^{-3x}$, $y_0 = 10$, $x_0 = 0$.

We are looking for a function $y = y(x)$ such that $y(0) = 10$, and $y'(x) = e^{-3x}$. The antiderivative is

$$\int e^{-3x}\, dx = -\frac{1}{3}e^{-3x} + C,$$

so the solution must be of the form $y(x) = -\frac{1}{3}e^{-3x} + C$. Using the initial data, we get

$$10 = y_0 = y(x_0) = y(0) = -\frac{1}{3}e^0 + C = C - \frac{1}{3}.$$

Thus $C = 10 + \frac{1}{3} = \frac{31}{3}$, so our final answer is

$$y(x) = -\frac{1}{3}e^{-3x} + \frac{31}{3}.$$

Section 8.1, # 10) *The amount of phosphorus in a lake at time* t*, denoted* $P(t)$*, satisfies, follows the equation* $\frac{dP}{dt} = 3t + 1$*, with* $P(0) = 0$*. Find the amount of phosphorus at time* $t = 10$*.*

We can find a formula for $P(t)$ by finding an antiderivative as in the previous problem. ($P(t) = \frac{3}{2}t^2 + t + C$, solving for C gives $C = 0$.) Alternatively, we can use the Fundamental Theorem of Calculus to write

$$P(10) - P(0) = \int_0^{10} P'(t)\,dt.$$

Hence,

$$P(10) = P(0) + \int_0^{10} P'(t)\,dt = 0 + \int_0^{10} 3t + 1\,dt = \frac{3}{2}t^2 + t\Big|_0^{10} = \frac{3}{2}(100) + 10 = 160.$$

Equations of the form $\frac{dy}{dx} = f(y)$

We use separation of varaiables: Put all y's on one side of the equation and all x's on the other side, including differentials, and integrate, to get

$$\int \frac{dy}{f(y)} = \int dx.$$

Then integrate. The above is an informal manipulation, but is valid as long as the integrals are well-behaved.

On that note, it's a good policy to check for *equilibria* first, i.e. values of y such that $f(y) = 0$. Note that since we are dividing by $f(y)$, the equilibria are exactly the points where the integral on the left

Section 8.1, #12) $\frac{dy}{dx} = 2(1 - y)$, $y_0 = 2$, $x_0 = 0$.

$2(1 - y) = 0$ implies $y = 1$, so $y = 1$ is the only equilibrium point. Since we don't start at the equilibrium point, we proceed analytically.

We rearrange formally to get

$$\frac{dy}{1 - y} = 2dx,$$

and then integrate:

$$\begin{aligned}
\int \frac{dy}{1-y} &= \int 2\,dx \\
-\ln|1-y| &= 2x + C \\
\ln|1-y| &= -2x + C \\
|1-y| &= e^{-2x+C} = e^{-2x}e^{C} \\
\pm(1-y) &= Ce^{-2x} && (C > 0) \\
1-y &= Ce^{-2x} && (C \neq 0) \\
y &= 1 - Ce^{-2x} && (C \neq 0)
\end{aligned}$$

Note that we often change to a different constant C without giving it a new name. The notes on the right are to indicate what is know about the constant C. For example, when we make e^C the new C, we know that C can not be ≤ 0.

Plugging in our intial data, we get

$$2 = 1 - Ce^0 = 1 - C,$$

so $C = -1$. Thus our final solution is

$$y(x) = 1 + e^{-2x}.$$

We can check that

$$\frac{dy}{dx} = -2e^{-2x} = 2(1 - (1 + e^{-2x}) = 2(1 - y).$$

Section 8.1, #36) *Find a solution of* $\frac{dy}{dx} = y^2 + 4$ *that passes through (0,2).*

In other words, we want the solution to satisfy $y(0) = 2$. Note that 2 is not an equilibrium point, and in fact there are no equilibria since $y^2 + 4$ is never equal to 0. Thus we proceed analytically.

Separating variables and integrating, we get

$$\begin{aligned}
\int \frac{dy}{y^2+4} &= \int dx \\
\frac{1}{4}\int \frac{dy}{(y/2)^2+1}\,dy &= x + C \\
\frac{1}{4}\frac{1}{1/2}\arctan(y/2) &= x + C \\
\arctan(y/2) &= 2x + C \\
\frac{y}{2} &= \tan(2x + C) \\
y &= 2\tan(2x + C).
\end{aligned}$$

The initial data gives

$$2 = 2\tan(C)$$
$$1 = \tan(C)$$

so we may choose $C = \arctan(1) = \pi/4$. Our final solution is

$$y = 2\tan(2x + \pi/4).$$

To check our answer, we see that $\frac{dy}{dx} = 4\sec^2(2x+\pi/4) = 4(\tan^2(2x+\pi/4+1) = (2\tan(2x+\pi/4))^2 + 4 = y^2 + 4$, so we indeed have a solution.

Section 8.1, #40)(b) We have the equation $\frac{dN}{dt} = 0.7N(1 - \frac{N}{35})$. We wish to find a formula for $N(t)$ given various initial conditions, and in each case find $\lim_{t\to\infty} N(t)$.

First we investigate equilibria: $0.7N(1 - \frac{N}{35}) = 0$ when $N = 0$ or $N = 35$, so we have two equilibria. Solutions that start at these points will remain there forever. Assuming the solution does not start at an equilibrium point, we separate variables and integrate to get

$$\int \frac{dN}{N(1 - \frac{N}{35})} = \int 0.7\,dx$$
$$\int \frac{35}{N(35 - N)}\,dN = \int 0.7\,dx.$$

For the N integral we use partial fractions decomposition:

$$\frac{35}{N(35 - N)} = \frac{A}{N} + \frac{B}{35 - N}$$
$$35 = A(35 - N) + BN.$$

Setting $N = 35$ gives $B = 1$, and setting $N = 0$ gives $A = 1$. Thus the integral is

$$\int \frac{35}{N(35 - N)}\,dN = \int \frac{1}{N} + \frac{1}{35 - N}\,dN$$
$$= \ln|N| - \ln|35 - N| + C$$

So we get

$$\ln|N| - \ln|35 - N| = 0.7t + C$$
$$\ln\left|\frac{N}{35 - N}\right| = 0.7t + C$$
$$\left|\frac{N}{35 - N}\right| = e^{0.7t+C}$$
$$\frac{N}{35 - N} = Ce^{0.7t}, \qquad (C \neq 0)$$
$$N = 35Ce^{0.7t} - NCe^{0.7t}$$
$$N(1 + Ce^{0.7t}) = 35Ce^{0.7t}$$

so

$$N(t) = \frac{35Ce^{0.7t}}{1 + Ce^{0.7t}} = \frac{35}{\frac{1}{C}e^{-0.7t} + 1}.$$

Note that $\frac{1}{C}$ is okay to write since we know $C \neq 0$.

Now we investigate what happens with various initial conditions:

- $N(0) = 10$. Since 10 is not an equilibrium, we have the formula above, and use the intial data to find C:

  $$10 = \frac{35}{\frac{1}{C}e^{0} + 1}$$

  which gives $C = \frac{2}{5}$. Then

  $$N(t) = \frac{35}{\frac{5}{2}e^{-0.7t} + 1},$$

  and we find

  $$\lim_{t \to \infty} N(t) = \frac{35}{0 + 1} = 35.$$

- $N(0) = 35$

  Since $N = 35$ is an equilibrium, the analysis above is unnecessary, and we know the solution is

  $$N(t) = 35.$$

  Also, $\lim_{t \to \infty} N(t) = \lim_{t \to \infty} 35 = 35$.

- $N(0) = 50$

  50 is not an equilibrium, so we have the formula above, and use the intial data to find C:

  $$50 = \frac{35}{\frac{1}{C}e^{0} + 1}$$

  so $C = \frac{7}{3}$. Thus the solution is

  $$N(t) = \frac{35}{\frac{3}{7}e^{-0.7t} + 1}.$$

  and

  $$\lim_{t \to \infty} N(t) = \frac{35}{0 + 1} = 35.$$

- $N(0) = 0$

  Since $N = 0$ is the other equilibrium, the solution is simply

  $$N(t) = 0,$$

  and $\lim_{t \to \infty} N(t) = 0$.